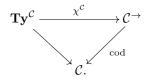
THE TWISTED GLUE MODEL

CHRISTIAN SATTLER

1. The twisted glue model

For the rest of this section, fix a discrete comprehension category



We use established notation for working with discrete comprehension categories.

Let us require that C is the initial discrete comprehension category (with some type constructors, to be specified). Then C is a contextual category.

Definition 1.1. The *category of weakenings* \mathcal{W} is defined as a wide subcategory of \mathcal{C} . The morphisms of \mathcal{W} are inductively specified as follows.

- The identity morphism on the empty context ϵ is in \mathcal{W} .
- Given $\sigma: \Delta \to \Gamma$ in \mathcal{W} and $A \in \mathbf{Ty}(\Gamma)$, then $\sigma.A: \Delta.A[\sigma] \to \Gamma.A$ is in \mathcal{W} .
- Given $\sigma: \Delta \to \Gamma$ in \mathcal{W} and $B \in \mathbf{Ty}(\Delta)$, then the composite

$$\Delta . B \xrightarrow{p_B} \Delta \xrightarrow{\sigma} \Gamma$$

is in \mathcal{W} .

Note that this category is smaller than the category of renamings considered by Altenkirch and Kaposi. There are two differences:

- A renaming can make use of a variable more than once.
- A renaming can make use of variables out of order.

In theory, based on all combinations of these points, one could form four different wide categories of C. We will see later what is required of W to make the twisted glueing model work.

The association of a length to any context lifts to a functor $\mathcal{W} \to \Delta_{\text{aug},+}^{\text{op}}$. This functor is faithful.

We have the restricted Yoneda functor $F: \mathcal{C} \to \widehat{\mathcal{W}}$.

1.1. Neutral terms and normal forms. For a context $\Gamma \in C$ and a type $A \in \mathbf{Ty}(\Gamma)$, we define sets of *neutral terms* NE(Γ, A) and *normal forms* NF(Γ, A) together with interpretation functions

$$NE(\Gamma, A) \to Tm(\Gamma, A)$$

and

$$NF(\Gamma, A) \to Tm(\Gamma, A)$$

by mutual induction.

Neutral terms:

- Given $\Gamma \in \mathcal{C}$ and $A \in \mathbf{Ty}(\Gamma)$, then $\operatorname{var}_0 \in \operatorname{NE}(\Gamma, A, A[p_A])$.
- Given $n \in NE(\Gamma, A)$ and $B \in \mathbf{Ty}(\Gamma)$, then $s(n) \in NE(\Gamma.B, A[p_B])$.

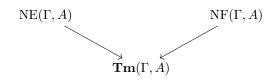
- Given $\Gamma \in \mathcal{C}$, $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma,A)$, and $n \in NE(\Gamma, \Sigma_A B)$ then $fst(n) \in NE(\Gamma,A)$ and $snd(n) \in NE(\Gamma, B[fst(n)])$.
- Given $\Gamma \in C$, $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma,A)$, $n \in NE(\Gamma,\Pi_A B)$, and $t \in NF(\Gamma,A)$, then $app(n,t) \in NE(\Gamma, B[n])$.

Normal forms:

- Given $\Gamma \in \mathcal{C}$, $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma,A)$, $a \in NF(\Gamma,A)$, and $b \in NF(\Gamma, B[a])$, then $pair(a,b) \in NF(\Gamma, \Sigma_A B)$.
- Given $\Gamma \in \mathcal{C}$, $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma, A)$, and $b \in NF(\Gamma, A, B)$, then $\lambda(b) \in NF(\Gamma, \Pi_A B)$.
- Given $\Gamma \in \mathcal{C}$, $A \in \mathbf{Ty}(\Gamma)$, and $n \in NE(\Gamma, A)$, then neut $(n) \in NF(\Gamma, A)$.

Note that normally people require A to be a base type in the last clause. Let us see if we can keep it like it is, but then not having η -equalities.

The interpretation functions



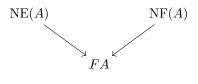
are clear.

One can show that NE(Γ , A) and NF(Γ , A) are contravariantly functorial in $\Gamma \in \mathcal{W}$. Formally, they are presheaves over $\mathcal{W} \times_{\mathcal{C}} \mathbf{Ty}$.

Given $\Gamma \in \mathcal{C}$ and $A \in \mathbf{Ty}(\Gamma)$, we write NE(A) and NF(A) for the induced presheaves over $\int (F\Gamma)$ via precomposition with the functor

$$\int (F\Gamma) \xrightarrow{\simeq} \mathcal{W} \downarrow \Gamma \xrightarrow{\simeq} \mathcal{W} \times_{\mathcal{C}} (\mathcal{C}/\Gamma) \xrightarrow{\simeq} \mathcal{W} \times_{\mathcal{C}} (\mathbf{Ty}/A) \longrightarrow \mathcal{W} \times_{\mathcal{C}} \mathbf{Ty}.$$

We thus obtain a diagram



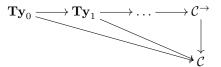
in $\widehat{\int (F\Gamma)}$.

1.2. What is a universe? Let U be a context and $El \in \mathbf{Ty}(U)$ a type over it. This induces a new cwf \mathcal{C}_U as follows.

The contexts are the same as before. The types over $\Gamma \in \mathcal{C}$ are maps $\lceil A \rceil$: $\Gamma \to U$. The context extension of a type $\lceil A \rceil$ is Γ . $\text{El}[\lceil A \rceil]$.

We have a strict morphism of cwf's $\mathcal{C}_U \to \mathcal{C}$ that is the identity on contexts. Note that \mathbf{Ty}_U has a terminal object. We can reconstruct (U, El) from this data as the image of this terminal object in \mathbf{Ty} .

So an ω -hierarchy of universes is a diagram

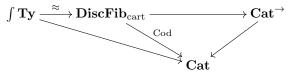


where each of the discrete fibration $\mathbf{Ty}_n \to \mathcal{C}$ has a terminal object in its total space.

1.3. The fundamental model. Let us build a discrete comprehension category. The category of contexts is **Cat**, the category of categories. The types **Ty** are given by the category of elements of the composite

$$\operatorname{Cat}^{\operatorname{op}} \xrightarrow{\widehat{(-)}} \operatorname{Cat} \longrightarrow \operatorname{Set}$$

where the second functor returns the set of objects of a category. Comprehension is given as in the diagram



where the first horizontal arrow is the well-known equivalence of presheaves and discrete fibrations given by the category of elements construction and the second horizontal arrow returns the underlying functor of a discrete fibration. By construction, the horizontal composite maps morphisms to cartesian squares in **Cat**.

Let **U** be a Grothendieck universe. We denote $(-)_{\mathbf{U}}$ the operation of forming presheaves in **U**.

Let $\mathcal{B} \in \mathbf{Cat}$ be a context. We define the type $U_{\mathcal{B}} \in \widehat{\mathcal{B}}$ as the composite

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}/-} \mathbf{Cat}^{\mathrm{op}} \xrightarrow{\widehat{(-)}_{\mathbf{U}}} \mathbf{Cat}$$

Sections to the type $U_{\mathcal{B}}$ are in bijection with U-valued presheaves over \mathcal{B} . Note that $U_{\mathcal{B}}$ is not stable under substitution. It is, however, pseudostable under weakening. [If we define contexts as categories with a specified model for taking slices, we can make it strictly stable.]

Recall that **Cat** has left and right adjoints to pullback along discrete fibrations. Discrete fibrations are

Let us specify dependent sums in this model. Given a context \mathcal{C} , a type $A \in \widehat{\mathcal{C}}$, and a type $B \in \widehat{\int A}$, we define the dependent sum of A and B as the type

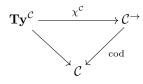
$$\Sigma_A B =_{\mathrm{def}} \in \widehat{\mathcal{C}}$$

Note that the category of elements of **Ty** is equivalent to



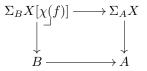
The context extension of a type F over a context C is given by $\int F$, the category of elements of F.

1.4. Discrete comprehension categories of types. Let



be a discrete comprehension category. We write $\mathbf{Tx}^{\mathcal{C}}$ for the category with objects those of \mathbf{Ty} and morphisms induced by χ relative to \mathcal{C} . If we replace \mathbf{Ty} by \mathbf{Tx} , we switch from the discrete comprehension category to the full split comprehension category point of view.

Fix a context $\Gamma \in C$. We build a new comprehension category C_{Γ} follows. The contexts are given by $\mathbf{Tx}(\Gamma)$. Given a context $A \in \mathbf{Tx}(\Gamma)$, the type over A are given by $\mathbf{Ty}(\Gamma,A)$. The context extension of $A \in \mathbf{Tx}(\Gamma)$ with $X \in \mathbf{Ty}(\Gamma,A)$ is given by the map $\Sigma_A X \to A$ in $\mathbf{Tx}(\Gamma)$. Observe that for a map $f: B \to A$ in $\mathbf{Tx}(\Gamma)$ and $X \in \mathbf{Ty}(\Gamma,A)$, we have a pullback



in $\mathbf{Tx}(\Gamma)$. This gives the functorial action of context extension.

Let us check that the new discrete comprehension category has dependent sums. Given a context $A \in \mathbf{Tx}(\Gamma)$, a type $X \in \mathbf{Ty}(\Gamma.A)$, and a type $Y \in \mathbf{Ty}(\Gamma.A.X)$, the dependent sum of X and Y is defined as $\Sigma_X Y$. Everything is induced from the original discrete comprehension category \mathcal{C} . Dependent products, finitary products, finitary coproducts work the same.

2. Type formers

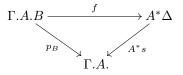
 $\mathbf{Ty}^{\mathcal{C}} \xrightarrow{\chi^{\mathcal{C}}} \mathcal{C}^{\rightarrow}$

Let



Consider a context $\Gamma \in \mathcal{C}$ and a type $A \in \mathbf{Ty}(\Gamma)$. Recall the functor $A^* : \mathcal{C}/\Gamma \to \mathcal{C}/\Gamma A$.

2.1. **Dependent sums.** Given a type $B \in \mathbf{Ty}(\Gamma.A)$, we can form the comma category $p_B \downarrow A^*$. Its objects are pairs (s, f) where $s: \Delta \to \Gamma$ is an object of \mathcal{C}/Γ and f is a map making the following diagram commute:



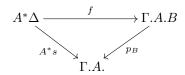
A dependent sum of A and B is an initial object (s, pair) in $p_B \downarrow A^*$ together with a type $\Sigma_{A,B} \in \mathbf{Ty}(\Gamma)$ such that $p_{\Sigma_A B} = s$.

For \mathcal{C} to have dependent sums, we require a dependent sum $(\Sigma_{A,B}, \operatorname{pair}_{A,B})$ for any input data (Γ, A, B) that is substitution stable in the sense that given a substitution $\sigma \colon \Gamma' \to \Gamma$, a map $m \colon A \to A'$ over σ , and a map $n \colon B \to B'$ over $\sigma.m$, we have a map $o \colon \Sigma_{A',B'} \to \Sigma_{A,B}$ such that

$$\begin{array}{c} \Gamma'.A'.B' \xrightarrow{\operatorname{pair}_{A',B'}} (A')^* (\Gamma'.\Sigma_{A',B'}) \\ \downarrow \\ \Gamma.m.n & \downarrow \\ \Gamma.A.B \xrightarrow{\operatorname{pair}_{A,B}} A^* (\Gamma.\Sigma_{A,B}) \end{array}$$

commutes.

2.2. Dependent products. Given a type $B \in \mathbf{Ty}(\Gamma.A)$, we can form the comma category $A^* \downarrow p_B$. Its objects are pairs (s, f) where $s: \Delta \to \Gamma$ is an object of \mathcal{C}/Γ and f is a map making the following diagram commute:



A dependent product of A and B is an initial object (s, eval) in $A^* \downarrow p_B$ together with a type $\Pi_A B \in \mathbf{Ty}(\Gamma)$ such that $p_{\Pi_A B} = s$.

2.3. Identity types. Let's say we have $\Gamma \vdash A$. In the syntax, we have an identity type $\Gamma.A.A \vdash Id_A$.

Let's build the identity type in the glueing. Suppose we have a context $(\Gamma, |\Gamma|, |\Gamma| \xrightarrow{\alpha} F\Gamma)$. And we have a type $(A, |A|, |A| \xrightarrow{\beta} FA[\alpha])$ over it.

Recall that $\Gamma \vdash A$ and |A| is a presheaf over $\int |\Gamma|$.

Now we need a type over

$$(\Gamma, |\Gamma|, \alpha).(A, |A|, \beta).(A, |A|, \beta).$$

This rewrites up to isomorphism to

$$(\Gamma.A.A, |\Gamma|.|A|.|A| \rightarrow F\Gamma.FA.FA).$$

We have to give the identity type over this. We let it be

$$(\mathrm{Id}_A, \mathrm{Id}_{|A|} \to F \mathrm{Id}_A[\ldots]).$$

Now we need an unquote map. Need to add neutral elements. So the new identity type is

$$(\mathrm{Id}_A, \mathrm{Id}_{|A|} + \mathrm{NE}(\mathrm{Id}_A) \to F \mathrm{Id}_A[\ldots]).$$

The constructor is not affected. For the quote map, we use the embedding of NE(Id_A) into NF(Id_A). For the eliminator, given a type C over $|\Gamma| \cdot |A| \cdot |A| \cdot (Id_{|A|} + NE(Id_A))$, we are given a map

$$c \colon |\Gamma|.|A| \to |\Gamma|.|A|.C[\mathsf{refl}].$$

We need to produce a map

$$|\Gamma|.|A|.|A|.(\mathrm{Id}_{|A|} + NE(\mathrm{Id}_A)) \to |\Gamma|.|A|.|A|.(\mathrm{Id}_{|A|} + NE(\mathrm{Id}_A)).C$$

We perform case distinction. On the left summand, we use the eliminator of $Id_{|A|}$. On the right summand, we use postcomposition with the map $NE(C) \rightarrow C$. We build an element of NE(C):

Id-elim(witness, equality)

The equality has to be neutral. It is given by the input NE(Id_A). The witness has to be normal. We use postcomposition with $C[refl] \rightarrow NF(C[refl])$. So it remains to construct an approxiate element of C[refl]. We use c applied to a variable.

So identity types work just like coproducts.

2.4. Reflections on types in twisted glueing. Categories of types

 $\mathcal{C}, \mathcal{C}_{\rm NF}, \mathcal{C}_{\rm NF, NE}.$

Have full embeddings

$$\mathcal{C}_{\mathrm{NF},\mathrm{NE}} \hookrightarrow \mathcal{C}_{\mathrm{NF}} \hookrightarrow \mathcal{C}.$$

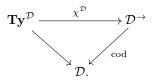
Let us try to construct a left adjoint to the first map.

We have a category of glue types C and a category of twisted glue types D. The forgetful functor $D \to C$ is fully faithful (since, by construction, the glue and twisted glue model have the same category of contexts, i.e. maps between types are the same).

Given $A \in \mathcal{C}$, let us try to build a free object $X \in \mathcal{D}$ on it.

A -¿ Y — A + NE(A) -¿ Y

2.5. The orginary glue model. We now build a new discrete comprehension category



The category of contexts is given by $\mathcal{D} =_{def} \widehat{\mathcal{W}} \downarrow F$. We write its objects as pairs $(\Gamma, |\Gamma| \xrightarrow{\alpha} F\Gamma)$.

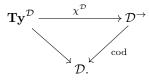
A type over this context, i.e. an element of $\mathbf{Ty}(\Gamma, |\Gamma| \xrightarrow{\alpha} F\Gamma)$, is given by a type $A \in \mathbf{Ty}(\Gamma)$ and a presheaf $|A| \in \widehat{\int |\Gamma|}$ with a map $|A| \to (FA)[\alpha]$.

$$\mathbf{T}\mathbf{y}^{\mathcal{D}} = F$$

Let us describe the context extension of this type.

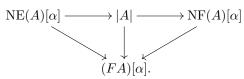
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2.6. The model. We now build a new discrete comprehension category

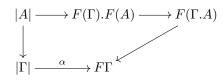


The category of contexts is given by $\mathcal{D} =_{\operatorname{def}} \widehat{\mathcal{W}} \downarrow F$. We write its objects as pairs $(\Gamma, |\Gamma| \xrightarrow{\alpha} F\Gamma)$.

A type over $(\Gamma, |\Gamma| \xrightarrow{\alpha} F\Gamma)$ is given by a type $A \in \mathbf{Ty}(\Gamma)$, a presheaf $|A| \in \widehat{\int |\Gamma|}$ fitting into a diagram



The context extension of this type is given on the first component by $p_A \colon \Gamma.A \to \Gamma$ and on the second component by the diagram



References