ON SIMPLICIAL COLIMITS

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Abstract. We show that any simplicial colimit in a higher category can be equivalently computed as a colimit over a cube of infinite dimension with terminal vertex missing.

Let Δ denote the simplex category, the full subcategory of posets consisting of non-empty finite total linear orders; a skeleton is given by the posets

$$[n] =_{\mathrm{def}} \{0 \to 1 \to \ldots \to n\}$$

for $n \geq 0$. Let $\Delta_+ \subseteq \Delta$ denote the semisimplex category, the wide subcategory of Δ of monomorphisms (this notation is non-standard: Δ_+ is frequently used to denote the augmented simplex category). Given $n \geq 0$, we denote $\Delta^{\leq n}$ the restriction of Δ to posets of height $n+1$; a skeleton is given by the restriction to objects $[0], \ldots, [n]$.

Recall that a map $f : \mathcal{C} \to \mathcal{D}$ between quasicategories is *final* if precomposition with f preserves colimiting cocones. Dually, f is called *initial* if precomposition with f preserves limiting cones (note that 'final' used to be called 'cofinal' in older literature). Note that f is initial exactly if $f^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is final. If $\mathcal{C}$ and $\mathcal{D}$ are categories, we speak of *homotopy finality* and *homotopy initiality* instead of finality and initiality to avoid confusion with the weaker categorical notions of finality and initiality.

A useful criterion for homotopy initiality is the following.

Proposition 0.1 (Joyal?). *A functor $F : \mathcal{C} \to \mathcal{D}$ between categories is homotopy initial exactly if for every $Y \in D$, the comma category $F \downarrow Y$ has weakly contractible nerve.* □

We can now state our main observation (probably known?).

Proposition 0.2. *For any $n \geq 0$, the composite*

$$\Delta_+/[n] \longrightarrow \Delta_+^{\leq n} \longrightarrow \Delta^{\leq n}$$

is homotopy initial (here, the first arrow is the forgetful functor).

Example 0.3. In any suitably cocomplete $(\infty, 1)$-category, we have the following.

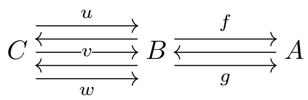
- The reflexive coequalizer of

$$B \;\substack{\xrightarrow{\;f\;} \\ \xleftarrow{} \\ \xrightarrow[\;g\;]{}}\; A$$

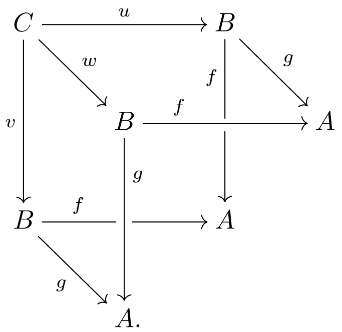
is given by the pushout of

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle g}\big\downarrow & & \\ A. & & \end{array}$$

- The colimit of a 2-truncated simplicial object



  is given by the colimit of the cubical diagram



□

Proof of Proposition 0.2. Let u denote the composite functor of the statement. Appealing to Proposition 0.1, we have to show that $u \downarrow [k]$ has weakly contractible nerve for $0 \leq k \leq n$. Since $\Delta_+/[n]$ is a poset, so is $u \downarrow [k]$.

Note that $u \downarrow [k]$ is equivalently the full subposet $\mathcal{A}$ of $\mathbf{Fun}([k], \mathcal{P}([n]))$ consisting of those F such that $F(i) < F(j)$ for $i < j$ in $[k]$ and $F(i)$ is inhabited for some $i \in [k]$.[1] It remains to show that the nerve of $\mathcal{A}$ is contractible.

We define a filtration

$$\mathcal{B}_k \subseteq \mathcal{C}_{k-1} \subseteq \ldots \subseteq \mathcal{B}_1 \subseteq \mathcal{C}_0 \subseteq \mathcal{B}_0 \subseteq \mathcal{C}_{-1} = \mathcal{A}$$

of full inclusions mutally recursively as follows:

- for $0 \leq t < k$, let $\mathcal{B}_t \subseteq \mathcal{C}_{t-1}$ consist of those F such that $F(t) \leq t$ or $t \in F(t)$.[2]
- for $0 \leq t < k$, let $C_t \subseteq B_t$ consist of those F such that $F(t) \leq t$.
- let $\mathcal{B}_k \subseteq \mathcal{C}_{k-1}$ consist of those F such that $k \in F(k)$.

For $0 \leq t < k$, the inclusion $C_t \subseteq B_t$ coreflective. The coreflector sends F to F' where

$$F'(i) = \begin{cases} F(t) \cap [t] & \text{if } i = t \\ F(i) & \text{else.} \end{cases}$$

By the definition of $\mathcal{B}_t$, we have that $F'(t)$ is inhabited if $F(t)$ is. Thus, the coreflector is valued in $\mathcal{A}$; one then checks easily that it is valued in $\mathcal{C}_t$. It is clearly functorial, stationary on $\mathcal{C}_t$, and admits a counit.

For $0 \leq t < k$, the inclusion $\mathcal{B}_t \subseteq \mathcal{C}_{t-1}$ is reflective. The reflector sends F to F' where

$$F'(i) = \begin{cases} F(t) \cup \min(t, F(t)) & \text{if } i = t,^3 \\ F(i) & \text{else.} \end{cases}$$

Note that for $t < i$ we have

$$F(t) \cup \min(t, F(t)) < F(i)$$

as $F(t) < F(i)$, and for $i < t$ we have

$$F(i) < F(t) \cup \min(t, F(t))$$

[1] Here, $F(i) < F(j)$ means $x < j$ for all $x \in F(i)$ and $y \in F(j)$.

[2] Here, $F(t) \leq t$ means $x \leq t$ for all $x \in F(t)$.

[3] Here, $\min(t, F(t))$ means $\{\min(t, x) \mid x \in F(t)\}$.

as $F(i) < F(t)$ and $F(i) \leq i < t$ since $F \in \mathcal{C}_i$. This shows that the reflector is valued in $\mathcal{A}$; one then checks easily that it is valued in $\mathcal{B}_t$. It is clearly functorial, stationary on $\mathcal{B}_t$, and admits a unit.

Finally, the inclusion $\mathcal{B}_k \subseteq \mathcal{C}_{k-1}$ is reflective. The reflector sends F to F' where

$$F'(i) = \begin{cases} F(k) \cup \{k\} & \text{if } i = k, \\ F(i) & \text{else.} \end{cases}$$

It is here that we use $k \leq n$ so that $F' \in \mathbf{Fun}([k], \mathcal{P}([n])$. The reflector is checked to be valued in and stationary on $\mathcal{B}_t$ as well as admitting a unit as in the previous paragraph.

Note that $\mathcal{B}_k$ has an initial object given by F where

$$F(i) = \begin{cases} \{k\} & \text{if } i = k, \\ \emptyset & \text{else.} \end{cases}$$

Its nerve is thus weakly contractible. Recall that adjunctions induce weak equivalences on nerves. By 2-out-of-3, it thus follows that $\mathcal{A}$ has weakly contractible nerve. □

Given a set A, let $\mathcal{P}_{\mathsf{fin,inhab}}(A)$ denote the poset of inhabited finite subsets of A. If A is itself a total poset, then $\mathcal{P}_{\mathsf{fin,inhab}}(A)$ is the comma category over A of the embedding from Δ_+ to the wide subcategory of posets on monomorphisms. In that case, we have a forgetful map from $\mathcal{P}_{\mathsf{fin,inhab}}(A)$ to Δ_+.

Corollary 0.4. *The composite $\mathcal{P}_{\mathsf{fin,inhab}}(\omega) \to \Delta_+ \to \Delta$ is homotopy initial.*

Proof. The map in question is a sequential colimit of the map of Proposition 0.2. The step maps are embeddings, hence this is also a colimit of $(\infty, 1)$-categories. The claim thus follows by accessibility in $(\infty, 1)$-categories of initial maps, in particular closure under sequential colimits. □

Remark 0.5. As is well-known, the inclusion $\Delta_+ \to \Delta$ is also homotopy initial. However, the map $\mathcal{P}_{\mathsf{fin,inhab}}(\omega) \to \Delta_+$ is not homotopy initial. □