THE TWISTED GLUE MODEL

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1. The twisted glue model

For the rest of this section, fix a discrete comprehension category

$$\begin{tikzcd} \mathbf{Ty}^{\mathcal{C}} \arrow[rr, "\chi^{\mathcal{C}}"] \arrow[dr] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C}. & \end{tikzcd}$$

We use established notation for working with discrete comprehension categories.

Let us require that $\mathcal{C}$ is the initial discrete comprehension category (with some type constructors, to be specified). Then $\mathcal{C}$ is a contextual category.

Definition 1.1. The *category of weakenings* $\mathcal{W}$ is defined as a wide subcategory of $\mathcal{C}$. The morphisms of $\mathcal{W}$ are inductively specified as follows.

- The identity morphism on the empty context ϵ is in $\mathcal{W}$.
- Given $\sigma\colon \Delta \to \Gamma$ in $\mathcal{W}$ and $A \in \mathbf{Ty}(\Gamma)$, then $\sigma.A\colon \Delta.A[\sigma] \to \Gamma.A$ is in $\mathcal{W}$.
- Given $\sigma\colon \Delta \to \Gamma$ in $\mathcal{W}$ and $B \in \mathbf{Ty}(\Delta)$, then the composite

$$\Delta.B \xrightarrow{p_B} \Delta \xrightarrow{\sigma} \Gamma$$

  is in $\mathcal{W}$.

Note that this category is smaller than the category of renamings considered by Altenkirch and Kaposi. There are two differences:

- A renaming can make use of a variable more than once.
- A renaming can make use of variables out of order.

In theory, based on all combinations of these points, one could form four different wide categories of $\mathcal{C}$. We will see later what is required of $\mathcal{W}$ to make the twisted glueing model work.

The association of a length to any context lifts to a functor $\mathcal{W} \to \Delta^{\mathrm{op}}_{\mathrm{aug},+}$. This functor is faithful.

We have the restricted Yoneda functor $F\colon \mathcal{C} \to \widehat{\mathcal{W}}$.

1.1. Neutral terms and normal forms.

For a context $\Gamma \in \mathcal{C}$ and a type $A \in \mathbf{Ty}(\Gamma)$, we define sets of *neutral terms* $\mathrm{NE}(\Gamma, A)$ and *normal forms* $\mathrm{NF}(\Gamma, A)$ together with interpretation functions

$$\mathrm{NE}(\Gamma, A) \to \mathbf{Tm}(\Gamma, A)$$

and

$$\mathrm{NF}(\Gamma, A) \to \mathbf{Tm}(\Gamma, A)$$

by mutual induction.

Neutral terms:

- Given $\Gamma \in \mathcal{C}$ and $A \in \mathbf{Ty}(\Gamma)$, then $\mathrm{var}_0 \in \mathrm{NE}(\Gamma.A, A[p_A])$.
- Given $n \in \mathrm{NE}(\Gamma, A)$ and $B \in \mathbf{Ty}(\Gamma)$, then $s(n) \in \mathrm{NE}(\Gamma.B, A[p_B])$.

- Given $\Gamma \in \mathcal{C}$, $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma.A)$, and $n \in \mathrm{NE}(\Gamma, \Sigma_A B)$ then $\mathrm{fst}(n) \in \mathrm{NE}(\Gamma, A)$ and $\mathrm{snd}(n) \in \mathrm{NE}(\Gamma, B[\mathrm{fst}(n)])$.
- Given $\Gamma \in \mathcal{C}$, $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma.A)$, $n \in \mathrm{NE}(\Gamma, \Pi_A B)$, and $t \in \mathrm{NF}(\Gamma, A)$, then $\mathrm{app}(n, t) \in \mathrm{NE}(\Gamma, B[n])$.

Normal forms:

- Given $\Gamma \in \mathcal{C}$, $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma.A)$, $a \in \mathrm{NF}(\Gamma, A)$, and $b \in \mathrm{NF}(\Gamma, B[a])$, then $\mathrm{pair}(a, b) \in \mathrm{NF}(\Gamma, \Sigma_A B)$.
- Given $\Gamma \in \mathcal{C}$, $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma.A)$, and $b \in \mathrm{NF}(\Gamma.A, B)$, then $\lambda(b) \in \mathrm{NF}(\Gamma, \Pi_A B)$.
- Given $\Gamma \in \mathcal{C}$, $A \in \mathbf{Ty}(\Gamma)$, and $n \in \mathrm{NE}(\Gamma, A)$, then $\mathrm{neut}(n) \in \mathrm{NF}(\Gamma, A)$.

Note that normally people require A to be a base type in the last clause. Let us see if we can keep it like it is, but then not having η-equalities.

The interpretation functions

$$\begin{tikzcd} \mathrm{NE}(\Gamma, A) \arrow[dr] & & \mathrm{NF}(\Gamma, A) \arrow[dl] \\ & \mathbf{Tm}(\Gamma, A) & \end{tikzcd}$$

are clear.

One can show that $\mathrm{NE}(\Gamma, A)$ and $\mathrm{NF}(\Gamma, A)$ are contravariantly functorial in $\Gamma \in \mathcal{W}$. Formally, they are presheaves over $\mathcal{W} \times_{\mathcal{C}} \mathbf{Ty}$.

Given $\Gamma \in \mathcal{C}$ and $A \in \mathbf{Ty}(\Gamma)$, we write $NE(A)$ and $NF(A)$ for the induced presheaves over $\int(F\Gamma)$ via precomposition with the functor

$$\textstyle\int(F\Gamma) \xrightarrow{\simeq} \mathcal{W} \downarrow \Gamma \xrightarrow{\simeq} \mathcal{W} \times_{\mathcal{C}} (\mathcal{C}/\Gamma) \xrightarrow{\simeq} \mathcal{W} \times_{\mathcal{C}} (\mathbf{Ty}/A) \longrightarrow \mathcal{W} \times_{\mathcal{C}} \mathbf{Ty}.$$

We thus obtain a diagram

$$\begin{tikzcd} \mathrm{NE}(A) \arrow[dr] & & \mathrm{NF}(A) \arrow[dl] \\ & FA & \end{tikzcd}$$

in $\widehat{\int(F\Gamma)}$.

1.2. **What is a universe?** Let U be a context and $\mathrm{El} \in \mathbf{Ty}(U)$ a type over it. This induces a new cwf $\mathcal{C}_U$ as follows.

The contexts are the same as before. The types over $\Gamma \in \mathcal{C}$ are maps $\ulcorner A \urcorner \colon \Gamma \to U$. The context extension of a type $\ulcorner A \urcorner$ is $\Gamma.\,\mathrm{El}[\ulcorner A \urcorner]$.

We have a strict morphism of cwf's $\mathcal{C}_U \to \mathcal{C}$ that is the identity on contexts. Note that $\mathbf{Ty}_U$ has a terminal object. We can reconstruct (U, El) from this data as the image of this terminal object in $\mathbf{Ty}$.

So an ω-hierarchy of universes is a diagram

$$\begin{tikzcd} \mathbf{Ty}_0 \arrow[r] \arrow[drrr] & \mathbf{Ty}_1 \arrow[r] \arrow[drr] & \ldots \arrow[r] & \mathcal{C}^{\to} \arrow[d] \\ & & & \mathcal{C} \end{tikzcd}$$

where each of the discrete fibration $\mathbf{Ty}_n \to \mathcal{C}$ has a terminal object in its total space.

1.3. **The fundamental model.** Let us build a discrete comprehension category. The category of contexts is **Cat**, the category of categories. The types **Ty** are given by the category of elements of the composite

$$\mathbf{Cat}^{\mathrm{op}} \xrightarrow{\widehat{(-)}} \mathbf{Cat} \longrightarrow \mathbf{Set}$$

where the second functor returns the set of objects of a category. Comprehension is given as in the diagram

$$\begin{tikzcd} \int \mathbf{Ty} \arrow[r, "\approx"] \arrow[drr] & \mathbf{DiscFib}_{\mathrm{cart}} \arrow[rr] \arrow[dr, "\mathrm{Cod}"] & & \mathbf{Cat}^{\to} \arrow[dl] \\ & & \mathbf{Cat} & \end{tikzcd}$$

where the first horizontal arrow is the well-known equivalence of presheaves and discrete fibrations given by the category of elements construction and the second horizontal arrow returns the underlying functor of a discrete fibration. By construction, the horizontal composite maps morphisms to cartesian squares in **Cat**.

Let **U** be a Grothendieck universe. We denote $\widehat{(-)}_{\mathbf{U}}$ the operation of forming presheaves in **U**.

Let $\mathcal{B} \in \mathbf{Cat}$ be a context. We define the type $U_{\mathcal{B}} \in \widehat{\mathcal{B}}$ as the composite

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}/-} \mathbf{Cat}^{\mathrm{op}} \xrightarrow{\widehat{(-)}_{\mathbf{U}}} \mathbf{Cat}.$$

Sections to the type $U_{\mathcal{B}}$ are in bijection with **U**-valued presheaves over $\mathcal{B}$. Note that $U_{\mathcal{B}}$ is not stable under substitution. It is, however, pseudostable under weakening. [If we define contexts as categories with a specified model for taking slices, we can make it strictly stable.]

Recall that **Cat** has left and right adjoints to pullback along discrete fibrations. Discrete fibrations are

Let us specify dependent sums in this model. Given a context $\mathcal{C}$, a type $A \in \widehat{\mathcal{C}}$, and a type $B \in \widehat{\int A}$, we define the dependent sum of A and B as the type

$$\Sigma_A B =_{\mathrm{def}} \in \widehat{\mathcal{C}}$$

Note that the category of elements of **Ty** is equivalent to

$$\begin{tikzcd} \mathbf{DiscFib}_{\mathrm{cart}} \arrow[d, "\mathrm{Cod}"] \\ \mathbf{Cat}. \end{tikzcd}$$

The context extension of a type F over a context $\mathcal{C}$ is given by $\int F$, the category of elements of F.

1.4. **Discrete comprehension categories of types.** Let

$$\begin{tikzcd} \mathbf{Ty}^{\mathcal{C}} \arrow[rr, "\chi^{\mathcal{C}}"] \arrow[dr] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

be a discrete comprehension category. We write $\mathbf{Tx}^{\mathcal{C}}$ for the category with objects those of **Ty** and morphisms induced by χ relative to $\mathcal{C}$. If we replace **Ty** by **Tx**, we switch from the discrete comprehension category to the full split comprehension category point of view.

Fix a context $\Gamma \in \mathcal{C}$. We build a new comprehension category $\mathcal{C}_\Gamma$ follows. The contexts are given by $\mathbf{Tx}(\Gamma)$. Given a context $A \in \mathbf{Tx}(\Gamma)$, the type over A are given by $\mathbf{Ty}(\Gamma.A)$. The context extension of $A \in \mathbf{Tx}(\Gamma)$ with $X \in \mathbf{Ty}(\Gamma.A)$ is given by the map $\Sigma_A X \to A$ in $\mathbf{Tx}(\Gamma)$. Observe that for a map $f\colon B \to A$ in $\mathbf{Tx}(\Gamma)$ and $X \in \mathbf{Ty}(\Gamma.A)$, we have a pullback

$$\begin{tikzcd}
\Sigma_B X[\chi(f)] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma_A X \arrow[d] \\
B \arrow[r] & A
\end{tikzcd}$$

in $\mathbf{Tx}(\Gamma)$. This gives the functorial action of context extension.

Let us check that the new discrete comprehension category has dependent sums. Given a context $A \in \mathbf{Tx}(\Gamma)$, a type $X \in \mathbf{Ty}(\Gamma.A)$, and a type $Y \in \mathbf{Ty}(\Gamma.A.X)$, the dependent sum of X and Y is defined as $\Sigma_X Y$. Everything is induced from the original discrete comprehension category $\mathcal{C}$. Dependent products, finitary products, finitary coproducts work the same.

2. Type formers

Let

$$\begin{tikzcd}
\mathbf{Ty}^{\mathcal{C}} \arrow[rr, "\chi^{\mathcal{C}}"] \arrow[dr] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{C} &
\end{tikzcd}$$

be a discrete comprehension category.

Consider a context $\Gamma \in \mathcal{C}$ and a type $A \in \mathbf{Ty}(\Gamma)$. Recall the functor $A^*\colon \mathcal{C}/\Gamma \to \mathcal{C}/\Gamma.A$.

2.1. **Dependent sums.** Given a type $B \in \mathbf{Ty}(\Gamma.A)$, we can form the comma category $p_B \downarrow A^*$. Its objects are pairs (s, f) where $s\colon \Delta \to \Gamma$ is an object of $\mathcal{C}/\Gamma$ and f is a map making the following diagram commute:

$$\begin{tikzcd}
\Gamma.A.B \arrow[rr, "f"] \arrow[dr, "p_B"'] & & A^*\Delta \arrow[dl, "A^*s"] \\
& \Gamma.A. &
\end{tikzcd}$$

A *dependent sum* of A and B is an initial object (s, pair) in $p_B \downarrow A^*$ together with a type $\Sigma_{A,B} \in \mathbf{Ty}(\Gamma)$ such that $p_{\Sigma_A B} = s$.

For $\mathcal{C}$ to have dependent sums, we require a dependent sum $(\Sigma_{A,B}, \mathrm{pair}_{A,B})$ for any input data (Γ, A, B) that is substitution stable in the sense that given a substitution $\sigma\colon \Gamma' \to \Gamma$, a map $m\colon A \to A'$ over σ, and a map $n\colon B \to B'$ over $\sigma.m$, we have a map $o\colon \Sigma_{A',B'} \to \Sigma_{A,B}$ such that

$$\begin{tikzcd}
\Gamma'.A'.B' \arrow[r, "\mathrm{pair}_{A',B'}"] \arrow[d, "\Gamma.m.n"] & (A')^*(\Gamma'.\Sigma_{A',B'}) \arrow[d, "m^*(\Gamma.o)"] \\
\Gamma.A.B \arrow[r, "\mathrm{pair}_{A,B}"] & A^*(\Gamma.\Sigma_{A,B})
\end{tikzcd}$$

commutes.

2.2. **Dependent products.** Given a type $B \in \mathbf{Ty}(\Gamma.A)$, we can form the comma category $A^* \downarrow p_B$. Its objects are pairs (s, f) where $s\colon \Delta \to \Gamma$ is an object of $\mathcal{C}/\Gamma$ and f is a map making the following diagram commute:

$$\begin{tikzcd} A^*\Delta \arrow[rr, "f"] \arrow[dr, "A^*s"'] & & \Gamma.A.B \arrow[dl, "p_B"] \\ & \Gamma.A. & \end{tikzcd}$$

A *dependent product* of A and B is an initial object (s, eval) in $A^* \downarrow p_B$ together with a type $\Pi_A B \in \mathbf{Ty}(\Gamma)$ such that $p_{\Pi_A B} = s$.

2.3. **Identity types.** Let's say we have $\Gamma \vdash A$. In the syntax, we have an identity type $\Gamma.A.A \vdash \mathrm{Id}_A$.

Let's build the identity type in the glueing. Suppose we have a context $(\Gamma, |\Gamma|, |\Gamma| \xrightarrow{\alpha} F\Gamma)$. And we have a type $(A, |A|, |A| \xrightarrow{\beta} FA[\alpha])$ over it.

Recall that $\Gamma \vdash A$ and $|A|$ is a presheaf over $\int|\Gamma|$.

Now we need a type over

$$(\Gamma, |\Gamma|, \alpha).(A, |A|, \beta).(A, |A|, \beta).$$

This rewrites up to isomorphism to

$$(\Gamma.A.A, |\Gamma|.|A|.|A| \to F\Gamma.FA.FA).$$

We have to give the identity type over this. We let it be

$$(\mathrm{Id}_A, \mathrm{Id}_{|A|} \to F\,\mathrm{Id}_A[\ldots]).$$

Now we need an unquote map. Need to add neutral elements. So the new identity type is

$$(\mathrm{Id}_A, \mathrm{Id}_{|A|} + \mathrm{NE}(\mathrm{Id}_A) \to F\,\mathrm{Id}_A[\ldots]).$$

The constructor is not affected. For the quote map, we use the embedding of $\mathrm{NE}(\mathrm{Id}_A)$ into $\mathrm{NF}(\mathrm{Id}_A)$. For the eliminator, given a type C over $|\Gamma|.|A|.|A|.(\mathrm{Id}_{|A|} + NE(\mathrm{Id}_A))$, we are given a map

$$c\colon |\Gamma|.|A| \to |\Gamma|.|A|.C[\mathsf{refl}].$$

We need to produce a map

$$|\Gamma|.|A|.|A|.(\mathrm{Id}_{|A|} + NE(\mathrm{Id}_A)) \to |\Gamma|.|A|.|A|.(\mathrm{Id}_{|A|} + NE(\mathrm{Id}_A)).C.$$

We perform case distinction. On the left summand, we use the eliminator of $\mathrm{Id}_{|A|}$. On the right summand, we use postcomposition with the map $\mathrm{NE}(C) \to C$. We build an element of $\mathrm{NE}(C)$:

$$\mathsf{Id\text{-}elim}(\mathsf{witness}, \mathsf{equality})$$

The $\mathsf{equality}$ has to be neutral. It is given by the input $\mathrm{NE}(\mathrm{Id}_A)$. The $\mathsf{witness}$ has to be normal. We use postcomposition with $C[\mathsf{refl}] \to \mathrm{NF}(C[\mathsf{refl}])$. So it remains to construct an approriate element of $C[\mathsf{refl}]$. We use c applied to a variable.

So identity types work just like coproducts.

2.4. **Reflections on types in twisted glueing.** Categories of types

$$\mathcal{C}, \mathcal{C}_{\mathrm{NF}}, \mathcal{C}_{\mathrm{NF,NE}}.$$

Have full embeddings

$$\mathcal{C}_{\mathrm{NF,NE}} \hookrightarrow \mathcal{C}_{\mathrm{NF}} \hookrightarrow \mathcal{C}.$$

Let us try to construct a left adjoint to the first map.

We have a category of glue types $\mathcal{C}$ and a category of twisted glue types $\mathcal{D}$. The forgetful functor $\mathcal{D} \to \mathcal{C}$ is fully faithful (since, by construction, the glue and twisted glue model have the same category of contexts, i.e. maps between types are the same).

Given $A \in \mathcal{C}$, let us try to build a free object $X \in \mathcal{D}$ on it.

A -¿ Y ———— A + NE(A) -¿ Y

2.5. **The orginary glue model.** We now build a new discrete comprehension category

$$\begin{tikzcd} \mathbf{Ty}^{\mathcal{D}} \arrow[rr, "\chi^{\mathcal{D}}"] \arrow[dr] & & \mathcal{D}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{D}. & \end{tikzcd}$$

The category of contexts is given by $\mathcal{D} =_{\text{def}} \widehat{\mathcal{W}} \downarrow F$. We write its objects as pairs $(\Gamma, |\Gamma| \xrightarrow{\alpha} F\Gamma)$.

A type over this context, i.e. an element of $\mathbf{Ty}(\Gamma, |\Gamma| \xrightarrow{\alpha} F\Gamma)$, is given by a type $A \in \mathbf{Ty}(\Gamma)$ and a presheaf $|A| \in \widehat{\int |\Gamma|}$ with a map $|A| \to (FA)[\alpha]$.

$$\mathbf{Ty}^{\mathcal{D}} = F$$

[

Let us describe the context extension of this type.

2.6. **The model.** We now build a new discrete comprehension category

$$\begin{tikzcd} \mathbf{Ty}^{\mathcal{D}} \arrow[rr, "\chi^{\mathcal{D}}"] \arrow[dr] & & \mathcal{D}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{D}. & \end{tikzcd}$$

The category of contexts is given by $\mathcal{D} =_{\text{def}} \widehat{\mathcal{W}} \downarrow F$. We write its objects as pairs $(\Gamma, |\Gamma| \xrightarrow{\alpha} F\Gamma)$.

A type over $(\Gamma, |\Gamma| \xrightarrow{\alpha} F\Gamma)$ is given by a type $A \in \mathbf{Ty}(\Gamma)$, a presheaf $|A| \in \widehat{\int |\Gamma|}$ fitting into a diagram

$$\begin{tikzcd} \mathrm{NE}(A)[\alpha] \arrow[r] \arrow[dr] & {|A|} \arrow[r] \arrow[d] & \mathrm{NF}(A)[\alpha] \arrow[dl] \\ & (FA)[\alpha]. & \end{tikzcd}$$

The context extension of this type is given on the first component by $p_A \colon \Gamma.A \to \Gamma$ and on the second component by the diagram

$$\begin{tikzcd} {|A|} \arrow[r] \arrow[d] & F(\Gamma).F(A) \arrow[r] & F(\Gamma.A) \arrow[dl] \\ {|\Gamma|} \arrow[r, "\alpha"] & F\Gamma & \end{tikzcd}$$

References