PUSHFORWARD OF FIBRATIONS "ON THE RIGHT"

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ABSTRACT. We work in cartesian cubical sets with the ABCFHL notion of fibrations (called unbiased fibrations by Awodey). We give a diagrammatic presentation of Thierry's proof of fibrancy of dependent products "on the right", i.e. avoiding speaking about (generating) left categories as far as possible (by adjointness, one also obtains a presentation "on the left", but there one has to speak about coherence). Unfolding it reproduces Thierry's original proof, of which it was derived.

When saying that a map is a (trivial) fibration, we mean that we have a (trivial) fibration structure on it (note that trivial fibration structures are called plus-algebras by Awodey). Recall that fibration structures on a map $Y \to X$ are naturally isomorphic to trivial fibration structures in the slice over I on the pullback exponential with $\Delta : (I, \mathrm{id}) \to (I \times I, \pi_1)$ of the base change of $Y \to X$ to I:

$$Y^I \longrightarrow X^I \times_X Y.$$

Below, we will use the following facts:

(a) trivial fibrations have sections,
(b) trivial fibrations are closed under pushforward.
(c) trivial fibrations are closed under retracts.

Let $Z \to Y \to X$ be maps with fibration structures. We wish to produce a fibration structure on $\Pi_Y Z \to X$. We work in the slice over I, leaving the base change of objects to I implicit; for example, we have $\Delta \colon 1 \to I$. We have that

$$Y^I \longrightarrow X^I \times_X Y, \tag{0.1}$$

$$Z^I \longrightarrow Y^I \times_Y Z \tag{0.2}$$

are trivial fibrations and wish to show that

$$(\Pi_Y Z)^I \longrightarrow X^I \times_X \Pi_Y Z \tag{0.3}$$

is a trivial fibration.

In the following, we use the language of extensional dependent type theory in the locally cartesian closed category of cartesian cubical sets to shorten the presentation. There, we write the given objects as a type X and families Y over X and Z over $\Sigma_{x:X} Y(x)$. We work in context $\Delta : I$. The map (0.3) that we wish to exhibit as a trivial fibration is

$$\begin{array}{c} \displaystyle\sum_{x:X^I} \prod_{i:I} \prod_{y:Y(x(i))} Z(x(i), y(i)) \\ \big\downarrow \\ \displaystyle\sum_{x:X^I} \prod_{y:Y(x(\Delta))} Z(x(\Delta), y(\Delta)). \end{array} \tag{0.4}$$

By (a), we have a section to (0.1):

$$
\begin{tikzcd}
\sum_{x:X^I} Y(x(\Delta)) \arrow[r, dotted] \arrow[rr, bend left, "\mathrm{id}"] & \sum_{x:X^I} \prod_{i:I} Y(x(i)) \arrow[r] & \sum_{x:X^I} Y(x(\Delta)).
\end{tikzcd}
\tag{0.5}
$$

This allows us to write (0.4) as a retract

$$
\begin{tikzcd}
\sum_{x:X^I} \prod_{i:I} \prod_{y:Y(x(i))} Z(x(i), y) \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{id}"] & \sum_{x:X^I} \prod_{i:I} \prod_{y:\prod_{j:I} Y(x(j))} Z(x(i), y(i)) \arrow[r, dotted] \arrow[d] & \sum_{x:X^I} \prod_{i:I} \prod_{y:Y(x(i))} Z(x(i), y) \arrow[d] \\
\sum_{x:X^I} \prod_{y:Y(x(\Delta))} Z(x(\Delta), y) \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \sum_{x:X^I} \prod_{y:\prod_{j:I} Y(x(j))} Z(x(\Delta), y(\Delta)) \arrow[r, dotted] & \sum_{x:X^I} \prod_{y:Y(x(\Delta))} Z(x(\Delta), y)
\end{tikzcd}
$$

where the dotted maps are induced by the dotted map in (0.5). By (c), it thus suffices to show that the middle vertical map in this diagram is a trivial fibration. Commuting the dependent products in its domain, that map is isomorphic to

$$
\begin{tikzcd}
\sum_{x:X^I} \prod_{y:\prod_{j:I} Y(x(j))} \prod_{i:I} Z(x(i), y(i)) \arrow[d] \\
\sum_{x:X^I} \prod_{y:\prod_{j:I} Y(x(j))} Z(x(\Delta), y(\Delta)).
\end{tikzcd}
$$

This is (the underlying map of) the pushforward of (0.2) along the map

$$
\sum_{x:X^I} \prod_{j:I} Y(x(j)) \longrightarrow X^I.
$$

By ((b)), it is a trivial fibration.