Constructive Completness. An application.

Thierry Coquand Jan M. Smith

May 2, 1995

This is a first

DRAFT

Introduction

The completeness theorem for first-order logic is one of the basic result of classical model theory. It states that a first-order sentence is provable if and only if it holds in any possible models, or, in a relativised form, that a first-order sentence is derivable in a theory if and only if it holds in any model of this theory. This expresses a strong relation between syntax and semantics and can be used to give elegant semantical proofs of purely syntactical properties. One typical use is for proving conservativity results. To prove for instance that a first-order theory T_2 is a conservative extension of a theory T_1, it is enough to show that any model of T_1 can be extended to a model of T_2. It follows then directly that a formula derivable in T_2 holds in any model of T_1, and hence, by the completeness theorem, is also derivable in T_1.

The usual proofs of the completeness theorem are non-constructive. In this paper, we explore one possible effective version of this theorem, that uses topological models in a point-free setting, following Sambin [6]. The truth-values, instead of being simply booleans, can be arbitrary open of a given topological space. There are two advantages with considering this more abstract notion of models. The first is that, by using point-free topology, we get a remarkably simple completeness proof; it seems indeed simpler than the usual classical completeness proof by Henkin. The second is that this completeness proof is now constructive, and can be done in intuitionistic type theory.

In view of the extreme simplicity of this proof, it might be feared that this version is essentially weaker than its classical counterpart. It can be thought that the strength, and elegance, of the classical completeness theorem is connected to its non-effective character. We show that this is perhaps not the case, by analysing a conservativity theorem due to Dragalin [3]. We can transpose directly the usual model theoretic conservativity argument, that we sketched above, in our framework. It seems likely that a direct syntactical proof of this result would have to be more involved.

The first part of this paper presents a definition of topological models, and a completeness proof, based on Sambin [6]. The second part shows how to use this in order to give a proof of Dragalin's conservativity result; our proof is different from his and, we believe, simpler.

1 Intuitionistic Model Theory

A topology $T = \langle S, \cdot, 1, \lhd \rangle$ is a commutative idempotent monoid $\langle S, \cdot, 1 \rangle$ with a covering relation $\lhd$ which satisfies the following rules.

Reflexivity

$$\frac{a \in U}{a \lhd U}$$

Transitivity

$$\frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

Right

$$\frac{a \lhd U}{a \cdot b \lhd U}$$

Stability

$$\frac{a \lhd U \qquad b \lhd V}{a \cdot b \ \lhd \ U \cdot V}$$

where $U \lhd V$ means that every element of U is covered by V. Intuitively, the elements of S are the basic opens and the multiplication $\cdot$ corresponds to the intersection of basic open. An open is represented by the set of basic open it contains. For the details of point-free topology in a type theoretic setting, we refer to Sambin [5].

1.1 Topological interpretations

A *point-free topological interpretation* of a first order language consists of the following.

- A topology $T = \langle S, \cdot, 1, \lhd \rangle$.
- A set D, the domain of the interpretation.
- To each individual constant a an element $\overline{a}$ in D and to each function constant f^n of arity n a function $\overline{f}^n$ in $D^n \to D$.
- To each relation R^n of arity n a function $\overline{R}^n$ that gives an open set to each element in D^n.

Given an assignment σ of an element in D to each variable, we associate an element t^σ in D to each term t by

$$\begin{aligned} x^\sigma &= \sigma(x) \\ a^\sigma &= \overline{a} \\ f^n(t_1, \ldots, t_n)^\sigma &= \overline{f}^n({t_1}^\sigma, \ldots, {t_n}^\sigma) \end{aligned}$$

We can now associate an open set $[\![A]\!]^\sigma$ to each formula A by induction as follows.

1. $[\![R(t_1, \ldots, t_n)]\!]_\sigma \;=\; \overline{R}^n({t_1}^\sigma, \ldots, {t_n}^\sigma)$
2. $[\![\top]\!]_\sigma \;=\; 1$
3. $[\![\bot]\!]_\sigma \;=\; \emptyset$
4. $[\![A \& B]\!]_\sigma \;=\; [\![A]\!]_\sigma \cdot [\![B]\!]_\sigma$
5. $[\![A \vee B]\!]_\sigma \;=\; [\![A]\!]_\sigma \cup [\![B]\!]_\sigma$
6. $[\![A \supset B]\!]_\sigma \;=\; \{s \in S \,:\, \{s\} \cdot [\![A]\!]_\sigma \lhd [\![B]\!]_\sigma\}$
7. $[\![\exists x B(x)]\!]_\sigma \;= \bigcup_{d \in D} [\![B(d)]\!]_\sigma$
8. $[\![\forall x B(x)]\!]_\sigma \;=\; \{s \in S \,:\, (\forall d \in D)(s \lhd [\![B(d)]\!]_\sigma)\}$

In this definition, $[\![B(d)]\!]_\sigma$ is an abbreviation of $[\![B(x)]\!]_{\sigma[x:=d]}$, where the assignment $\sigma[x := d]$ is obtained from σ by giving the variable x the value d.

We say that a formula A is *valid* in an interpretation if $1 \lhd [\![A]\!]_\sigma$ holds for every assignment σ. A *model* of a set Γ of formulas is an interpretation in which all formulas of Γ are valid.

A topological interpretation can be seen as a generalization of the ordinary classical notion of interpretation in the sense that the the two truth values *true* and *false* are replaced by the much richer structure of an arbitrary topological space. Topological interpretations of intuitionistic propositional logic were considered already in the thirties by Tarski [7].

Gödel and Kreisel [4] have showed that a constructive completeness proof for predicate logic is impossible with the usual intuitive definition of validity in all structures, which, in fact, turns out to be the same as validity in all Beth models; see Troelstra [8]. But by weakening the interpretation of absurdity by allowing it to possibly hold in some nodes of a Beth model, it is possible to prove completeness [10, 2]. Models where $k \Vdash \bot$ is not ruled out are called *exploding*; and the reason why Gödel and Kreisel's argument is not applicable on exploding models is the fact that $k \Vdash \bot$ is in general not decidable.

The point-free formulation of topological models automatically includes the exploding ones: there might be basic opens which are covered by the empty set, that is, the interpretation of absurdity; and in general it is not decidable whether $a \lhd \emptyset$ holds. We will describe in more detail below how an exploding Beth model can easily be formulated in a point-free setting. This is in contrast to the traditional approach to topological models where it seems difficult to directly introduce exploding models: in order to represent the forcing relation, a covering relation must be introduced [9].

We can now prove the following soundness theorem, where $A_1, \ldots, A_n \vdash A$ means that A is derivable from $A_1, \ldots, A_n$ in intuitionistic predicate logic, formulated in natural deduction.

Theorem 1 *If $A_1, \ldots, A_n \vdash A$, then $[\![A_1]\!]_\sigma \cdot \ldots \cdot [\![A_n]\!]_\sigma \lhd [\![A]\!]_\sigma$ holds for every topological interpretation and for every assignment σ.*

Proof. Straightforward by induction on the length of the derivation. □

When $n = 0$, $[\![A_1]\!]_\sigma \cdot \ldots \cdot [\![A_n]\!]_\sigma$ should be understood as the full space S; hence we get as a corollary that if A is a logical truth, then $[\![A]\!]_\sigma$ covers the whole space, or equivalently, $1 \lhd [\![A]\!]_\sigma$.

1.2 The completeness proof

In this section we let $\vdash$ denote the derivability relation in some arbitrary first-order theory. The next theorem expresses completeness.

Theorem 2 *If $[\![A_1]\!]_\sigma \cdot \ldots \cdot [\![A_n]\!]_\sigma \lhd [\![A]\!]_\sigma$ holds for every topological interpretation and every assignment σ, then $A_1, \ldots, A_n \vdash A$.*

The proof of the completeness is by constructing a topology that is universal in the sense that $[\![A_1]\!]_\sigma \cdot \ldots \cdot [\![A_n]\!]_\sigma \lhd [\![A]\!]_\sigma$ holds for the topology and every assignment σ if and only if $A_1, \ldots, A_n \vdash A$.

The universal model is a term model, that is, the domain is the set of terms of the language and a term is interpreted as itself. The topology of the interpretation is obtained from the monoid $\langle L, \cdot, 1\rangle$ of formulas with provable equivalence as equality

$$A = B \text{ if and only if } \vdash A \leftrightarrow B,$$

and the operation $\cdot$ defined by

$$A \cdot B \;=\; A \& B.$$

The unit of the monoid is defined by $1 = \top$. Clearly, the operation $\cdot$ is well defined, that is, if $A = A'$ and $B = B'$ then $A \cdot B = A' \cdot B'$.

A is covered by a set U if and only if every proposition C that can be proved from each of the formulas in U can also be proved from A:

$$A \lhd U \;=\; (\forall C)((\forall D \in U)(D \vdash C) \Rightarrow A \vdash C).$$

Note that this definition respects equality: if $A \lhd U$ and $A = B$ then $B \lhd U$.

The open set associated with an atomic proposition is defined to be the set of formulas which proves it:

$$\overline{R}^n(t_1, \ldots, t_n) \; = \; \{A \, : \, A \vdash R^n(t_1, \ldots, t_n)\}.$$

We call the topology $\mathcal{L} = \langle L, \cdot, 1, \lhd \rangle$, the *Tarski-Lindenbaum topology.* We will write $[\![A]\!]$ for $[\![A]\!]_\sigma$ when σ is the identity assignment.

From the definition of covering and that $[\![\bot]\!] = \emptyset$ we get that $A \lhd \emptyset$ if and only if $A \vdash \bot$; hence, $A \lhd \emptyset$ is in general not decidable.

The open set $[\![A]\!]$ associated to a formula A is the set of all formulas which prove it:

Proposition 1 *For the Tarski-Lindenbaum topology,*
$B \lhd [\![A]\!]$ *if and only if* $B \vdash A$.

Proof: By a straightforward induction on the length of the formula A. □

Note that completeness follows: if for every topology $[\![A_1]\!]_\sigma \cdot \ldots \cdot [\![A_n]\!]_\sigma \lhd [\![A]\!]_\sigma$ holds for all assignments σ, then, in particular, it holds for the Tarski-Lindenbaum topology and the identity assignment; hence, by the proposition 1, $A_1, \ldots, A_n \vdash A$. This remarkably simple proof should be compared with Henkin's proof for first order classical logic; but, of course, the notion of model used here is weaker than the usual one. Tarski's completeness proof for ordinary topological interpretations of intuitionistic logic is based on similar ideas as Henkin's proof.

1.3 Connection with Beth models

There are two versions of exploding models, one for Kripke models [10] and one for Beth models [2]. We shall explain how exploding Beth models can be seen as particular cases of topological models.

First we reformulate the notion of *spread* in the context of formal topology. We use variables $u, v, \ldots$ for finite sequences of integers. We write concatenation by juxtaposition, and $v \leq u$ means that v is of the form $un_1 \ldots n_p$, where possibly p may be 0 (in which case $u = v$.) We say that u, v are incompatible if and only if neither of the relation $u \leq v$ nor $v \leq u$ hold. A spread S is then a decidable inhabited set of sequences of integers such that

1. if $v \leq u$ and $u \in S$, then $v \in S$,
2. if $u \in S$, then there exists n such that $un \in S$.

To each spread S we associate a formal space $X(S)$ defined as follows. As a semi-lattice, $X(S)$ is formed by the disjoint union of S and an extra element Δ. The product operation is defined such that

- $u \cdot v = v$ if $v \leq u$,

- $u \cdot v = \Delta$ if u, v are incompatible,
- $x \cdot y = \Delta$ if x or y is Δ.

The covering relation $u \lhd U$ is inductively defined by the clauses

- $\Delta \lhd \emptyset$,
- $v \lhd U$ if $v \leq u$ and $u \in U$,
- $u \lhd U$ if $un \lhd U$ for all n.

Notice that all elements of S are *positive* for this notion of covering: if $u \lhd U$ and $u \in S$, then U is inhabited. This follows from the second clause of the definition of a spread.

A non-exploding Beth model over the spread S corresponds then exactly to a topological model over the space $X(S)$ and we have $u \in [\![A]\!]$ if and only if $u \Vdash A$. An exploding Beth model is a model where we allow $u \Vdash \bot$ for some $u \in S$. The corresponding notion of space is obtained as follows. Given a subset $E \subseteq S$, we defined a new space $Y(S)$ by adding to the inductive definition of $u \lhd U$ the clause that $u \lhd \emptyset$ if $u \in E$. The semilattice structure of $Y(S)$ is the same as the one of $X(S)$. We can then verify that the equivalence $u \in [\![A]\!]$ if and only if $u \Vdash A$ still holds. The result of Gödel on the impossibility of getting a completeness theorem with non-exploding Beth models can then be formulated as the fact that in order to get a completeness theorem, it is essential to consider *non*-decidable subsets E.

2 Application to a Conservativity Result

The conservativity result we present is due to Dragalin [3].

2.1 Non-standard arithmetic

We first extend the language of HA with a new constant ∞ and for each numeral $\overline{n}$ add the axiom

1. $\overline{n} < \infty$

Clearly, this extension is conservative over HA: only a finite number of the new axioms can appear in a derivation; so if we just replace ∞ with a sufficiently big numeral in the derivation, we obtain a derivation in HA. We denote this extension by HA^∞.

We next extend HA^∞ with a new predicate $\mathsf{F}(x)$, expressing that x is a standard, or feasible, number. For the new predicate we add the axiom

2. $\mathsf{F}(0)$

3. $\neg\mathsf{F}(\infty)$

4. $x = y \wedge \mathsf{F}(x) \supset \mathsf{F}(y)$

5. $y < x \wedge \mathsf{F}(x) \supset \mathsf{F}(y)$

6. Let f be an arbitrary function constant of HA. Then

$$\mathsf{F}(x_1) \wedge \cdots \wedge \mathsf{F}(x_n) \supset \mathsf{F}(f(x_1, \ldots, x_n))$$

7. Induction scheme for standard numbers:

$$A(0) \wedge \forall x(\mathsf{F}(x) \wedge A(x) \supset A(s(x))) \supset \forall x(\mathsf{F}(x) \supset A(x))$$

where $A(x)$ is an arbitrary formula of the extended language.

From now on we let $\lhd$ denote the covering relation of an arbitrary model for HA^∞. We define an interpretation of F by

$$[\![\mathsf{F}(t)]\!]_\sigma \;=\; \bigcup_{n \in N} [\![t_\sigma = \overline{n}]\!]$$

In this interpretation the above axioms are all validated:

Lemma 1 *Let P be any of the above axioms about F. Then $1 \lhd [\![P]\!]$ in any model of HA^∞.*

Proof. The proof is straightforward and almost the same as the corresponding part of the proof by Dragalin [3]. □

We let $\mathsf{HA}^\infty\mathsf{F}$ denote HA extended with these axioms. Our main result is the following.

Theorem 3 *$\mathsf{HA}^\infty\mathsf{F}$ is conservative over HA.*

Proof. It is enough to prove that $\mathsf{HA}^\infty\mathsf{F}$ is conservative over HA^∞. So let A be formula of HA which is derivable in $\mathsf{HA}^\infty\mathsf{F}$, that is, there is a finite conjunction Γ of axioms about F such that $\mathsf{HA}^\infty + \Gamma \vdash A$. By theorem 1 we have that $\Gamma \lhd [\![A]\!]$ holds in every model of HA^∞. By lemma 1 we then obtain $1 \lhd [\![A]\!]$ in every model of HA^∞; hence, by theorem 1, $\mathsf{HA}^\infty \vdash A$. □

Let $\mathsf{PA}^\infty\mathsf{F}$ denote PA extended by the above axioms. We then have

Corollary 1 *$\mathsf{PA}^\infty\mathsf{F}$ is conservative over PA.*

The proof of the corollary is by Gödel's double negation interpretation. Let A^* denote the double negation interpretation of the formula A and P the set of axioms above about ∞ and F. Let A be a formula in the language of PA such that $PA + P \vdash A$. Since $\Gamma \vdash B$ implies $\Gamma^* \vdash B^*$ for all sets Γ of formulas and formulas B, we get $\mathsf{HA} + P^* \vdash A^*$. Hence, by the lemma below, $\mathsf{HA}^\infty\mathsf{F} \vdash A^*$. By the theorem, we then get $\mathsf{HA} \vdash A^*$. Since $\mathsf{PA} \vdash A^* \supset A$ we obtain $\mathsf{PA} \vdash A$. □

Lemma 2 *Let P be any of the axioms in* $\mathsf{PA}^\infty\mathsf{F}$. *Then* $\mathsf{HA}^\infty\mathsf{F} \vdash P^*$.

The only non-trivial case is when F is an instance of the induction scheme in $\mathsf{HA}^\infty\mathsf{F}$. So F^* is

$$A^*(0) \wedge \forall x(\neg\neg\mathsf{F}(x) \wedge A^*(x) \supset A^*(s(x))) \supset \forall x(\neg\neg\mathsf{F}(x) \supset A^*(x))$$

for some formula $A(x)$. Since all double negated interpreted formulas are stable, we have $\vdash \neg\neg A^*(x) \supset A^*(x)$ which implies $\vdash (\mathsf{F}(x) \supset A^*(x)) \supset (\neg\neg\mathsf{F}(x) \supset A^*(x))$. Hence, F^* follows from F. $\square$

Since the arguments of the paper are effective, the proof of this corollary can be seen directly as an algorithm that, given a proof of a sentence in $\mathsf{PA}^\infty\mathsf{F}$, transforms this into a proof of the same sentence in PA. Cederquist [1] has developed point-free topology in the computer system ALF and, based on this, Henrik Persson has expressed the above completeness proof in ALF; a proof in ALF of the conservativity result will then give a computer implementation of the transformation algorithm.

Acknowledgement

We want to thank Erik Palmgren for suggesting us to look at Dragalin's paper.

References

[1] Jan Cederquist. A machine assisted formalization of pointfree topology in type theory. Licentiate thesis, Chalmers University of Technology and University of Göteborg, Sweden, 1994.

[2] H.C.M. de Swart. An intuitionistically plausible interpretation of intuitionistic logic. *Journal of Symbolic Logic*, 42:564–578, 1977.

[3] Albert G. Dragalin. An explicit boolean-valued model for non-standard arithmetic. *Publ. Math. Drebecen*, 42:369–389, 1993.

[4] G. Kreisel. On weak completeness of intutionistic predicate logic. *Journal of Symbolic logic.*, 27:139–158, 1962.

[5] Giovanni Sambin. Intuitionistic Formal Spaces - A First Communication. In *The Proceedings of Conference on Logic and its Applications, Bulgaria.* Plenum Press, 1986.

[6] Giovanni Sambin. Pretopologies and completeness proofs. Technical report, Dip.di Matematica Pura e Appl., Univ. di Padova, August 1992.

[7] A. Tarski. Der Aussagenkalkül und die Topologie. *Fundamenta Mathematicae*, 31:103–134, 1938.

[8] A. S. Troelstra. *Choice Sequences, a Chapter of Intuitionistic Mathematics*. Clarendon Press, 1977.

[9] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics. An Introduction*, volume II. North-Holland, 1988.

[10] Wim Veldman. An intuitionistic completeness theorem for intuitionistic predicate logic. *Journal of Symbolic Logic*, 41:159–166, 1976.