A constructive ∞-groupoid model of homotopy type theory

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Outline

Background: what are ∞-groupoids?

Problem: interpret HoTT *effectively* in ∞-groupoids.

- Current state: no model does this!

A solution:

- intuition
- properties
- applications

What are ∞-groupoids?

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If we are already in a univalent metatheory:
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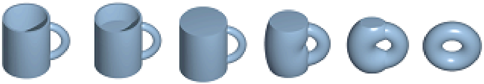
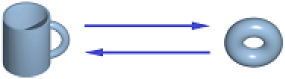
But if we have to start from **sets**?

∞-groupoids from sets

Historically: homotopy types, i.e.,

- "spaces" up to "homotopy equivalence".





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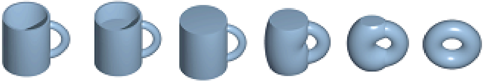
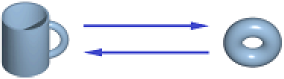
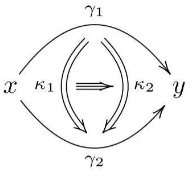




Brings point-set topology into the picture. :(

∞-groupoids from sets: higher setoids

Better: sets with a higher-dimensional equivalence relation.



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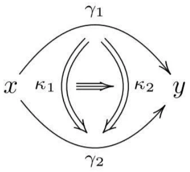

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- a set X_0 of *points*,

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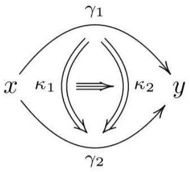

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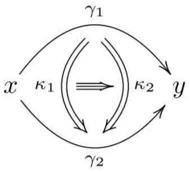

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- a set X_0 of *points*,
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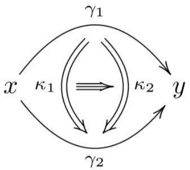

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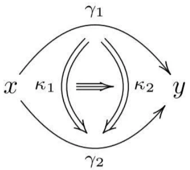

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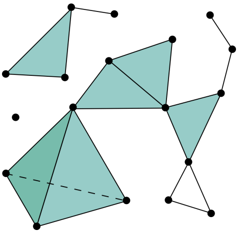
The ∞-exact completion of sets.

∞-groupoids from sets: Kan semisimplicial sets

A precise definition:
Kan semisimplicial sets up to homotopy equivalence.



Kan operation:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

"horn filling"

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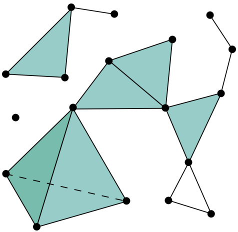

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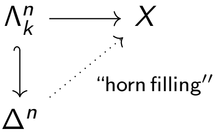

- Semisimplex category Δ_+: inhabited finite linear orders.
- Kan semisimplicial sets: full subcategory of $\mathcal{P}(\Delta_+)$ of objects with Kan operation.

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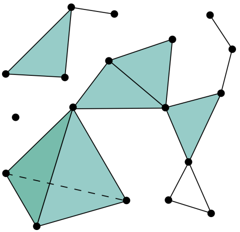

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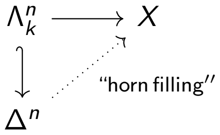

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Semisimplicial sets carry Kan weak model structure (Henry, 2020).
Quillen equivalent to simplicial sets (Henry, 2020).

∞-groupoids from sets: desired properties

A correct notion of ∞-groupoids should satisfy:

- Restricting to 0-truncated ∞-groupoids recovers setoids.
- A family over an ∞-groupoid X is contractible exactly if the fiber over each point of X is contractible over each point of X.

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Consequences:

- choice for discrete ∞-groupoids,
- presentation: every ∞-groupoid is covered by a projective one.

Types and ∞-groupoids

(Lumsdaine, 2010) and (van den Berg and Garner, 2011):

- Martin-Löf identity types endow every type with the structure of an ∞-groupoid!

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But wait, hasn't this problem been solved already?

State of the art: constructive models of HoTT

By HoTT, I mean "book HoTT": all type formers, fully split.

Ref	Setting	Model of HoTT?	Presents ∞-groupoids?
[KL21]	simplicial sets	non-constructive	✓
[BCH15]	semisimplicial sets	no	✓
[vdBF22]	effective Kan fibrations	not known	non-constructive
[GH22]	cofibrant simplicial sets	no	✓
[BCH14]	1st-generation cubical model	✓	no
[CCHM18]	2nd-generation cubical model	✓	no[a] or not known[b]
[ABC+21]	3rd-generation cubical model	✓	no
[ACC+24]	equivariant cubical model	✓	non-constructive
[CS22]	one-connection cubical sets	✓	non-constructive

[a] with reversals
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Core problem:

- Pushforward closure of fibrations requires uniformity.
- Hard to get uniformity from a higher setoid fibration.

Effective interpretation in ∞-groupoids

Starting point.

Build a cubical model ([CCHM18] or [ABC$^+$21]) in $\mathcal{P}(\Box)$ for:

- fully faithful extension $j \colon \Delta \to \Box$ of simplex category,
- induced interval object in $\Box$ has connections,
- cofibration classifier in $\mathcal{P}(\Box)$ classifies simplex boundaries.

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Build a cubical model ([CCHM18] or [ABC^{+}21]) in $\mathcal{P}(\square)$ for:

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- cofibration classifier in $\mathcal{P}(\square)$ classifies simplex boundaries.

For example:

- presheaves over inhabited finite complete posets,
- presheaves over inhabited finite posets.

Effective interpretation in ∞-groupoids

We obtain adjunctions:

$$\mathcal{P}(\Delta_+)_{\mathsf{Kan}} \underset{i_*}{\overset{i^*}{\leftrightarrows}} \mathcal{P}(\Delta)_{\mathsf{Kan}} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathcal{P}(\square)_{\mathsf{uniform\ Kan}}$$

with $i^* \dashv i_*$ and $j^* \dashv j_*$.

Write m for the composite

$$\Delta_+ \xrightarrow{\ i\ } \Delta \xrightarrow{\ j\ } \square.$$

Effective interpretation in ∞-groupoids

Consider the adjunction

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We claim:

(1) the functors m^* and m_* in (1) are (algebraic) right Quillen,

(2) the Quillen adjunction (1) is a Quillen reflection,

(3) the induced lex operation M on $\mathcal{P}(\square)$ is a lex modality (Rijke, Spitters, Shulman; 2020).

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The surprising part is (1) (see next pages).

Preservation of trivial fibrations

Key: semisimplicial sets are a "garden of uniformity".

$$
\begin{array}{c}
\mathcal{P}(\square) \\
m^* \Big\downarrow \dashv \Big\uparrow m_* \\
\mathcal{P}(\Delta_+)
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{TF}_{\mathsf{unif}} & \longrightarrow & \mathsf{TF}_{\mathsf{Kan}} & & \mathsf{TF}_{\mathsf{unif}} \\
 & & \Big\downarrow m^* & & \Big\uparrow m_* \\
 & & \mathsf{TF}_{\mathsf{Kan}} & \xrightarrow[\text{garden of uniformity}]{} & \mathsf{TF}_{\mathsf{unif}}
\end{array}
$$

Terminology:

- $\mathsf{TF}_{\mathsf{Kan}}$ is trivial Kan fibrations.
  These are maps with fillers for simplex boundaries.
- $\mathsf{TF}_{\mathsf{unif}}$ is uniform trivial fibrations.

Preservation of fibrations

$$
\begin{array}{ccccccc}
F_{\mathrm{unif}} & \longrightarrow & F_{\mathrm{Kan}} & & & & F_{\mathrm{unif}} \\
& & \Big\downarrow{\scriptstyle m^*} & & & & \Big\uparrow{\scriptstyle m_*} \\
& & F_{\mathrm{Kan}} & \longrightarrow & F_{\mathrm{fill}} \xrightarrow[\text{garden of uniformity}]{} F_{\mathrm{unif\,fill}} & \longrightarrow & F_{\mathrm{unif\,comp}}
\end{array}
$$

Terminology:

- F_{Kan} is Kan fibrations.
  These are maps with fillers for horn inclusions.
- F_{fill} is prism filling fibrations in semisimplicial sets.
  These are created from trivial Kan fibrations by pullback monoidal hom with interval endpoints.
- $F_{\mathrm{unif\,fill}}$ is uniform filling fibrations.
  These are created from uniform trivial fibrations by pullback monoidal hom with interval endpoints.
- $F_{\mathrm{unif\,comp}}$ is uniform composition fibrations.

Axioms validated

Inherited from the homotopy theory of $\mathcal{P}(\Delta)_+$:

- pointwise principle,
- discrete choice,
- dependent choice,
- presentation.

Also:

- higher inductive types (justified similar to [CRS21]).

Applications

With a homotopically correct base model, we finally have a road to constructive higher topos models of HoTT!

Idea: combine the model with the construction of [CRS21].

Ongoing project: constructive model of synthetic stone duality and better constructive models for synthetic algebraic geometry (joint work with Thierry Coquand and Jonas Höfer).

Another application:

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Also:

- higher realizability (Swan).

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