Mapping spaces of pushouts along fully faithful functors

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Extracted from joint work with David Wärn on the categorical zig-zag construction mapping spaces of pushouts of catgories.

Adjunctions and slices

We recall how adjunctions descend to slices.

Lemma 1. *Consider a functor $F\colon \mathcal{C} \to \mathcal{D}$. If F has a right adjoint and $\mathcal{C}$ has pullbacks, then F slicewise has right adjoints. That is, given $c \in \mathcal{C}$, the slice action*

$$\mathcal{C} \downarrow c \xrightarrow{F_{\downarrow c}} \mathcal{D} \downarrow F(c)$$

has a right adjoint.

In detail, consider $(d, g\colon d \to F(c)) \in \mathcal{D} \downarrow F(c)$. Consider right adjoints of F at $F(c)$, denoted $(RF(c), \varepsilon_{F(c)})$, and at d, denoted $(R(d), \varepsilon_d)$. Consider a pullback

$$\begin{array}{ccc} c' & \xrightarrow{h} & R(d) \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle Rg} \\ c & \xrightarrow{\eta_c} & RF(c). \end{array}$$

Then $(c', f) \in \mathcal{C} \downarrow c$ together with the transposed square

$$\begin{array}{ccc} F(c') & \xrightarrow{\varepsilon_d \circ Fh} & d \\ \downarrow & & \downarrow{\scriptstyle Ff} \\ F(c) & = & F(c) \end{array}$$

forms a right adjoint of $F_{\downarrow c}$ at (d, g). □

Lemma 2. *Assume that $\mathcal{C}$ has and $F\colon \mathcal{C} \to \mathcal{D}$ preserves pullbacks. If F has a fully faithful right adjoint, then F slicewise has fully faithful right adjoints.*

In detail, consider right adjoints at $F(c)$ and d as before. Assume that $\varepsilon_{F(c)}$ and ε_d are invertible and F preserves the pullback defining c'. Then the counit $F_{\downarrow c}\colon (c', f) \to (c, f)$ is invertible.

Proof. Since $\varepsilon_{F(c)}$ is invertible, so is $F\eta_c$ by 2-out-of-3. Then so is its pullback Fh, hence also the composite $\varepsilon_d \circ Fh$. □

Pullbacks of adjunctions

Consider a pullback of categories

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{G} & \mathcal{C} \\ {\scriptstyle V}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ \mathcal{B} & \xrightarrow{F} & \mathcal{A}. \end{array}$$

We wish to describe right adjoints of G and V in terms of (slicewise) right adjoints of F and U. In general, this problem is solved by the zig-zag construction using a sequential colimit. The below statements are the special cases where the sequential colimit converges already after one and two steps, respectively.

We recall pullback stability of reflectors.

Lemma 3. *If F has a fully faithful right adjoint, so does G and Beck–Chevalley holds.*

In detail, consider $c \in \mathcal{C}$. Assume a right adjoint of F at $U(c)$ given by $b \in \mathcal{B}$ with invertible counit $\varepsilon_b \colon F(b) \to U(c)$. Then G has a right adjoint at c given by the induced object $d \in \mathcal{D}$ with invertible counit $G(d) \simeq c$.

Proof. Recall that a functor $P \colon M \to N$ has a right adjoint at $n \in N$ with invertible counit exactly if there is m over n such P that the action

$$M(-, m) \xrightarrow{P} N(P(-), n)$$

is invertible (that is, P is fully faithful relative to mapping into m).

We have an induced pullback of presheaves:

$$\begin{array}{ccc} \mathcal{D}(-, d) & \longrightarrow & \mathcal{C}(G(-), c) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{B}(V(-), b) & \longrightarrow & \mathcal{A}(FV(-), U(c)). \end{array}$$

By assumption, the bottom map is invertible. Therefore, so is the top map, which is the claim. ☐

Lemma 4. *Assume U has a right adjoint and F slicewise has fully faithful right adjoints. Then V has a right adjoint and the Beck-Chevalley condition holds.*

In detail, consider $b \in \mathcal{B}$. Assume a right adjoint of U at $F(b)$ given by $c \in \mathcal{C}$ with $\varepsilon_c \colon U(c) \to F(b)$. Assume a right adjoint of $F_{\downarrow b}$ at $(U(c), \varepsilon_c)$ given by $(b', g \colon b' \to b) \in \mathcal{B} \downarrow b$ with invertible counit. Then V has a right adjoint at b given by the object $d \in \mathcal{D}$ induced by $b' \in \mathcal{B}$ and $c \in \mathcal{C}$ with $F(b') \simeq U(c)$ with counit $\varepsilon_d \colon V(d) \to b$ given by $g \colon b' \to b$.

Proof. We have an induced pullback of categories:

$$\begin{array}{ccc} V \downarrow b & \longrightarrow & U \downarrow F(b) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{B} \downarrow b & \xrightarrow{F_{\downarrow b}} & \mathcal{A} \downarrow F(b), \end{array} \qquad \begin{array}{ccc} (d, \varepsilon_d) & \longmapsto & (c, \varepsilon_c) \\ \downarrow & & \downarrow \\ (b', g) & \longmapsto & (U(c), \varepsilon_c). \end{array}$$

By Lemma 3 applied to this pullback, $(d, \varepsilon_d) \in V \downarrow b$ is a right adjoint with invertible counit of the top functor at $(c, \varepsilon_c) \in U \downarrow F(b)$. It is terminal as right adjoints preserve terminal objects. ☐

Corollary 5. *Consider a pullback of categories*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\bar{p}^*} & \mathcal{Y} \\ {\scriptstyle \bar{q}^*}\downarrow & \lrcorner & \downarrow{\scriptstyle q^*} \\ \mathcal{X} & \xrightarrow{p^*} & \mathcal{Z} \end{array}$$

with left adjoints $p_! \dashv p^*$ and $q_! \dashv q^*$. Assume that $p_!$ is fully faithful and that $\mathcal{X}$ has and p^* preserves pushouts. Then we have a left adjoint $\bar{q}_! \dashv \bar{q}^*$ and the square

$$\begin{array}{ccc} p_! p^* & \longrightarrow & \mathrm{id} \\ \downarrow & & \downarrow \\ p_! q^* q_! p^* & \longrightarrow & \bar{q}^* \bar{q}_! \end{array}$$

is a pointwise pushout.

In detail, consider $x \in \mathcal{X}$. Take the pushout

$$\begin{array}{ccc} p_! p^* x & \longrightarrow & x \\ \downarrow & & \downarrow \\ p_! q^* q_! p^* x & \longrightarrow & x'. \end{array}$$

Apply p^* and use the unit $\mathrm{id} \simeq p^* p_!$ to obtain a pushout

$$\begin{array}{ccc} p^* x & = & p^* x \\ \downarrow & & \downarrow \\ q^* q_! p^* x & \xrightarrow{\alpha} & p^* x', \end{array}$$

making α invertible. This induces $\bar{q}_! x = \langle q_! p^* x \,{,_\alpha}\, x' \rangle \in \mathcal{W}$ with unit

$$\eta_x \colon x \to x' \simeq \bar{q}^* \bar{q}_! x.$$

Proof. This is the combination and dualization of Lemma 4 and Lemma 2. The claimed description of the right adjoint is that of Lemma 4 with the description of the slicewise right adjoint given by Lemma 1 unfolded. □

Mapping spaces of pushouts

Corollary 6. *Consider a pushout of categories*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{g} & \mathcal{C} \\ \downarrow{\scriptstyle f} & \ulcorner & \downarrow{\scriptstyle \bar{f}} \\ \mathcal{B} & \xrightarrow[\bar{g}]{} & \mathcal{D} \end{array}$$

with f fully faithful. The maps

$$\begin{aligned} \mathcal{C}(c_0, c_1) &\longrightarrow \mathcal{D}(\bar{f}(c_0), \bar{f}(c_1)), \\ |(c_0 \downarrow \mathcal{C}) \times_{\mathcal{C}} \mathcal{A} \times_{\mathcal{B}} (\mathcal{B} \downarrow b_1)| &\longrightarrow \mathcal{D}(\bar{f}(c_0), \bar{g}(b_1)), \\ |(b_0 \downarrow \mathcal{B}) \times_{\mathcal{B}} \mathcal{A} \times_{\mathcal{C}} (\mathcal{C} \downarrow c_1)| &\longrightarrow \mathcal{D}(\bar{g}(b_0), \bar{f}(c_1)) \end{aligned}$$

are invertible and the square

$$\begin{array}{ccc} |(b_0 \downarrow \mathcal{B}) \times_{\mathcal{A}} (\mathcal{B} \downarrow b_1)| & \longrightarrow & \mathcal{B}(b_0, b_1) \\ \downarrow & & \downarrow \\ |(b_0 \downarrow \mathcal{B}) \times_{\mathcal{B}} \mathcal{A} \times_{\mathcal{C}} \mathcal{C}^{[1]} \times_{\mathcal{C}} \mathcal{A} \times_{\mathcal{B}} (\mathcal{B} \downarrow b_1)| & \longrightarrow & \mathcal{D}(\bar{g}(b_0), \bar{g}(b_1)) \end{array}$$

is a pushout.

Proof. We have an induced pullback of categories

$$\begin{array}{ccc} \mathsf{Psh}(\mathcal{D}) & \xrightarrow{\bar{f}^*} & \mathsf{Psh}(\mathcal{B}) \\ {\scriptstyle \bar{g}^*}\downarrow & \lrcorner & \downarrow{\scriptstyle g^*} \\ \mathsf{Psh}(\mathcal{C}) & \xrightarrow{f^*} & \mathsf{Psh}(\mathcal{A}). \end{array}$$

Since f is fully faithful, so is the left adjoint $f_!$ of f^*. Lemma 3 applied dualized to the mirrored pullback shows that the unit $\mathrm{id} \to \bar{f}^* \bar{f}_!$ and the Beck–Chevalley map $f_! g^* \to \bar{g}^* \bar{f}_!$ are invertible. Corollary 5 applied to the pullback shows that the square

$$\begin{array}{ccc} f_! f^* & \longrightarrow & \mathrm{id} \\ \downarrow & & \downarrow \\ f_! g^* g_! f^* & \longrightarrow & \bar{g}^* \bar{g}_! \end{array}$$

is a pointwise pushout and the Beck–Chevalley map $g_! f^* \to \bar{f}^* \bar{g}_!$ is invertible. The mapping space claims correspond to the evaluations of these statements at representables. □