NORMALIZATION BY EVALUATION FOR CATEGORIES WITH FAMILIES

CHRISTIAN SATTLER

Abstract. Some unfinished notes on a categorical presentation of normalization by evaluation We choose a presentation of cwf's close to natural models [Awo16].

1. Categories with families

Given a category $\mathcal{C}$, we write $\widehat{\mathcal{C}}$ for the category of presheaves on $\mathcal{C}$. We will freely use type-theoretic language for working with presheaves, referring to $\widehat{\mathcal{C}}$ as a *presheaf model*. Here, a context Γ means a presheaf $\Gamma \in \widehat{\mathcal{C}}$, a type $\Gamma \vdash A$ means a presheaf $A \in \widehat{\int \Gamma}$, and a term $\Gamma \vdash t : A$ means a section t of A. Given types $\Gamma \vdash A$ and $\Gamma, a : A \vdash B(a)$, we write

$$\Gamma \vdash (a : A) \times B(a) =_{\mathrm{def}} \sum_{a:A} B(a)$$

for the dependent sum

$$\Gamma \vdash (a : A) \to B(a) =_{\mathrm{def}} \prod_{a:A} B(a)$$

for the dependent product. We also use list notation for iterated dependent sums. For example, given types $\Gamma \vdash A$, $\Gamma.A \vdash B$, $\Gamma.A.B \vdash C$, we write

$$\Gamma \vdash [a : A, b : B(a), C(a, b)] =_{\mathrm{def}} \sum_{a:A} \sum_{b:B(a)} C(a, b) =_{\mathrm{def}} \Sigma_A \Sigma_B C.$$

The extensional equality type is denoted Eq.

We allow ourselves to identify and freely mix contexts and global types.

Definition 1.1. A *category with families* (cwf) is a category $\mathcal{C}$ together with, in the presheaf model $\widehat{\mathcal{C}}$, a context $\mathbf{Ty}$, a type $\mathbf{Ty} \vdash \mathbf{Tm}$, and a representing object of $\mathbf{Tm}[A]$ for each element $A : y\Gamma \to \mathbf{Ty}$ of $\mathbf{Ty}$. □

The last condition in the above definition is equivalent to a right adjoint to the first projection $\mathbf{Ty}.\mathbf{Tm} \to \mathbf{Ty}$.

Remark 1.2. We adopt the usual notation for a cwf $\mathcal{C}$. The category $\mathcal{C}$ is referred to as the *category of contexts and substitutions*. The elements of $\mathbf{Ty}$ and $\mathbf{Tm}$ are called *types* and *terms*, respectively. Given a type $A \in \mathbf{Ty}(\Gamma)$, note that $\mathbf{Tm}[A]$ is a presheaf over $\int y\Gamma \simeq \mathcal{C}/\Gamma$; its representing object is denoted $q_A \in \mathbf{Tm}[A](p_A) = \mathbf{Tm}[A[p_A]]$ where $p_A : \Gamma.A \to \Gamma$; we call $\Gamma.A$ the *context extension* of Γ by A and respectively call p_A and q_A the *context projection* and *last variable* of the context $\Gamma.A$.

Consider a substitution $\sigma : \Delta \to \Gamma$. Given a type $A \in \mathbf{Ty}(\Gamma)$, we write $A[\sigma] \in \mathbf{Ty}(\Delta)$ for the image of A under the action of $\mathbf{Ty}$ on σ; we have an induced substitution $\sigma.A$ between the associated context extensions, forming a pullback as below:

$$\begin{array}{ccc} \Delta.A[\sigma] & \xrightarrow{\sigma.A} & \Gamma.A \\ {\scriptstyle p_{A[\sigma]}}\downarrow & \lrcorner & \downarrow{\scriptstyle p_A} \\ \Delta & \xrightarrow{\sigma} & A. \end{array}$$

Given a term $t \in \mathbf{Tm}(\Gamma, A)$, we write $t[\sigma] \in \mathbf{Tm}(\Delta, A[\sigma])$ for the image of t under the action of $\mathbf{Tm}$ on (σ, A); note that $q_{A[\sigma]} = q_A[\sigma.A]$.

We omit subscripts if they are evident from the surrounding context. □

Remark 1.3. Following Cartmell, Cwf's are the models for a generalized algebraic theory; we do not describe it in detail here. Recall that the models for a generalized algebraic theory form a category. We thus obtain a *category of cwf's*. A morphism

$$(F, u, v) \colon (\mathcal{C}, \mathbf{Ty}_{\mathcal{C}}, \mathbf{Tm}_{\mathcal{C}}) \to (\mathcal{D}, \mathbf{Ty}_{\mathcal{D}}, \mathbf{Tm}_{\mathcal{D}})$$

consists of a functor $F \colon \mathcal{C} \to \mathcal{D}$, a natural transformation $u \colon \mathbf{Ty}_0 \to \mathbf{Ty}_1 F$, and an isomorphism $v \colon \mathbf{Tm}_0 \simeq (\mathbf{Tm}_1 F)[u]$. Note that precomposition with F, denoted here by postfixing F, induces a morphism from the presheaf model over $\mathcal{C}$ to the presheaf model over $\mathcal{D}$. □

We will be interested also in the category of cwf's with category of contexts $\mathcal{C}$ fixed, i.e. the category of cwf structures on $\mathcal{C}$. We see that a morphism from $(\mathbf{Ty}_0, \mathbf{Tm}_0)$ to $(\mathbf{Ty}_1, \mathbf{Tm}_1)$ is a pair (u, v) with a natural transformation $u \colon \mathbf{Ty}_0 \to \mathbf{Ty}_1$ and an isomorphism $v \colon \mathbf{Tm}_0 \simeq \mathbf{Tm}_1[u]$. Note that this can also be represented by a pullback square

$$\begin{array}{ccc} \mathbf{Ty}_0.\mathbf{Tm}_0 & \longrightarrow & \mathbf{Ty}_1.\mathbf{Tm}_1 \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{Ty}_0 & \longrightarrow & \mathbf{Ty}_1. \end{array}$$

We will now specify what it means for a cwf $\mathcal{C}$ as above to have certain type formers.

1.1. Extensional type formers.

Definition 1.4. *Unit types* are given by a term $\mathbf{1} : \mathbf{Ty}$ with an isomorphism $1 \simeq \mathbf{Tm}(\mathbf{1})$, all in the empty context. □

Definition 1.5. *Dependent sums* are given by a term $\mathbf{\Sigma}_{A,B} : \mathbf{Ty}$ with an isomorphism

$$\sum_{a : \mathbf{Tm}(A)} \mathbf{Tm}(B(a)) \simeq \mathbf{Tm}(\mathbf{\Sigma}_{A,B}),$$

all in context $A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}$. □

Definition 1.6. *Dependent products* are given by a term $\mathbf{\Pi}_{A,B} : \mathbf{Ty}$ with an isomorphism

$$\prod_{a : \mathbf{Tm}(A)} \mathbf{Tm}(B(a)) \simeq \mathbf{Tm}(\mathbf{\Pi}_{A,B}),$$

all in context $A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}$. □

1.2. **Empty types.** Before we start defining non-extensional type formers, we give a definition that will help us characterize many of the common eliminators.

Definition 1.7. Given a function $X \vdash f : A \to B$, *elimination* consists of a section of the pullback exponential in context X with f of $\mathsf{fst} : \Sigma_{\mathbf{Ty}}\mathbf{Tm} \to \mathbf{Ty}$. □

In more concrete terms, elimination for $X \vdash f : A \to B$ is a term

$$X, C : B \to \mathbf{Ty}, c_A : (a : A) \to \mathbf{Tm}(C(f(x))) \vdash c_B : (b : B) \to \mathbf{Tm}(C(b))$$

such that $c_B \circ f = c_A$.

Definition 1.8. *Empty types* are given by a term $\mathbf{0} : \mathbf{Ty}$ and a function $0 \to \mathbf{Tm}(\mathbf{0})$ with elimination, all in the empty context. □

In more concrete terms, empty types consist of a term $\vdash \mathbf{0} : \mathbf{Ty}$ with a term

$$C : \mathbf{0} \to \mathbf{Ty} \vdash \mathbf{0\text{-}elim}_C : (t : \mathbf{0}) \to \mathbf{Tm}(C(t)).$$

1.3. **Coproduct types.**

Definition 1.9. *Coproduct types* are given by a term $A \boldsymbol{+} B : \mathbf{Ty}$ and a function

$$\boldsymbol{\tau} : \mathbf{Tm}(A) + \mathbf{Tm}(B) \to \mathbf{Tm}(A \boldsymbol{+} B)$$

with elimination, all in context $A : \mathbf{Ty}, B : \mathbf{Ty}$. □

In more concrete terms, coproduct types consist of a term $A \boldsymbol{+} B : \mathbf{Ty}$ (we use infix notation) with terms

$$\begin{aligned} \boldsymbol{\tau}_0 &: \mathbf{Tm}(A) \to \mathbf{Tm}(A \boldsymbol{+} B), \\ \boldsymbol{\tau}_1 &: \mathbf{Tm}(B) \to \mathbf{Tm}(A \boldsymbol{+} B), \end{aligned}$$

all in context $A : \mathbf{Ty}, B : \mathbf{Ty}$, and a term

$$\boldsymbol{+}\text{-}\mathbf{elim}_{C,c_0,c_1} : (t : A \boldsymbol{+} B) \to C(t)$$

in context

$$\begin{aligned} A &: \mathbf{Ty}, \\ B &: \mathbf{Ty}, \\ C &: A \boldsymbol{+} B \to \mathbf{Ty}, \\ c_0 &: (a : \mathbf{Tm}(A)) \to \mathbf{Tm}(C(\boldsymbol{\tau}_0(a))), \\ c_1 &: (b : \mathbf{Tm}(B)) \to \mathbf{Tm}(C(\boldsymbol{\tau}_1(b))) \end{aligned}$$

such that

$$\begin{aligned} \boldsymbol{+}\text{-}\mathbf{elim}_{C,c_0,c_1}(\boldsymbol{\tau}_{\mathbf{0}}(a)) &= c_0(a), \\ \boldsymbol{+}\text{-}\mathbf{elim}_{C,c_0,c_1}(\boldsymbol{\tau}_{\mathbf{1}}(b)) &= c_1(b), \end{aligned}$$

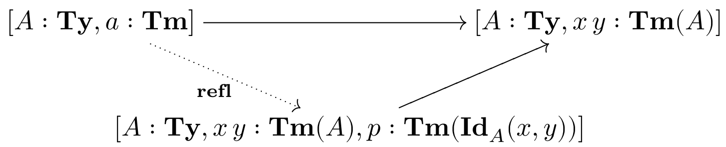
in additional contexts $a : \mathbf{Tm}(A)$ and $b : \mathbf{Tm}(B)$, respectively.

1.4. **Identity types.**

Definition 1.10. *Identity types* are given by a term

$$A : \mathbf{Ty}, x\,y : \mathbf{Tm}(A) \vdash \mathbf{Id}_A(x, y) : \mathbf{Ty}$$

with a lift of the diagonal through the context projection



that has elimination in context $\mathbf{Ty}$. □

In more concrete terms, identity types consist of a term

$$A : \mathbf{Ty}, x\,y : \mathbf{Tm}(A) \vdash \mathbf{Id}_A(x, y) : \mathbf{Ty}$$

as above and a term

$$A : \mathbf{Ty}, a : \mathbf{Tm}(A) \vdash \mathbf{refl}_A(a) : \mathbf{Tm}(\mathbf{Id}_A(a, a))$$

such that the function

$$A : \mathbf{Ty} \vdash \langle \mathrm{id}, \mathrm{id}, \mathbf{refl} \rangle : \mathbf{Tm}(A) \to [x\,y : \mathbf{Tm}(A), \mathbf{Tm}(\mathbf{Id}_A(x, y))]$$

has elimination; this means a term

$$J_{C,d} : [x\,y : \mathbf{Tm}(A), p : \mathbf{Id}_A(x, y)] \to \mathbf{Tm}(C(x, y, p))$$

in context

$$\begin{aligned} &A : \mathbf{Ty}, \\ &C : [x\,y : \mathbf{Tm}(A), p : \mathbf{Id}_A(x, y))] \to \mathbf{Ty}, \\ &d : (a : \mathbf{Tm}(A)) \to \mathbf{Tm}(C(a, a, \mathbf{refl}_A(a))) \end{aligned}$$

such that $J_{C,d}(a, a, \mathbf{refl}_A(a)) = d$ in additional context $a : \mathbf{Tm}(A)$.

1.5. W**-types.**

Observe that in context $A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}$, we have an internal endofunctor $P_{A,B}$ sending X to $[a : \mathbf{Tm}(A), f : \mathbf{Tm}(B(a)) \to X]$. We have the type

$$[T : \mathbf{Ty}, s : [a : \mathbf{Tm}(A), f : \mathbf{Tm}(T)^{\mathbf{Tm}(B(a))}] \to \mathbf{Tm}(T)]$$

of *algebra types*. This is isomorphically the type of algebras of $P_{A,B}$ with carrier classified by $\mathbf{Ty} \vdash \mathbf{Tm}$. In additional context of an algebra (X, s), we have the type

$$[C : X \to \mathbf{Ty}, [a : \mathbf{Tm}(A), f : X^{\mathbf{Tm}(B(a))}, u : (b : \mathbf{Tm}(B(a))) \to \mathbf{Tm}(C(f(b)))] \to C(s(a, f))$$

of *algebra families* over (X, s). This is isomorphically the type of objects of the slice over (X, s) of the internal category of algebra over $P_{A,B}$ with morphism on carriers classified by $\mathbf{Ty} \vdash \mathbf{Tm}$. A *section* of such an algebra family is a section of the morphism to (X, s) it induces.

Definition 1.11. W*-types* are given by an algebra type $(\mathbf{W}_{A,B}, \mathbf{sup})$ in context $A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}$ and in additional context of an algebra family over its induced algebra, a section. □

In more concrete terms, W-types consist of a term $\mathbf{W}_{A,B}$ and a function

$$\mathbf{sup}\colon [a : \mathbf{Tm}(A), f : \mathbf{Tm}(B(a)) \to \mathbf{Tm}(\mathbf{W}_{A,B})] \to \mathbf{Tm}(\mathbf{W}_{A,B}),$$

both in context $A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}$, and a term

$$\mathbf{W\text{-}elim}_{C,d} : (t : \mathbf{W}_{A,B}) \to \mathbf{Tm}(C(t))$$

in additional context

$$\begin{aligned}
&C : \mathbf{W}_{A,B} \to \mathbf{Ty},\\
&d : [a : \mathbf{Tm}(A), f : \mathbf{Tm}(B(a)) \to \mathbf{W}_{A,B}, u : (b : \mathbf{Tm}(B(a))) \to \mathbf{Tm}(C(f(b)))]\\
&\qquad \to \mathbf{Tm}(C(\mathbf{sup}(a, f)))
\end{aligned}$$

such that

$$\mathbf{W\text{-}elim}_{C,d}(\mathbf{sup}(a, f)) = d(a, f, \lambda b.\ \mathbf{W\text{-}elim}_{C,d}(f(b)))$$

in additional context $a : \mathbf{Tm}(A), f : \mathbf{Tm}(B(a)) \to \mathbf{W}_{A,B}$.

1.6. **Preservation of type formers.** Just as cwf's are the models for a generalized algebraic theory, so are cwf's with any choice of the above type formers. We thus obtain categories of cwf's with empty types, dependent sums, etc. Given a collection T of type formers, we write $\mathbf{CwF}^T$ for the category of cwf's with type formers T. These project to the category of cwf's via faithful forgetful functors. Thus, their morphisms can be described as morphisms $F = (F, u, v)\colon \mathcal{C} \to \mathcal{D}$ of cwf's that satisfy a condition modelling preservation of the respective type formers.

Let the cwf's $\mathcal{C}$ and $\mathcal{D}$ have dependent sums. Then F preserves dependent sums if: [ugly unfolding; it must be possible tot write this in a nice form]

1.7. **Universes.**

Definition 1.12. A *universe* (or *universe types*) is given by a term $\vdash \mathbf{U} : \mathbf{Ty}$ and a map $\mathbf{El}\colon \mathbf{Tm}(\mathbf{U}) \to \mathbf{Ty}$. □

The *associated cwf* of a universe as above has contexts $\mathcal{C}$, types $\mathbf{Tm}(\mathbf{U})$, and terms $\mathbf{Tm}(\mathbf{U}) \vdash \mathbf{Tm}[\mathbf{El}]$. We have a morphism of cwf's over $\mathcal{C}$ from the associated cwf $(\mathcal{C}, \mathbf{Tm}(\mathbf{U}), \mathbf{Tm}[\mathbf{El}])$ to the original cwf $(\mathcal{C}, \mathbf{Ty}, \mathbf{Tm})$ given by the natural transformation $\mathbf{El}\colon \mathbf{Tm}(\mathbf{U}) \to \mathbf{Ty}$ between presheaves of types and the identity isomorphism for presheaves of terms.

A universe is *closed under a type former* if that type former is present in the associated cwf and preserved by the morphism to the original cwf.

A *morphism of universes* from $(\mathbf{U}', \mathbf{El}')$ to $(\mathbf{U}, \mathbf{El})$ is a map

$$\mathbf{lift}\colon \mathbf{Tm}(\mathbf{U}') \to \mathbf{Tm}(\mathbf{U})$$

such that

$$\mathbf{Tm}[\mathbf{El} \circ \mathbf{lift}] = \mathbf{Tm}[\mathbf{El}'].$$

Note that this gives rise to an *associated morphism of cwf's* over $\mathcal{C}$ between the associated cwf's of $\mathbf{U}'$ and $\mathbf{U}$.

This morphism of universes is *closed under a type former* if both the associated cwf's have that type former and it is preserved by the morphism of cwf's between them.

Remark 1.13. A universe $(\mathbf{U}, \mathbf{El})$ is called *injective* if the components of El are injective.

Applying these definitions to the universe and morphism of universe type formers itself, one we may define arbitrary (cumulative as well as non-cumulative) hierarchies of universes (for circular hierarchies, the definition of cwf's becomes inductive). We will only consider a wellformed hierarchy indexed by ω. □

2. Renamings

A morphism of cwf's

$$(F, u, v) \colon (\mathcal{C}, \mathbf{Ty}_{\mathcal{C}}, \mathbf{Tm}_{\mathcal{C}}) \to (\mathcal{D}, \mathbf{Ty}_{\mathcal{D}}, \mathbf{Tm}_{\mathcal{D}})$$

is *creating types* if u is an isomorphism. We denote

$$\mathbf{CwF}^{\textsf{iso-on-types}} \hookrightarrow \mathbf{CwF}.$$

the wide subcategory of those morphisms of cwf's that create types.

There must be a categorical way to define renamings. Given a cwf $\mathcal{C}$, the renamings are essentially the morphisms of $\mathcal{C}$ that are constructible from zero. Can we express this as a free construction?

Renamings form a cwf, but without any type constructors. Should objects of $\mathcal{C}$ stay the same? Maybe we should integrate it with the construction that freely constructs objects as iterated context extensions. So then the entire category $\mathcal{C}$ is freely reconstructed.

Lemma 2.1. *Let $\mathcal{C}$ be a cwf. The slice category*

$$\mathbf{CwF}^{\textsf{iso-on-types}}/\mathcal{C}$$

has an initial object.

Proof. The category $\mathbf{CwF}^{\textsf{iso-on-types}}/\mathcal{C}$ is equivalent to the category of models of the following essentially algebraic theory:

- Sorts are obj_{Γ} indexed by objects Γ of $\mathcal{C}$ and hom_{σ} indexed by morphisms σ of $\mathcal{C}$.
- We have domain and codomain, identity, and composition operations over those of $\mathcal{C}$, satisfying the laws of a category.
- Given $\Gamma \in \mathcal{C}$ and $A \in \mathbf{Ty}(\Gamma)$, an operations that take $\Gamma' \in \mathsf{obj}_{\Gamma}$ and return an object $\Gamma'.A \in \mathsf{obj}_{\Gamma.A}$ and a map $p'_A \colon \Gamma'.A \to \Gamma'$ in hom_{p_A} with laws specifying its domain and codomain as indicated. Furthermore, operations and laws that express functoriality of this operation and operations and laws that express that the action on morphisms is valued in cartesian squares.

Thus, by standard universal algebra, we have an inital object. □

Definition 2.2. The *cwf of renamings* $\mathsf{Ren}(\mathcal{C})$ of a cwf $\mathcal{C}$ is defined with an associated morphism of cwf's $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ as follows. In the following, types and terms by default refer to those of $\mathsf{Ren}(\mathcal{C})$; those of $\mathcal{C}$ will be explicitly annotated by a subscript.

The objects of $\mathsf{Ren}(\mathcal{C})$ are tuples (m, Γ, A) with $m \in \mathbb{N}$, $\Gamma_i \in \mathcal{C}$ for $0 \leq i \leq m$ with $\Gamma_0 = \varepsilon$, and $A_i \in \mathbf{Ty}_{\mathcal{C}}(\Gamma_i)$ and $\Gamma_{i+1} = \Gamma_i.A_i$ for $0 \leq i < m$. We call m the *degree* of the object (m, Γ, A). The action of $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ on objects sends (m, Γ, A) to Γ_m.

The types of $\mathsf{Ren}(\mathcal{C})$ are created from those of $\mathcal{C}$ via $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$. That is, we set

$$\mathbf{Ty}(m, \Gamma, A) =_{\mathrm{def}} \mathbf{Ty}_{\mathcal{C}}(A_m)$$

and the action of $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ on types is the identity.

The terms of $\mathsf{Ren}(\mathcal{C})$ are defined as follows: given a context (m, Γ, A) with a type $X \in \mathbf{Ty}_{\mathcal{C}}(A_m)$, we set

$$\mathbf{Tm}((m, \Gamma, A); X) =_{\mathrm{def}} \{0 \leq i < m \mid X = A_i[p^{m-i}]]\}.$$

The action of $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ on terms sends i to $q_{A_i}[p^{m-(i+1)}] \in \mathbf{Tm}_{\mathcal{C}}(\Gamma_m, X)$.

The morphisms of $\mathsf{Ren}(\mathcal{C})$ are defined together with the action of $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ on morphisms as follows. We will define $(n, \Delta, B) \to (m, \Gamma, A)$ by recursion on m. If $m = 0$, we let there be a unique morphism; the action of F on it is uniquely determined as (m, Γ, A) is sent to a terminal object of $\mathcal{C}$. If $m = m' + 1$, we let a morphism be pair (u, i) where $u \colon (n, \Delta, B) \to (m, \Gamma, A)$ and $i \in \mathbf{Tm}((n, \Delta, B); A_{m'}[Fu])$; the action of F sends it to $\langle Fu, Fi \rangle$.

Substitution of types of $\mathsf{Ren}(\mathcal{C})$ is created from $\mathcal{C}$ via $\mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$. That is, given $A \in \mathbf{Ty}(m, \Gamma, A)$ and $\sigma \colon (n, \Delta, B) \to (m, \Gamma, A)$, we set $A[\sigma] = A[F\sigma]$. This is clearly compatible with $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$.

Substitution of terms of $\mathsf{Ren}(\mathcal{C})$ is defined as follows. Given a type $X \in \mathbf{Ty}(m, \Gamma, A)$, a term $i \in \mathbf{Tm}((m, \Gamma, A); X)$, i.e. $0 \leq i < m$ such that $X = A_i[p^{m-i}]$, and $u \colon (n, \Delta, B) \to (m, \Gamma, A)$, we define $i[u] \in \mathbf{Tm}((n, \Delta, B); X[Fu])$, i.e. $0 \leq i[u] < n$ such that $X[Fu] = B_{i[u]}[p^{n-i[\sigma]}]$, by recursion on m.

Note that $m > i \geq 0$. Write $m = m' + 1$ and $u = (u', j)$ with $u' \colon (n, \Delta, B) \to (m', \Gamma', A')$ and $j \in \mathbf{Tm}((n, \Delta, B); A_{m'}[Fu'])$, i.e. $0 \leq j < n$ such that $A_{m'}[Fu'] = B_j[p^{n-j}]$.

If $m' = i$, we set $i[u] =_{\mathrm{def}} j$, using that

$$\begin{aligned} X[Fu] &= A_{m'}[p^{m-m'}][Fu] \\ &= A_{m'}[p][\langle Fu', Fj \rangle] \\ &= A_{m'}[Fu'] \\ &= B_j[p^{n-j}]. \end{aligned}$$

This is compatible with $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ since

$$\begin{aligned} q_{A_i}[p^{m-(i+1)}][Fu] &= q_{A_i}[Fu] \\ &= q_{A_i}[\langle Fu', Fj \rangle] \\ &= Fj \\ &= q_{B_j}[p^{n-(j+1)}]. \end{aligned}$$

Otherwise, we have $i < m'$ and $i \in \mathbf{Tm}((m', \Gamma', A'), A_i[p^{m'-i}])$, hence by recursion have $i[u'] \in \mathbf{Tm}((n, \Delta, B), A_i[p^{m'-i}][Fu'])$ already defined, i.e. $A_i[p^{m'-i}][Fu'] = B_{i[u]}[p^{n-i[u]}]$. We set $i[u] =_{\mathrm{def}} i[u']$, using that

$$\begin{aligned} X[Fu] &= A_i[p^{m-i}][Fu] \\ &= A_i[p^{m'-i}][p][\langle Fu', Fj \rangle] \\ &= A_i[p^{m'-i}][Fu'] \\ &= B_{i[u]}[p^{n-i[u]}]. \end{aligned}$$

This is compatible with $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ since

$$\begin{aligned} q_{A_i}[p^{m-(i+1)}][Fu] &= q_{A_i}[p^{m'-i}][p][\langle Fu', Fj \rangle] \\ &= q_{A_i}[p^{m'-i}][Fu'] \\ &= q_{B_{i[u]}}[p^{n-(i[u]+1)}] \end{aligned}$$

where we have used the induction hypothesis in the last step.

Identities in $\mathsf{Ren}(\mathcal{C})$ are defined as follows. Consider an object (m, Γ, A). By recursion on $k < m$, we define a morphism $w_{m,k} \colon (m, \Gamma, A) \to (k, \Gamma', A')$ such that $Fw_{m,k} = p^{m-k}$ as follows. For $k = 0$, there is nothing to define and the condition holds for free. For $k = k' + 1$, we let

$w_{m,k} =_{\mathrm{def}} (w_{m,k'}, k')$ using that $A_{k'}[Fw_{m,k'}] = A_{k'}[p^{m-k'}]$; we have

$$\begin{aligned} Fw_{m,k} &= \langle Fw_{m,k'}, Fk' \rangle \\ &= \langle p^{m-k'}, q_{A'_k}[p^{m-(k'+1)}] \rangle \\ &= \langle pp^{m-k}, q_{A'_k}[p^{m-k}] \rangle \\ &= p^{m-k} \end{aligned}$$

where we used the induction hypothesis in the second step.

We now let the identity on (m, Γ, A) be given by $w_{m,m}$. Note that F preserves identities since $Fw_{m,m} = p^{m-m} = \mathrm{id}$.

Compositions in $\mathsf{Ren}(\mathcal{C})$ are defined as follows. Consider a composable pair of morphisms

$$(o, \Theta, C) \xrightarrow{\ v\ } (n, \Delta, B) \xrightarrow{\ u\ } (m, \Gamma, A).$$

We define the composition $u \circ v$ in a way that is compatible with the action of $F\colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ by recursion on m.

For $m = 0$, there is nothing to define and compatibility with F holds for free.

Otherwise, we have $m = m' + 1$. Write $u = (u', i)$ with $u'\colon (n, \Delta, B) \to (m', \Gamma', A')$ and $i \in \mathbf{Tm}((n, \Delta, B); A_{m'}[Fu'])$, i.e. $0 \le i < n$ such that $A_{m'}[Fu'] = B_i[p^{n-i}]$. By recursion, we already have defined $u' \circ v\colon (o, \Theta, C) \to (m', \Gamma', A')$ compatible with F. We will define $u \circ v =_{\mathrm{def}} (u \circ v, j)$ with $0 \le j < o$ such that $B_i[p^{n-i}][Fv] = C_j[p^{o-j}]$, implying

$$\begin{aligned} A_{m'}[F(u' \circ v)] &= A_{m'}[Fu'][Fv] \\ &= B_i[p^{n-i}][Fv] \\ &= C_j[p^{o-j}]. \end{aligned}$$

This will then be compatible with F since

$$\begin{aligned} F(u \circ v) &= F(u' \circ v, j) \\ &= \langle F(u' \circ v), Fj \rangle \\ &= \langle Fu' \circ Fv, Fj \rangle \\ &= \langle Fu', Fj \rangle \circ Fv \\ &= F(u', j) \circ Fv \\ &= Fu \circ Fv. \end{aligned}$$

The definition of j depends only on v and i, so we can forget about u'. It proceeds by recursion on n.

Note that $n > i \ge 0$. Write $n = n' + 1$ and $v = (v', k)$ with $v'\colon (o, \Theta, C) \to (n', \Delta', B')$ and $k \in \mathbf{Tm}((o, \Theta, C); B_{n'}[Fv'])$, i.e. $0 \le k < o$ such that $B_{n'}[Fv'] = C_k[p^{o-k}]$.

If $n' = i$, we set $j =_{\mathrm{def}} k$, using that

$$\begin{aligned} B_i[p^{n-i}][Fv] &= B_{n'}[p][\langle Fv', Fk \rangle] \\ &= B_{n'}[Fv'] \\ &= C_k[p^{o-k}]. \end{aligned}$$

Otherwise, we have $i < n'$ and $i \in \mathbf{Tm}((n', \Delta', B'); B_i[p^{n'-i}])$, hence by recursion we have $0 \leq j < o$ such that $B_i[p^{n'-i}][Fv'] = C_j[p^{o-j}]$. Then we also have

$$\begin{aligned} B_i[p^{n-i}][Fv] &= B_i[p^{n'-i}][p][\langle Fv', Fk \rangle] \\ &= B_i[p^{n'-i}][Fv'] \\ &= C_j[p^{o-j}]. \end{aligned}$$

The empty context is given by the unique object of degree zero. The context extension of an object (m, Γ, A) by a type $A_m \in \mathbf{Ty}_{\mathcal{C}}(\Gamma_m)$ is given by $(m+1, \Gamma, A)$. The context projection $p_{A_m} : (m+1, \Gamma, A) \to (m, \Gamma, A)$ is given by $w_{m+1,m}$. The generic term $q_{A_m} : \mathbf{Tm}((m+1, \Gamma, A), A_m[p])$ is given by m. The substitution extension operation is given by the definition of morphisms. All of these are clearly compatible with $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$.

We have defined all the sorts and operations of a cwf, defined the action of $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ on the sorts and checked that F preserves the operations. It remains to check that the operation in $\mathsf{Ren}(\mathcal{C})$ satisfy the laws of a cwf. Here, we will take a shortcut and go via a faithful morphism to the cwf $\mathbf{FinSet}^{\mathrm{op}}$.

Recall that $\mathbf{FinSet}^{\mathrm{op}}$ forms a cwf. There is a unique type in every context, i.e. the presheaf of types is terminal. The terms in context S of the unique type are given by the elements of S, i.e. the presheaf of terms is given by the embedding $\mathbf{FinSet} \to \mathbf{Set}$. The empty context is given by 0. The context extension of S by the unique type is $S+1$. The associated context projection is the inclusion $p_S \colon S \hookrightarrow S+1$. The representing term of the context extension is given by the element $q_S \colon 1 \hookrightarrow S+1$. The substitution extension operation is given by the universal property of coproducts.

Let us construct a morphism of cwf's $P \colon \mathsf{Ren}(\mathcal{C}) \to \mathbf{FinSet}^{\mathrm{op}}$ (this notion makes sense also when some of the laws of a cwf in the domain are not required to hold).

For an object (m, Γ, A), we define

$$P(m, \Gamma, A) =_{\mathrm{def}} \underline{m} =_{\mathrm{def}} \{0, 1, ..., m-1\}.$$

For a morphism $u \colon (n, \Delta, B) \to (m, \Gamma, A)$, we define the function $Pu \colon \underline{m} \to \underline{n}$ by recursion on m. If $m = 0$, there is nothing to define. If $m = m' + 1$, we write $u = (u', j)$ with $u' : (n, \Delta, B) \to (m', \Gamma', A')$ and $j \in \mathbf{Tm}((n, \Delta, B); A_{m'}[Fu'])$, i.e. $0 \leq j < n$ such that $A_{m'}[Fu'] = B_j[p^{n-j}]$; we define $Pu(i) =_{\mathrm{def}} Pu'(i)$ for $i < m'$ and $Pu(m') =_{\mathrm{def}} j$.

From the recursions defining identities and compositions in $\mathsf{Ren}(\mathcal{C})$, it is evident that P preserves these. Thus, we have shown P a functor.

The action of P on types is uniquely contrained.

The action of P on terms sends $i \in \mathbf{Tm}((m, \Delta, A); X)$ to $i \in \underline{m}$. From the recursion defining substitutions of terms in $\mathsf{Ren}(\mathcal{C})$, we see that P preserves these.

By inspecting the corresponding definitions in $\mathsf{Ren}(\mathcal{C})$, we also see that P preserves the empty context, context extension (if we take care to define $\underline{m}$ correctly), context projections, representing terms of context extensions, and extension of substitutions.

It follows that P is a morphism of cwf's. Furthermore, the assignment of a function to each morphism is clearly injective, making P a faithful morphism. From faithfulness, it follows that all the laws of a cwf hold in $\mathsf{Ren}(\mathcal{C})$ (except for preservation of identities and compositions by substitutions of types, which holds because types are created from $\mathcal{C}$).

This finishes the definition of the cwf $\mathsf{Ren}(\mathcal{C})$ of renamingsand its associated morphism of cwf's $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$. □

Remark 2.3. Note that the cwf $\mathbf{FinSet}^{\mathrm{op}}$ considered in the construction of Definition 2.2 is itself an instance of a cwf of renamings: we have $\mathbf{FinSet}^{\mathrm{op}} \simeq \mathsf{Ren}(1)$ where 1 denotes the terminal cwf. The morphism $\mathsf{Ren}(\mathcal{C}) \to \mathbf{FinSet}^{\mathrm{op}}$ can be seen as the action of Ren as an endofunctor on **CwF** applied to the unique morphism $\mathcal{C} \to 1$.

Alternatively, $\mathbf{FinSet}^{\mathrm{op}}$ can be seen as the free cwf with one global type. □

We can characterize the cwf $\mathsf{Ren}(\mathcal{C})$ of renamings by a universal property. Hopefully, this can be used later to gain some high-level understanding.

Lemma 2.4. *Let $\mathcal{C}$ be a cwf. Then $\mathsf{Ren}(\mathcal{C})$ together with the associated morphism $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ is an initial object of the slice category*

$$\mathbf{CwF}^{\textsf{iso-on-types}}/\mathcal{C}.$$

Proof. Let $\mathcal{D}$ be a cwf with a morphism $G \colon \mathcal{D} \to \mathcal{C}$ that creates types. We need to show that there is a unique morphism $H \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{D}$ that creates types such that $G = FH$.

Let us first check that the action of H on an object (m, Γ, A) is uniquely determined. We proceed by induction on m. If $m = 0$, then (m, Γ, A) is the empty context and thus needs to be sent to the empty context in $\mathcal{D}$. If $m = m' + 1$, then (m, Γ, A) is the context extension of (m', Γ', A') with $A_{m'}$; it must thus be sent to the context extension of $H(m', \Gamma', A')$ by $HA_{m'}$. Note that the action of H on objects defined thusly coheres with those of F and G.

The action of H on types is uniquely determined by the condition that all of F, G, H create types.

The terms of $\mathsf{Ren}(\mathcal{C})$ in context (m, Γ, A) are all of the form $q_{A_i}[p^{m-(i+1)}]$ with $0 \leq i < m$. Since H needs to preserve q, p, and substitution of terms, the action of H on terms is uniquely determined.

The morphisms of $\mathsf{Ren}(\mathcal{C})$ are of the form of iterated substitution extensions starting with the unique morphism to the empty context. Since H needs to preserve substitution extension and terminality of the empty context, the action of H on morphisms is uniquely determined.

It follows that H is uniquely determined. It remains to check that the operations of H thusly defined satisfy the laws of a morphism of cwf's and that $G = FH$. This is all straightforward, though tedious. □

Lemma 2.5. *Let $\mathcal{C}$ be a cwf. Then $\mathsf{Ren}(\mathcal{C})$ admits an orthogonal factorization system created from the (epi, mono) orthogonal factorizatiom system via the functor $\mathsf{Ren}(\mathcal{C}) \to \mathbf{FinSet}^{\mathrm{op}}$.*

Lemma 2.6. *Let $\mathcal{C}$ be a cwf. Then $\mathsf{Ren}(\mathcal{C})$ has pushouts, and they are preserved by the functor $\mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$.*

Proof. Consider a span

$$\begin{array}{ccc} (k, \Gamma, A) & \xrightarrow{f} & (m, \Delta, B) \\ \Big\downarrow{\scriptstyle g} & & \vdots \\ (n, \Theta, C) & \cdots\cdots\to & (p, \Xi, D). \end{array}$$

We will try to construct its pushout as indicated. For this, we perform induction on n.

If $n = 0$, then we let $p = 0$, and the dotted maps are uniquely determined. This maps to a pushout in $\mathbf{FinSet}^{\mathrm{op}}$ because pullbacks in **FinSet** preserve initial objects as **FinSet** as locally cartesian closed. It remains to see that, given any competing cocone in $\mathsf{Ren}(\mathcal{C})$, the induced map in $\mathbf{FinSet}^{\mathrm{op}}$ lifts to $\mathsf{Ren}(\mathcal{C})$. But this is evident as it is a copy of the map from (n, Θ, C).

Now let $n = n'+1$. Then $g = (g', i)$ where $g' \colon (k, \Gamma, A) \to (n', \Theta, C)$ and $i \in \mathbf{Tm}((k, \Gamma, A); C_{n'}[Fg'])$. By induction hypothesis, we have a pushout

$$\begin{array}{ccc} (k, \Gamma, A) & \xrightarrow{f} & (m, \Delta, B) \\ \big\downarrow{\scriptstyle g'} & & \big\downarrow \\ (n', \Theta, C) & \longrightarrow & \ulcorner (p', \Xi', D') \end{array}$$

that is preserved under $\mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$.

Given a map

$$u \colon (n, \Delta, B) \to (m, \Gamma, A)$$

and an index $i < n$, let's look at the set of $j < m$ that get mapped to i.

□

Lemma 2.7. *Let $\mathcal{C}$ be a cwf. Then $\mathsf{Ren}(\mathcal{C})$ has pullbacks, and they are preserved by the functor $\mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$.*

Proof. Consider a cospan

$$\begin{array}{ccc} & & (k, \Gamma, A) \\ & & \big\downarrow{\scriptstyle f} \\ (m, \Delta, B) & \xrightarrow{g} & (n, \Theta, C). \end{array}$$

We will try to construct its pullback.

□

3. Abstract Nonsense

Let $\mathcal{E}$ be a category.

Definition 3.1. A class of maps $\mathcal{R}$ in $\mathcal{E}$ is called *good* if:

- $\mathcal{R}$ is closed under identities and composition; we denote $\mathcal{E}^{\mathcal{R}}$ the wide subcategory of $\mathcal{E}$ with maps restricted to $\mathcal{R}$,
- $\mathcal{R}$ is closed under pullback,
- $\mathcal{R}$ is right cancellable: if $gf, g \in \mathcal{R}$, then $f \in \mathcal{R}$,
- for every $A \in \mathcal{E}$, the slice category $\mathcal{E}^{\mathcal{R}}/A$ has an initial object $(F(A), F(A) \xrightarrow{t_A} A)$.

□

Let $\mathcal{E}$ be a good class of maps in $\mathcal{E}$. For $A \in \mathcal{E}$, we call $F(A)$ the *$\mathcal{R}$-free object* on A. We say that A is *$\mathcal{R}$-free* if $t_A \colon F(A) \to A$ is an isomorphism, i.e. if (A, id_A) is initial in $\mathcal{E}^{\mathcal{R}}/A$.

The naming is justified by the following statement.

Lemma 3.2. *For every $A \in \mathcal{E}$, the $\mathcal{R}$-free object $F(A)$ on A is itself $\mathcal{R}$-free.*

Proof. The forgetful functors

$$\mathcal{E}^{\mathcal{R}}/F(A) \xleftarrow{\simeq} (\mathcal{E}^{\mathcal{R}}/A)/(F(A), t_A) \longrightarrow \mathcal{E}^{\mathcal{R}}/A$$

create initial objects. □

Lemma 3.3. *An object is $\mathcal{R}$-free exactly if it is left orthogonal to $\mathcal{R}$.*

Proof. If $A \in \mathcal{E}$ is left orthogonal to $\mathcal{R}$, then there in particular unique solutions to lifting problem of the form

$$\begin{array}{ccc} & & B \\ & \nearrow & \big\downarrow {\scriptstyle \in\mathcal{R}} \\ A & \xrightarrow[\mathrm{id}_A]{} & A, \end{array}$$

making (A, id_A) initial in $\mathcal{E}^{\mathcal{R}}/A$.

Conversely, let $A \in \mathcal{E}$ be $\mathcal{R}$-free and consider a lifting problem

$$\begin{array}{ccc} & & Y \\ & \nearrow & \big\downarrow {\scriptstyle \in\mathcal{R}} \\ A & \longrightarrow & X. \end{array}$$

Lifts as indicated are in bijection with sections to $p \colon A \times_X Y \to A$. Note that $p \in \mathcal{R}$ since $\mathcal{R}$ is closed under pullback. These can be seen as maps from (A, id_A) to $(A \times_X Y, p)$ in $\mathcal{E}^{\mathcal{R}}/A$. Since (A, id_A) is initial, there is a unique lift. □

Do we have an orthogonal factorization system on cwf's with right class those maps that are bijective on types? Yes, the left class is generated by a single map: the inclusion from the free cwf with a context to the free cwf with a context and a type over it.

Do we have an orthogonal factorization system on cwf's with right class those maps that are fully faithful? Yes, the left class is generated by a single map: the inclusion from the free cwf to the free cwf with a context.

4. Contextual cwf's

We write $\mathbf{Cat}^{\mathsf{ff}}$ for the wide subcategory of categories with morphisms restricted to fully faithful functors. Given a category $\mathcal{E}$ with an implicit forgetful functor $\mathcal{E} \to \mathbf{Cat}^{\mathsf{ff}}$, we similarly write $\mathcal{E}^{\mathsf{ff}}$ to denote the pullback of $\mathcal{E}$ along $\mathbf{Cat}^{\mathsf{ff}} \to \mathbf{Cat}$, i.e. the wide subcategory of $\mathcal{E}$ with morphisms restricted to those maps that project to a fully faithful functor in $\mathbf{Cat}$.

Definition 4.1. The *contextual cwf* $\mathsf{Ctx}(\mathcal{C})$ on a cwf $\mathcal{C}$ with an associated forgetful morphism $B_{\mathcal{C}} \colon \mathsf{Ctx}(\mathcal{C}) \to \mathcal{C}$ is the initial object of the slice category $\mathbf{CwF}^{\mathsf{iso\text{-}on\text{-}types,ff}}/\mathcal{C}$. □

In other words, $\mathsf{Ctx}(\mathcal{C})$ is the free cwf with a fully faithful morphism $B_{\mathcal{C}} \colon \mathsf{Ctx}(\mathcal{C}) \to \mathcal{C}$.

Definition 4.2. A cwf $\mathcal{C}$ is *contextual* if the morphism $B_{\mathcal{C}} \colon \mathsf{Ctx}(\mathcal{C}) \to \mathcal{C}$ is an isomorphism. □

In other words, $\mathcal{C}$ is contextual if $(\mathcal{C}, \mathcal{C} \xrightarrow{\mathrm{id}} \mathcal{C})$ is initial in $\mathbf{CwF}^{\mathsf{iso\text{-}on\text{-}types,ff}}/\mathcal{C}$.

Lemma 4.3. *The contextual cwf* $\mathsf{Ctx}(\mathcal{C})$ *of a cwf* $\mathcal{C}$ *is contextual.*

Proof. Abbreviate $\mathcal{E} =_{\mathrm{def}} \mathbf{CwF}^{\mathsf{iso\text{-}on\text{-}types,ff}}/\mathcal{C}$ and use that the forgetful functor

$$\mathbf{CwF}^{\mathsf{ff}}/\mathsf{Ctx}(\mathcal{C}) \simeq \mathcal{E}/(\mathsf{Ctx}(\mathcal{C}), B_{\mathcal{C}}) \to \mathcal{E}$$

reflects initial objects. □

Lemma 4.4. *The free cwf is contextual.*

Proof. Let $\mathcal{C}$ be the initial cwf. Recall that the forgetful functor $\mathbf{CwF}/\mathcal{C} \to \mathcal{C}$ reflects initial objects. By cancellation properties of fully faithful functors (one of the cases of 2-out-of-3), any section to a fully faithful functors is itself fully faithful. Morphisms of cwf's that are bijective on types also enjoy the same cancellation property, so any section to a morphism of cwf's bijective on types is also bijective on types. Thus, also the forgetful functor $\mathbf{CwF}^{\mathsf{iso\text{-}on\text{-}types,ff}}/\mathcal{C} \to \mathbf{CwF}/\mathcal{C}$ reflects initial objects. □

Let us give an explicit construction of the contextual cwf $\mathsf{Ctx}(\mathcal{C})$ on a cwf $\mathcal{C}$ and the functor $B_{\mathcal{C}} : \mathsf{Ctx}(\mathcal{C}) \to \mathcal{C}$.

Consider the directed graph G with vertices objects of $\mathcal{C}$ and an edge from Γ to $\Gamma.A$ given for every type $A \in \mathbf{Ty}(\Gamma)$. We have a distinguished vertex $v_0 \in G$ given by the empty context. The objects are $\mathsf{Ctx}(\mathcal{C})$ are the paths in G that start in v_0. We write an object as a tuple (n, Γ, A) where $n \geq 0$, $\Gamma_i \in \mathcal{C}$ for $i \leq n$, Γ_0 is the empty context, and $A_i \in \mathbf{Ty}(\Gamma_i)$ and $\Gamma_{i+1} = \Gamma_i.A_i$ for $i < n$. The action of $B_{\mathcal{C}}$ on objects is to return the endpoint of the path. The rest of the category structure of $\mathsf{Ctx}(\mathcal{C})$ and the functor structure on $B_{\mathcal{C}}$ is induced by the requirement that $B_{\mathcal{C}}$ be fully faithful.

The types of $\mathsf{Ctx}(\mathcal{C})$ are uniquely determined by the requirement that $B_{\mathcal{C}}$ is bijective on types. The terms of $\mathsf{Ctx}(\mathcal{C})$ are uniquely determined since $B_{\mathcal{C}}$ needs to be bijective on terms (by fully faithfulness).

The empty context is the unique path of lenth zero. The context extension of (n, Γ, A) with $X \in \mathbf{Ty}(\Gamma_n)$ is given by $(n+1, \Gamma', A')$ where $\Gamma'_i = \Gamma_i$ for $i \leq n$, $\Gamma_{n+1} = \Gamma_n.X$, $A'_i = A_i$ for $i < n$, and $A'_n = X$. The associated context projection is given by p_X. The associated representing term is given by q_X.

One easily verifies that this constitutes a cwf and that B_C is a fully faithful morphism of cwf's bijective on types. Initiality is verified in a straightforward way.

The situation is more complicated for free cwf's with type formers.

Lemma 4.5 (Free cwf is contextual)**.** *Let $\mathcal{C}$ be the free cwf with type formers T. Consider the directed graph G with vertices objects of $\mathcal{C}$ and edges indexed by $A \in \mathbf{Ty}(\Gamma)$, going from Γ to $\Gamma.A$. We have a distinguished vertex $v_0 \in G$ given by the empty context. Then the function from paths starting at v_0 to objects of $\mathcal{C}$ returning the last endpoint is a bijection.*

We need to build a cwf based on $\mathcal{C}$ that includes context building information. Let us try $\mathcal{D}$ as follows. An object consists of a list of types (n, Γ, A) and an object Θ of C with a map $\Gamma_n \to \Theta$. Functor to $\mathcal{C}$ is fully faithful and sends this object to Θ. Types and terms created from $\mathcal{C}$. Empty context and context extension defined in the evident way.

Again, why do we have dependent products?

Can we abstractly specify what T is? It should be some kind of signature S in a type theory with standard extensional type formers, a global type $\mathbf{Ty}$, a type $\mathbf{Ty} \vdash \mathbf{Tm}$. We cannot specify types, just terms and equations between them. Specifying types might be useful for extensions of cwf's, for example when defining neutral terms and normal forms.

A cwf with type formers T is then a category $\mathcal{C}$ with a presheaf $\mathbf{Ty}$ on $\mathcal{C}$ and a presheaf $\mathbf{Tm}$ on $\int \mathbf{Ty}$ with the usual representability property and furthermore an interpretation of S in $\widehat{\mathcal{C}}$.

Dependent sums.

$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty} \vdash \Sigma(A, B) : \mathbf{Ty}$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, a : \mathbf{Tm}(A), b : \mathbf{Tm}(B(a)) \vdash \mathsf{pair}(a, b) : \mathbf{Tm}(\Sigma(A, B))$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, s : \mathbf{Tm}(\Sigma(A, B)) \vdash \mathsf{fst}(s) : \mathbf{Tm}(A)$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, s : \mathbf{Tm}(\Sigma(A, B)) \vdash \mathsf{snd}(s) : \mathbf{Tm}(B(\mathsf{fst}(s)))$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, a : \mathbf{Tm}(A), b : \mathbf{Tm}(B(a)) \vdash \mathsf{fst}(\mathsf{pair}(a, b)) = a$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, a : \mathbf{Tm}(A), b : \mathbf{Tm}(B(a)) \vdash \mathsf{snd}(\mathsf{pair}(a, b)) = b$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, s : \mathbf{Tm}(\Sigma(A, B)) \vdash s = \mathsf{pair}(\mathsf{fst}(s), \mathsf{snd}(s))$$

We could have chosen to use dependent sums in the specifying language. Then the specification, except for the first line, would really just be an isomorphism.

Dependent products.

$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty} \vdash \Pi(A, B) : \mathbf{Ty}$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, b : (a : \mathbf{Tm}(A)) \to \mathbf{Tm}(B(a)) \vdash \mathsf{lam}(b) : \mathbf{Tm}(\Pi(A, B))$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, f : \mathbf{Tm}(\Pi(A, B)), a : \mathbf{Tm}(A) \vdash \mathsf{app}(f, a) : \mathbf{Tm}(A)$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, f : \mathbf{Tm}(\Pi(A, B)) \vdash f = \mathsf{lam}(\lambda a.\, \mathsf{app}(f, a))$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, b : (a : \mathbf{Tm}(A)) \to \mathbf{Tm}(B(a)), a : \mathbf{Tm}(A) \vdash \mathsf{app}(\mathsf{lam}(b), a) = b(a)$$

We could have chosen to use dependent products in the specifying language. Then the specification, except for the first line, would really just be an isomorphism.

Unit types.

$$\vdash 1 : \mathbf{Ty}$$
$$\vdash \mathsf{unit} : \mathbf{Tm}(1)$$
$$x : \mathbf{Tm}(1) \vdash x = \mathsf{unit}$$

We could have chosen to use empty types in the specifying language. Then the specification, except for the first line, would really just be an isomorphism.

Empty types.

$$\vdash 0 : \mathbf{Ty}$$
$$C : 0 \to \mathbf{Ty}, x : \mathbf{Tm}(0) \vdash \mathsf{absurd} : \mathbf{Tm}(C(x))$$

Coproduct types.

$$A : \mathbf{Ty}, B : \mathbf{Ty} \vdash A + B : \mathbf{Ty}$$
$$A : \mathbf{Ty}, B : \mathbf{Ty}, a : \mathbf{Tm}(A) \vdash \tau_0(a) : \mathbf{Tm}(A + B)$$
$$A : \mathbf{Ty}, B : \mathbf{Ty}, b : \mathbf{Tm}(B) \vdash \tau_1(b) : \mathbf{Tm}(A + B)$$
$$C : \mathbf{Tm}(A + B) \to \mathbf{Ty}, u : (a : \mathbf{Tm}(A)) \to C(\tau_0(a)), v : (b : \mathbf{Tm}(B)) \to C(\tau_1(b)), x : \mathbf{Tm}(A + B) \vdash \mathsf{case}(u, v, x) : \mathbf{Tm}$$
$$C : \mathbf{Tm}(A + B) \to \mathbf{Ty}, u : (a : \mathbf{Tm}(A)) \to C(\tau_0(a)), v : (b : \mathbf{Tm}(B)) \to C(\tau_1(b)), a : \mathbf{Tm}(A) \vdash \mathsf{case}(u, v, \tau_0(a)) = u(a$$
$$C : \mathbf{Tm}(A + B) \to \mathbf{Ty}, u : (a : \mathbf{Tm}(A)) \to C(\tau_0(a)), v : (b : \mathbf{Tm}(B)) \to C(\tau_1(b)), b : \mathbf{Tm}(B) \vdash \mathsf{case}(u, v, \tau_1(b)) = v(b)$$

W-types.

$$A : \mathbf{Ty}, B : \mathbf{Ty} \to \mathbf{Ty} \vdash W(A, B) : \mathbf{Ty}$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, a : \mathbf{Tm}(A), f : \mathbf{Tm}(B(a)) \to W(A, B) \vdash \mathsf{sup}(a, f) : \mathbf{Tm}(W(A, B))$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, C : \mathbf{Tm}(W(A, B)) \to \mathbf{Ty}, d : ((a : \mathbf{Tm}(A)) \times (f : (b : \mathbf{Tm}(B(a))) \to \mathbf{Tm}(W(A, B))) \times (r$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, C : \mathbf{Tm}(W(A, B)) \to \mathbf{Ty}, d : ((a : \mathbf{Tm}(A)) \times (f : (b : \mathbf{Tm}(B(a))) \to \mathbf{Tm}(W(A, B))) \times (r$$

Universe.

$$\vdash U : \mathbf{Ty}$$
$$A : \mathbf{Tm}(U) \vdash \mathsf{El_n}(A) : \mathbf{Ty}$$
$$A : \mathbf{Tm}(U), B : \mathbf{Tm}(\mathsf{El}(A)) \to \mathbf{Tm}(U) \vdash \mathsf{U\text{-}\Sigma} : \mathbf{Tm}(U)$$
$$A : \mathbf{Tm}(U), B : \mathbf{Tm}(\mathsf{El}(A)) \to \mathbf{Tm}(U) \vdash \mathsf{El}(\mathsf{U\text{-}\Sigma})) = \Sigma(\mathsf{El}(A), \lambda a.\, \mathsf{El}(B(a)))$$

etc.

5. Neutral terms and normal forms

Let $\mathcal{C}$ be a cwf.

We define normal forms and neutral terms. These are defined mutually inductive with natural interpretation maps to terms. That is, we define

$$\mathsf{NF} \in \widehat{\mathbf{Ty}_{\mathsf{Ren}(\mathcal{C})}}, \qquad \mathsf{tm}\colon \mathsf{NF}F \to \mathbf{Tm}_{\mathcal{C}},$$
$$\mathsf{NE} \in \widehat{\mathbf{Ty}_{\mathsf{Ren}(\mathcal{C})}}, \qquad \mathsf{tm}\colon \mathsf{NE}F \to \mathbf{Tm}_{\mathcal{C}},$$

all by mutual induction.

$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \vdash \Sigma(A, B) : \mathbf{Ty}$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, a : \mathbf{Tm}(A), b : \mathbf{Tm}(B(a)), \mathsf{isNf}(a), \mathsf{isNf}(b) \vdash \mathsf{pair\text{-}nf} : \mathsf{isNf}(\mathsf{pair}(a, b))$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, s : \mathbf{Tm}(\Sigma(A, B)), \mathsf{isNe}(s) \vdash \mathsf{fst\text{-}ne} : \mathsf{isNe}(\mathsf{fst}(s))$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, s : \mathbf{Tm}(\Sigma(A, B)), \mathsf{isNe}(s) \vdash \mathsf{fst\text{-}ne} : \mathsf{isNe}(\mathsf{snd}(s))$$

We could have chosen to use dependent sums in the specifying language. Then the specification, except for the first line, would really just be an isomorphism.

Dependent products.

$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \vdash \Pi(A, B) : \mathbf{Ty}$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, b : (a : \mathbf{Tm}(A)) \to \mathbf{Tm}(B(a)) \vdash \mathsf{lam}(b) : \mathbf{Tm}(\Pi(A, B))$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty}, f : \mathbf{Tm}(\Pi(A, B)), a : \mathbf{Tm}(A) \vdash \mathsf{app}(f, a) : \mathbf{Tm}(A)$$

The basic neutral terms are just the terms of the cwf of renamings.

$$\frac{A \in \mathbf{Ty}(\Gamma) \qquad a \in \mathbf{Tm}_{\mathsf{Ren}(\mathcal{C})}(\Gamma, A)}{\mathsf{var}(a) \in \mathsf{NF}(\Gamma, a) \qquad \mathsf{tm}(\mathsf{var}(a)) = F(a)}$$

Dependent sums.

$$\frac{A \in \mathbf{Ty}(\Gamma) \qquad B \in \mathbf{Ty}(\Gamma.A) \qquad s \in \mathsf{NE}(\Gamma, \boldsymbol{\Sigma}_{A,B})}{\begin{array}{c}\mathsf{fst}(s) \in \mathsf{NE}(\Gamma, A) \qquad \mathsf{tm}(\mathsf{fst}(s)) = \mathbf{fst}(\mathsf{tm}(s)) \\ \mathsf{snd}(s) \in \mathsf{NE}(\Gamma, B(\mathbf{fst}(\mathsf{tm}(s)))) \qquad \mathsf{tm}(\mathsf{snd}(s)) = \mathbf{snd}(\mathsf{tm}(s))\end{array}}$$

$$\frac{A \in \mathbf{Ty}(\Gamma) \qquad B \in \mathbf{Ty}(\Gamma.A) \qquad a \in \mathsf{NF}(\Gamma, A) \qquad b \in \mathsf{NF}(\Gamma, B(\mathsf{tm}(a)))}{\mathsf{pair}(a, b) \in \mathsf{NF}(\Gamma, \boldsymbol{\Sigma}_{A,B}) \qquad \mathsf{tm}(\mathsf{pair}(a, b)) = \mathbf{pair}(\mathsf{tm}(a), \mathsf{tm}(b))}$$

Dependent products.

$$\frac{A \in \mathbf{Ty}(\Gamma) \qquad B \in \mathbf{Ty}(\Gamma.A) \qquad f \in \mathsf{NE}(\Gamma, \boldsymbol{\Pi}_{A,B}) \qquad a \in \mathsf{NF}(\Gamma, A)}{\mathsf{app}(f, a) \in \mathsf{NF}(\Gamma, B(\mathsf{tm}(a))) \qquad \mathsf{tm}(\mathsf{app}(f, a)) = \mathbf{app}(\mathsf{tm}(f), \mathsf{tm}(a))}$$

$$\frac{A \in \mathbf{Ty}(\Gamma) \qquad B \in \mathbf{Ty}(\Gamma.A) \qquad f \in \mathsf{NF}(\Gamma.A, B)}{\mathsf{lam}(b) \in \mathsf{NF}(\Gamma, \mathbf{\Pi}_{A,B}) \qquad \mathsf{tm}(\mathsf{lam}(b)) = \mathbf{lam}(\mathsf{tm}(b))}$$

Unit types.

$$\frac{}{\mathsf{unit} \in \mathsf{NF}(\Gamma, \mathbf{1}) \qquad \mathsf{tm}(\mathsf{unit}) = \mathbf{unit}}$$

Empty types.

$$\frac{C \in \mathbf{Ty}(\Gamma.\mathbf{0}) \qquad x \in \mathsf{NE}(\Gamma, A + B)}{\mathsf{0\text{-}elim}(C, x) \in \mathsf{NF}(\Gamma.\mathbf{0}, C) \qquad \mathsf{tm}(\mathsf{0\text{-}elim}(\mathrm{C}, \mathrm{x})) = \mathbf{0\text{-}elim}_C(\mathsf{tm}(x))}$$

Coproduct types.

$$\frac{\begin{array}{c} A, B \in \mathbf{Ty}(\Gamma) \qquad C \in \mathbf{Ty}(\Gamma.(A + B)) \\ c_0 \in \mathsf{NF}(\Gamma.(a : A), C(\boldsymbol{\tau}_0(a))) \qquad c_1 \in \mathsf{NF}(\Gamma.(b : B), C(\boldsymbol{\tau}_1(b))) \qquad x \in \mathsf{NE}(\Gamma, A + B) \end{array}}{\begin{array}{c} +\mathsf{\text{-}elim}(C, c_0, c_1, x) \in \mathsf{NF}(\Gamma.(A + B), C) \\ \mathsf{tm}(+\mathsf{\text{-}elim}(C, c_0, c_1, x)) = \mathbf{+\text{-}elim}_{C, \mathsf{tm}(c_0), \mathsf{tm}(c_1)}(\mathsf{tm}(x)) \end{array}}$$

$$\frac{A, B \in \mathbf{Ty}(\Gamma) \qquad a \in \mathsf{NF}(\Gamma, A)}{\tau_0(a) \in \mathsf{NF}(\Gamma, A + B) \qquad \mathsf{tm}(\ _0(a)) = \boldsymbol{\tau}_0(\mathsf{tm}(a))}$$

$$\frac{A, B \in \mathbf{Ty}(\Gamma) \qquad b \in \mathsf{NF}(\Gamma, A)}{\tau_1(b) \in \mathsf{NF}(\Gamma, A + B) \qquad \mathsf{tm}(\ _1(b)) = \boldsymbol{\tau}_1(\mathsf{tm}(b))}$$

Identity types.

$$\frac{\begin{array}{c} A \in \mathbf{Ty}(\Gamma) \qquad C \in \mathbf{Ty}(\Gamma.(x\, y : A).\mathbf{Id}_A(x, y)) \\ d \in \mathsf{NF}(\Gamma.(a : A), C(a, a, \mathbf{refl(a)})) \qquad x, y \in \mathsf{NF}(\Gamma, A) \qquad p \in \mathsf{NE}(\Gamma, \mathbf{Id}_A(x, y)) \end{array}}{\mathsf{J}_{C,d}(x, y, p) \in \mathsf{NF}(\Gamma.(x\, y : A).\mathbf{Id}_A(x, y), C) \qquad \mathsf{tm}(\mathsf{J}_{C,d}(x, y, p)) = \mathbf{J}_{C,d}(\mathsf{tm}(x), \mathsf{tm}(y), \mathsf{tm}(p))}$$

$$\frac{A \in \mathbf{Ty}(\Gamma) \qquad a \in \mathsf{NF}(\Gamma, A)}{\mathsf{refl}(a) \in \mathsf{NF}(\Gamma, \mathbf{Id}_A(\mathsf{tm}(a), \mathsf{tm}(a))) \qquad \mathsf{tm}(\mathsf{refl}(a)) = \mathbf{refl}(\mathsf{tm}(a))}$$

Universe.

$$\frac{A \in \mathsf{NF}(\Gamma, U) \qquad B \in \mathsf{NF}(\Gamma, U)}{A + B \in \mathsf{NF}(\Gamma, U) \qquad \mathsf{tm}(A + B) = \mathsf{tm}(A) + \mathsf{tm}(B)}$$

$$\frac{A \in \mathsf{NF}(\Gamma, U) \qquad B \in \mathsf{NF}(\Gamma.\mathbf{El}(A), U)}{\Sigma_{A,B} \in \mathsf{NF}(\Gamma, U) \qquad \mathsf{tm}(\Sigma_{A,B}) = \mathbf{\Sigma}_{\mathsf{tm}(A), \mathsf{tm}(B)}}$$

etc.

Neutral to normal. We need a way to go from neutral terms to normal forms. However, this should only be possible if we are not in an extensional type former, i.e. dependent product, dependent sum, or unit type. How do we enforce this? We could be in a type $\mathbf{El}(x)$ where x reduces to the code of a dependent sum.

The situation might be easier if we do not have any η-laws at all.

$$\frac{A \in \mathbf{Ty}(\Gamma) \qquad A \neq \mathbf{\Sigma}_{\bullet,\bullet}, \mathbf{\Pi}_{\bullet,\bullet}, 1 \qquad a \in \mathsf{NE}(\Gamma, A)}{\mathsf{ne}(a) \in \mathsf{NF}(\Gamma, A) \qquad \mathsf{tm}(\mathsf{ne}(a)) = \mathsf{tm}(a)}$$

This rule makes the definition impossible for an arbitrary $\mathcal{C}$: it might be that a type becomes equal to a sigma type after substituting with a renaming. Can we show this is impossible for the syntactic category? That is, given a type $X \in \mathbf{Ty}(\Gamma)$ and a renaming $\sigma \colon \Delta \to \Gamma$ such that $X[\sigma] = \Sigma_{A',B'}$ with $A' \in \mathbf{Ty}(\Delta), B' \in \mathbf{Ty}(\Delta.A)$, are there $A \in \mathbf{Ty}(\Gamma), B \in \mathbf{Ty}(\Gamma.A')$ such that $X = \Sigma_{A,B}$ and $A' = A[\sigma], B' = B[\sigma]$?

Is the action of renamings on terms injective? It feels like it is; if it is, then it will be enough to have $A' = A[\sigma], B' = B[\sigma]$.

Another approach. Change the condition to: the type will never become a dependent sum after any renaming. This is stable under renaming by construction.

$$\frac{A \in \mathbf{Ty}(\Gamma) \qquad \forall \sigma \in \mathsf{Ren}(\mathcal{C})(\Delta, \Gamma). A[\sigma] \neq \mathbf{\Sigma}_{\bullet,\bullet}, \mathbf{\Pi}_{\bullet,\bullet}, 1 \qquad a \in \mathsf{NE}(\Gamma, A)}{\mathsf{ne}(a) \in \mathsf{NF}(\Gamma, A) \qquad \mathsf{tm}(\mathsf{ne}(a)) = \mathsf{tm}(a)}$$

Another idea (Thierry). Omit the condition on A entirely. The the interpretation function from normal forms to syntax will not be injective, but that may not be a problem: at the same time, the quote function of the interpretation of any type will not be surjective. Since it is type-directed, only the truly-wellformed normal forms are taken as values.

This yields the nicest definition of NF and NE, possible for any cwf $\mathcal{C}$ with type formers. So let us adopt this approach for now.

6. The twisted glueing

Let $\mathcal{C}$ be a cwf with type formers as in the previous section. We construct a new cwf with type formers, the *twisted glueing* $\mathsf{Tw}(\mathcal{C})$ of $\mathcal{C}$.

Let $T \colon \mathcal{C} \to \widehat{\mathsf{Ren}(\mathcal{C})}$ denote the restricted Yoneda functor. So $T(\Gamma, \Delta) = \mathcal{C}(\Delta, \Gamma)$. It is a pseudomorphism of cwf's (defined in another note). On types, it sends $A \in \mathbf{Ty}(\Gamma)$ to the presheaf on $\int T\Gamma$ sending (Δ, σ) to $\mathbf{Tm}(\Delta, A[\sigma])$. On terms, it sends $t \in \mathbf{Tm}(\Gamma, A)$ to the presheaf section sending (Δ, σ) to $t[\sigma] \in \mathbf{Tm}(\Delta, A[\sigma])$.

Consider a type $A \in \mathbf{Ty}(\Gamma)$. We define types $\mathbf{NF}_A, \mathbf{NE}_A$ in context $T\Gamma$, i.e. presheaves on $\int T\Gamma$, as follows.

$$\mathbf{NF}_A(\Delta, \sigma) = \mathsf{NF}(\Delta, A[\sigma])$$
$$\mathbf{NE}_A(\Delta, \sigma) = \mathsf{NE}(\Delta, A[\sigma]).$$

From the previous section, we have maps

$$\mathbf{NE}_A \longrightarrow TA \longleftarrow \mathbf{NF}_A$$

in context $T\Gamma$.

The category of contexts $\mathsf{Tw}(\mathcal{C})$ is $\mathbf{Set} \downarrow T$.

The types over $|\Gamma| \xrightarrow{\alpha} T\Gamma$ consist of $A \in \mathbf{Ty}(\Gamma)$ and $|A| \in \mathbf{Ty}(|\Gamma|)$ with a map f making the following diagram commute:

$$\begin{array}{ccc} |\Gamma|.|A| & \xrightarrow{f} & T(\Gamma.A) \\ \downarrow & & \downarrow \\ |\Gamma| & \xrightarrow{\alpha} & T\Gamma. \end{array}$$

This is an internal function $|\Gamma| \vdash f : |A| \to TA[\alpha]$.

The terms over $|\Gamma| \xrightarrow{\alpha} T\Gamma$ of $(A, |A|, f)$ consist of $t \in \mathbf{Tm}(\Gamma, A)$ and $|t| \in \mathbf{Tm}(|\Gamma|, |A|)$ such that $f(|t|) = Tt[\alpha]$.

We modify the types as follows. In context $|\Gamma| \xrightarrow{\alpha} T\Gamma$, in addition to $A, |A|, f$ as above, require maps $|\Gamma| \vdash u : \mathbf{NE}(\Gamma, A)[\alpha] \to |A|$ and $|\Gamma| \vdash q : |A| \to \mathbf{NF}(\Gamma, A)[\alpha]$ commuting over $TA[\alpha]$.

We can give an interpretation of dependent sums, dependent products, and unit types.

We can give an interpretation of coproduct types.

Can we given an interpretation of equality types?

The interpretation of the universe uses normal forms.

6.1. **A universe.** The universe type in the empty context is given by $(|U|, T\mathbf{U}, f)$ as follows. Note that $|U|$ is a presheaf over $\mathsf{Ren}(\mathcal{C})$ and f is a map between such presheaves. We set $|U| = (A : TU) \times (\mathbf{Tm}(\mathbf{El}(A)) \to U)$.

Consider a context $|\Gamma| \xrightarrow{\alpha} T\Gamma$. The universe type in it is given by $(|U|, T\mathbf{U}, f)$ where

in it is given by the universe $\mathbf{U} \in \mathbf{Ty}(\Gamma)$, the set of sets $U \in \mathbf{Ty}(|\Gamma|)$, and the function $|\Gamma| \vdash U \to \mathbf{U}[\alpha]$ that sends

6.2. **Starting the normalization.** To start the normalization, we need an inhabitant of $|\Gamma|(\Gamma)$ where $[\![\Gamma]\!] = (\Gamma, |\Gamma|, f)$ is the interpretation of Γ in $\mathsf{Tw}(\mathcal{C})$. This will be easier to prove in generalized form: given a renaming $\Delta \to \Gamma$, there is an inhabitant of $|\Gamma|(\Delta)$ (really? does this not follow immediately since $|\Gamma|$ is a presheaf?)

An inhabitant of $|\Gamma|(\Gamma)$ is given by a map $y\Gamma \to |\Gamma|$. This is given by a map

$$(\Gamma, y\Gamma, c) \to (\Gamma, |\Gamma|, f)$$

(where $c \colon y\Gamma \to T\Gamma$ is the canonical inclusion) that maps to the identity in $\mathcal{C}$.

What properties does the map sending Γ to $(\Gamma, y\Gamma, c)$ have? It is a functor Q; is that really true? $y\Gamma$ is only functorial in $\mathsf{Ren}(C)$; the functor Q should start from $\mathsf{Ren}(C)$. Thus, the maps

$$(\Gamma, y\Gamma, c) \to (\Gamma, |\Gamma|, f)$$

should assemble into a natural transformation.

Functors $\mathcal{D} \to \mathsf{Tw}(C)$ are tuples (F, G, u) with a functor $F \colon \mathcal{D} \to \mathcal{C}$, a functor $G \colon \mathcal{D} \to \widehat{\mathsf{Ren}(\mathcal{C})}$, and a natural transformation $u \colon G \to TF$. Remember that $T = \mathsf{restrict} \circ y$. So we specify $Q \colon \mathsf{Ren}(\mathcal{C}) \to \mathsf{Tw}(\mathcal{C})$ as the tuple (F, G, u) where $F \colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ is embedding, $G \colon \mathsf{Ren}(\mathcal{C}) \to \widehat{\mathsf{Ren}(\mathcal{C})}$ is Yoneda, and $u \colon [\mathsf{Ren}(\mathcal{C}), \widehat{\mathsf{Ren}(\mathcal{C})}](y, \mathsf{restrict} \circ y \circ \mathsf{embed})$, equivalently $u \colon [\mathsf{Ren}(\mathcal{C} \times \mathcal{C}^{\mathrm{op}}, \mathbf{Set}](\hom, \hom \circ (\mathsf{embed} \times \mathsf{embed}))$ is the action of embedding on homs.

Want a natural transformation $Q \to [\![-]\!]$. Can we get this using freeness properties?

Consider $\Gamma \vdash A$. Assume we have an inhabitant of $|\Gamma|(\Gamma)$. We want an inhabitant of $|\Gamma.A|(\Gamma.A)$,

What is the exponential $\mathcal{D}^{\mathcal{C}}$ of cwf's of $\mathcal{D}$ with $\mathcal{C}$? Need a map

$$\mathcal{D}^{\mathcal{C}} \times \mathcal{C} \to \mathcal{D}$$

with a universal property. What is a type in $\mathcal{D}^{\mathcal{C}}$?

7. A clean specification of the toy type theory

Dependent sums.

$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty} \vdash \Sigma(A, B) : \mathbf{Ty}$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty} \vdash \mathbf{Tm}(\Sigma(A, B)) \simeq ((a : \mathbf{Tm}(A)) \times \mathbf{Tm}(B(a)))$$

Dependent products.

$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty} \vdash \Pi(A, B) : \mathbf{Ty}$$
$$A : \mathbf{Ty}, B : \mathbf{Tm}(A) \to \mathbf{Ty} \vdash \mathbf{Tm}(\Pi(A, B)) \simeq ((a : \mathbf{Tm}(A)) \to \mathbf{Tm}(B(a)))$$

Unit types.

$$\vdash \top : \mathbf{Ty}$$
$$\vdash \mathbf{Tm}(\top) \simeq 1$$

Empty types.

$$\vdash \bot : \mathbf{Ty}$$
$$C : \bot \to \mathbf{Ty}, x : \mathbf{Tm}(\bot) \vdash \mathsf{absurd}(C, x) : \mathbf{Tm}(C(x))$$

Coproduct types.

$$A : \mathbf{Ty}, B : \mathbf{Ty} \vdash A \oplus B : \mathbf{Ty}$$
$$A : \mathbf{Ty}, B : \mathbf{Ty}, x : \mathbf{Tm}(A) + \mathbf{Tm}(B) \vdash \tau(x) : \mathbf{Tm}(A \oplus B)$$
$$C : \mathbf{Tm}(A \oplus B) \to \mathbf{Ty}, d : (x : \mathbf{Tm}(A) + \mathbf{Tm}(B)) \to C(\tau(x)), x : \mathbf{Tm}(A \oplus B) \vdash \mathsf{case}(C, d, x) : \mathbf{Tm}(C(x))$$
$$C : \mathbf{Tm}(A \oplus B) \to \mathbf{Ty}, d : (x : \mathbf{Tm}(A) + \mathbf{Tm}(B)) \to C(\tau(x)), x : \mathbf{Tm}(A) + \mathbf{Tm}(B) \vdash \mathsf{case}(C, d, \tau(x)) = d(x)$$

Universe.

$$\vdash U_n : \mathbf{Ty}$$
$$A : \mathbf{Tm}(U_n) \vdash \mathsf{El}_n(A) : \mathbf{Ty}$$
$$A : \mathbf{Tm}(U_a) \vdash \mathsf{lift}_{a,b}(A) : \mathbf{Tm}(U_b)$$
$$A : \mathbf{Tm}(U_a) \vdash \mathsf{El}_b(\mathsf{lift}_{a,b}(A)) = \mathsf{El}_a(A)$$
$$A : \mathbf{Tm}(U_a) \vdash \mathsf{lift}_{b,c}(\mathsf{lift}_{a,b}(A)) = \mathsf{lift}_{a,c}(A)$$
$$\vdash \top_n : \mathbf{Tm}(U_n)$$
$$\vdash \mathsf{El}_n(\top_n) = \top$$
$$\vdash \mathsf{lift}_{a,b}(\top_a) = \top_n$$

etc.

8. A clean specification of the toy type theory, alternative

Dependent sums.

$$A : \mathbf{Ty}_n, B : \mathbf{Tm}_n(A) \to \mathbf{Ty}_n \vdash \Sigma_n(A, B) : \mathbf{Ty}_n$$
$$A : \mathbf{Ty}_n, B : \mathbf{Tm}_n(A) \to \mathbf{Ty}_n \vdash \mathbf{Tm}_n(\Sigma_n(A, B)) \simeq ((a : \mathbf{Tm}_n(A)) \times \mathbf{Tm}_n(B(a)))$$

Dependent products.

$$A : \mathbf{Ty}_n, B : \mathbf{Tm}_n(A) \to \mathbf{Ty}_n \vdash \Pi_n(A, B) : \mathbf{Ty}_n$$
$$A : \mathbf{Ty}_n, B : \mathbf{Tm}_n(A) \to \mathbf{Ty}_n \vdash \mathbf{Tm}_n(\Pi_n(A, B)) \simeq ((a : \mathbf{Tm}_n(A)) \to \mathbf{Tm}_n(B(a)))$$

Unit types.

$$\vdash \top_n : \mathbf{Ty}_n$$
$$\vdash \mathbf{Tm}_n(\top_n) \simeq 1$$

Empty types.

$$\vdash \bot_n : \mathbf{Ty}_n$$
$$C : \bot_n \to \mathbf{Ty}_n, x : \mathbf{Tm}_n(\bot_n) \vdash \mathsf{absurd}(C, x) : \mathbf{Tm}_n(C(x))$$

Coproduct types.

$A : \mathbf{Ty}_n, B : \mathbf{Ty}_n \vdash A \oplus_n B : \mathbf{Ty}_n$

$A : \mathbf{Ty}_n, B : \mathbf{Ty}_n, x : \mathbf{Tm}_n(A) + \mathbf{Tm}_n(B) \vdash \tau(x) : \mathbf{Tm}_n(A \oplus_n B)$

$C : \mathbf{Tm}_n(A \oplus_n B) \to \mathbf{Ty}_n, d : (x : \mathbf{Tm}_n(A) + \mathbf{Tm}_n(B)) \to C(\tau(x)), x : \mathbf{Tm}_n(A \oplus_n B) \vdash \mathsf{case}(C, d, x) : \mathbf{Tm}_n(C(x))$

$C : \mathbf{Tm}_n(A \oplus_n B) \to \mathbf{Ty}_n, d : (x : \mathbf{Tm}_n(A) + \mathbf{Tm}_n(B)) \to C(\tau(x)), x : \mathbf{Tm}_n(A) + \mathbf{Tm}_n(B) \vdash \mathsf{case}(C, d, \tau(x)) = d(x$

Universes.

$$\vdash U_{a,b} : \mathbf{Ty}_b$$
$$\vdash \mathbf{Tm}_b(U_{a,b}) \simeq \mathbf{Ty}_a$$

Cumulativity.

$A : \mathbf{Ty}_a \vdash \mathsf{lift}_{a,b}(A) : \mathbf{Ty}_b$

$A : \mathbf{Ty}_a \vdash \mathsf{lift}_{b,c}(\mathsf{lift}_{a,b}(A)) = \mathsf{lift}_{a,c}(A)$

$A : \mathbf{Ty}_a \vdash \mathbf{Tm}_b(\mathsf{lift}_{a,b}(A)) \simeq \mathbf{Tm}_a(A)$

coherence for the above isomorphism

optionally the requirement that lift is mono

equations specifying that lift preserves type formers, constructors, eliminators

9. A notion of type theory with only certain types exponentiable

Consider a functor $F : \mathcal{C} \to \mathcal{D}$.
Consider presheaves $A, B \in \widehat{\mathcal{D}}$.
Write precomposition with F as $F^* : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$.
We have a comparison map $F^*(B^A) \to (F^* B^{F^* A})$.
Ty Tm
Tm representable
category C, model of a certain generalized algebraic theory in Presheaf(C)
Presheaf(D) $\to$ Presheaf(C)

10. Manual specification of the type theory

Recall that we write $\mathbf{CwF}(\mathcal{C})$ for the category of cwf-structures on a category $\mathcal{C}$, i.e. the fiber of the forgetful functor $\mathbf{CwF} \to \mathbf{Cat}$ over $\mathcal{C}$. Note that a cwf morphism $(\mathsf{Id}, u, v) : (\mathcal{C}, \mathbf{Ty}_0, \mathbf{Tm}_0) \to (\mathcal{C}, \mathbf{Ty}_1, \mathbf{Tm}_1)$ that is the identity on underlying categories necessarily has v an isomorphism.

Definition 10.1. Let $\mathbb{I}$ be a non-empty category. An $\mathbb{I}$-*graded cwf* $\mathcal{C} = (\mathcal{C}, \mathbf{Ty}, \mathbf{Tm})$ consists of a category $\mathcal{C}$ and a diagram $\mathbb{I} \to \mathbf{CwF}(\mathcal{C})$. □

Let $\mathbb{I}$ be a category. An $\mathbb{I}$-graded cwf $\mathcal{C} = (\mathcal{C}, \mathbf{Ty}, \mathbf{Tm})$ consists of a category $\mathcal{C}$, for each $i \in \mathbb{I}$ presheafs $\mathbf{Ty}_i \in \widehat{C}$ and $\mathbf{Tm}_i \in \widehat{\int \mathbf{Ty}_i}$, and for each $f : i \to i'$ a map $\mathbf{Ty}_f : \mathbf{Ty}_i \to \mathbf{Ty}_{i'}$ with an isomorphism

Definition 10.2. *Dependent sums* for a cwf $\mathcal{C}$ consist of, for $\Gamma \in \mathcal{C}$:

- given $A \in \mathbf{Ty}(\Gamma)$ and $B \in \mathbf{Ty}(\Gamma.A)$, a type $\Sigma(A, B) \in \mathbf{Ty}(\Gamma)$,
- additionally, a bijection $\mathsf{pair}_{A,B}$ between $(a \in \mathbf{Tm}(A)) \times (b : \mathbf{Tm}(B(a)))$ and $\mathbf{Tm}(\Sigma(A, B))$

that are stable under substitution, i.e. given a substitution $\sigma : \Delta \to \Gamma$:

- given $A \in \mathbf{Ty}(\Gamma)$ and $B \in \mathbf{Ty}(\Gamma.A)$, we have $\Sigma(A, B)[\sigma] = \Sigma(A[\sigma], B[\sigma^+])$.
- given additionally $a \in \mathbf{Tm}(A)$ and $b \in \mathbf{Tm}(B(a))$, we have $\mathsf{pair}(a, b)[\sigma^+] = \mathsf{pair}(a[\sigma], b[\sigma])$.

□

Remark 10.3. Given dependent sums, the inverse to pair is written $(\mathsf{fst}, \mathsf{snd})$, i.e. for $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma.A)$, and $s \in \mathbf{Tm}(\Sigma(A, B))$, we have

$$\mathsf{fst}_{A,B}(s) \in \mathbf{Tm}(A),$$
$$\mathsf{snd}_{A,B}(s) \in \mathbf{Tm}(B(\mathsf{fst}(s))).$$

It follows that fst and snd are also stable under substitution, i.e. given additionally $\sigma : \Delta \to \Gamma$, we have

$$\mathsf{fst}(s)[\sigma] = \mathsf{fst}(s[\sigma]),$$
$$\mathsf{snd}(s)[\sigma] = \mathsf{snd}(s[\sigma]).$$

□

Definition 10.4. A *morphism of cwf's with dependent sums* is a morphism $F \colon \mathcal{C} \to \mathcal{D}$ of underlying cwf's such that for $\Gamma \in \mathcal{C}$:

- given $A \in \mathbf{Ty}(\Gamma)$ and $B \in \mathbf{Ty}(\Gamma.A)$, we have $F(\Sigma(A, B)) = \Sigma(F(A), F(B))$,
- additionally, we have $F(\mathsf{pair}_{A,B}) = \mathsf{pair}_{FA,FB}$.

□

Definition 10.5. *Dependent products* for a cwf $\mathcal{C}$ consist of, for $\Gamma \in \mathcal{C}$:

- given $A \in \mathbf{Ty}(\Gamma)$ and $B \in \mathbf{Ty}(\Gamma.A)$, a type $\Pi(A, B) \in \mathbf{Ty}(\Gamma)$,
- additionally, a bijection $\mathsf{lam}_{A,B}$ between $\mathbf{Tm}(B)$ and $\mathbf{Tm}(\Pi(A, B))$

that are stable under substitution, i.e. given a substitution $\sigma : \Delta \to \Gamma$:

- given $A \in \mathbf{Ty}(\Gamma)$ and $B \in \mathbf{Ty}(\Gamma.A)$, we have $\Pi(A, B)[\sigma] = \Pi(A[\sigma], B[\sigma^+])$.
- given additionally $b \in \mathbf{Tm}(B)$, we have $\mathsf{lam}(b)[\sigma] = \mathsf{lam}(b[\sigma^+])$.

□

Remark 10.6. Given dependent product, the inverse to lam is written eval, i.e. for $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma.A)$, and $f \in \mathbf{Tm}(\Pi(A, B))$, we have

$$\mathsf{eval}_{A,B}(f) \in \mathbf{Tm}(B).$$

It follows that eval is also stable under substitution, i.e. given additionally $\sigma : \Delta \to \Gamma$, we have

$$\mathsf{eval}(f)[\sigma^+] = \mathsf{eval}(f[\sigma]).$$

Given $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma.A)$, $f \in \mathbf{Tm}(\Pi(A, B))$, and $a \in \mathbf{Tm}(A)$, we define

$$\mathsf{app}_{A,B}(f, a) \in \mathbf{Tm}(B(a)),$$
$$\mathsf{app}_{A,B}(f, a) =_{\mathrm{def}} \mathsf{eval}(f)(a).$$

This is stable under substitution: given additionally $\sigma : \Delta \to \Gamma$, we have

$$\begin{aligned}\mathsf{app}(f,a)[\sigma] &= \mathsf{eval}(f)(a)[\sigma]\\ &= \mathsf{eval}(f)[\sigma^+](a[\sigma])\\ &= \mathsf{eval}(f[\sigma])(a[\sigma])\\ &= \mathsf{app}(f[\sigma],a[\sigma]).\end{aligned}$$

□

Definition 10.7. A *morphism of cwf's with dependent products* is a morphism $F\colon \mathcal{C} \to \mathcal{D}$ of underlying cwf's such that for $\Gamma \in \mathcal{C}$:

- given $A \in \mathbf{Ty}(\Gamma)$ and $B \in \mathbf{Ty}(\Gamma.A)$, we have $F(\Pi(A,B)) = \Pi(F(A),F(B))$,
- additionally, we have $F(\mathsf{lam}_{A,B}) = \mathsf{lam}_{FA,FB}$.

□

Definition 10.8. *Unit types* for a cwf $\mathcal{C}$ consist of, for $\Gamma \in \mathcal{C}$:

- a type $1_\Gamma \in \mathbf{Ty}(\Gamma)$,
- a unique inhabitant $\mathsf{pt}_\Gamma \in \mathbf{Tm}(1)$

that are stable under substitution, i.e. given a substitution $\sigma : \Delta \to \Gamma$:

- we have $1_\Gamma[\sigma] = 1_\Delta$,
- we have $\mathsf{pt}_\Gamma[\sigma] = \mathsf{pt}_\Delta$.

□

Definition 10.9. A *morphism of cwf's with unit types* is a morphism $F\colon \mathcal{C} \to \mathcal{D}$ of underlying cwf's such that for $\Gamma \in \mathcal{C}$:

- we have $F(1_\Gamma) = 1_{F\Gamma}$,
- additionally, we have $F(\mathsf{pt}_\Gamma) = \mathsf{pt}_{F\Gamma}$.

□

Definition 10.10. *Identity types* for a cwf $\mathcal{C}$ consist of, for $\Gamma \in \mathcal{C}$:

- given $A \in \mathbf{Ty}(\Gamma)$, a type $\mathsf{Id}_A \in \mathbf{Ty}(\Gamma.A.A)$,
- additionally, a term $\mathsf{refl}_A \in \mathbf{Tm}(\mathsf{Id}_A[\, p_A, q_A, q_A])$,
- given additionally $C \in \mathbf{Ty}(\Gamma.A.A.Id_A)$ and $d \in \mathbf{Tm}(C[\, p_A, q_A, q_A, \mathsf{refl}_A])$, a term $\mathsf{J}_{A,C,d} \in \mathbf{Tm}(C)$ such that $\mathsf{J}_{A,C,d}[\, p_A, q_A, q_A, \mathsf{refl}_A] = d$

that are stable under substitution, i.e. given a substitution $\sigma : \Delta \to \Gamma$:

- given $A \in \mathbf{Ty}(\Gamma)$, we have $\mathsf{Id}_A[\sigma^{++}] = \mathsf{Id}_{A[\sigma]}$,
- additionally, we have $\mathsf{refl}_A[\sigma^{++}] = \mathsf{refl}_{A[\sigma]}$
- given additionally $C \in \mathbf{Ty}(\Gamma.A.A.Id_A)$ and $d \in \mathbf{Tm}(C[\, p_A, q_A, q_A, \mathsf{refl}_A])$, we have $\mathsf{J}_{A,C,d}[\sigma^{+++}] = \mathsf{J}_{A[\sigma],C[\sigma^{+++}],d[\sigma^+]}$.

□

Definition 10.11. A *morphism of cwf's with identity types* is a morphism $F\colon \mathcal{C} \to \mathcal{D}$ of underlying cwf's such that for $\Gamma \in \mathcal{C}$:

- given $A \in \mathbf{Ty}(\Gamma)$, we have $F\mathsf{Id}_A = \mathsf{Id}_{FA}$,
- additionally, we have $F\mathsf{refl}_A = \mathsf{refl}_{FA}$,
- given additionally $C \in \mathbf{Ty}(\Gamma.A.A.Id_A)$ and $d \in \mathbf{Tm}(C[\, p_A, q_A, q_A, \mathsf{refl}_A])$, we have $F\mathsf{J}_{A,C,d} = \mathsf{J}_{FA,FC,Fd}$.

□

Definition 10.12. *Empty types* for a cwf $\mathcal{C}$ consist of, for $\Gamma \in \mathcal{C}$:

- a type $0_\Gamma \in \mathbf{Ty}(\Gamma)$,
- given $C \in \mathbf{Ty}(\Gamma.0)$, a term $\mathsf{0\text{-}elim}_C \in \mathbf{Tm}(C)$

that are stable under substitution, i.e. given a substitution $\sigma : \Delta \to \Gamma$:

- we have $0_\Gamma[\sigma] = 0_\Delta$,
- given additionally $C \in \mathbf{Ty}(\Gamma.0)$ we have $\mathsf{0\text{-}elim}_C[\sigma^+] = \mathsf{0\text{-}elim}_{C[\sigma^+]}$.

□

Definition 10.13. A *morphism of cwf's with empty types* is a morphism $F : \mathcal{C} \to \mathcal{D}$ of underlying cwf's such that for $\Gamma \in \mathcal{C}$:

- we have $F(0_\Gamma) = 0_{F\Gamma}$,
- given $C \in \mathbf{Ty}(\Gamma.0)$, we have $F\mathsf{0\text{-}elim}_C = \mathsf{0\text{-}elim}_{FC}$.

□

Definition 10.14. *Coproduct types* for a cwf $\mathcal{C}$ consist of, for $\Gamma \in \mathcal{C}$:

- given $A_0, A_1 \in \mathbf{Ty}(\Gamma)$, a type $+(A_0, A_1) \in \mathbf{Ty}(\Gamma)$,
- additionally, terms $\tau_{A_0,A_1,i} \in \mathbf{Tm}(\Gamma.A_i, +(A_0, A_1)[p])$ for $i \in \{0, 1\}$,
- given additionally $C \in \mathbf{Ty}(\Gamma. + (A_0, A_1))$ and $f_i \in \mathbf{Tm}(C[\, p, \tau_i])$ for $i \in \{0, 1\}$, a term $\mathsf{+\text{-}elim}_{C,f_0,f_1} \in \mathbf{Tm}(C)$ such that $\mathsf{+\text{-}elim}_{C,f_0,f_1}[\, p, \tau_i] = f_i$.

that are stable under substitution, i.e. given a substitution $\sigma : \Delta \to \Gamma$:

- given $A_0, A_1 \in \mathbf{Ty}(\Gamma)$, we have $+(A_0, A_1)[\sigma] = +(A_0[\sigma], A_1[\sigma])$.
- additionally, we have $\tau_i[\sigma^+] = \tau_i$ for $i \in \{0, 1\}$,
- given additionally $C \in \mathbf{Ty}(\Gamma. + (A_0, A_1))$ and $f_i \in \mathbf{Tm}(C[\, p, \tau_i])$ for $i \in \{0, 1\}$, we have $\mathsf{+\text{-}elim}_{C,f_0,f_1}[\sigma^+] = \mathsf{+\text{-}elim}_{C[\sigma^+],f_0[\sigma^+],f_1[\sigma^+]}$.

□

Definition 10.15. A *morphism of cwf's with coproduct types* is a morphism $F : \mathcal{C} \to \mathcal{D}$ of underlying cwf's such that for $\Gamma \in \mathcal{C}$:

- given $A_0, A_1 \in \mathbf{Ty}(\Gamma)$, we have $F(+(A_0, A_1)) = +(FA_0, FA_1)$,
- additionally, we have $F\tau_{A_0,A_1,i} = \tau_{FA_0,FA_1,i}$ for $i \in \{0, 1\}$,
- given additionally $C \in \mathbf{Ty}(\Gamma. + (A, B))$ and $f_i \in \mathbf{Tm}(C[\, p, \tau_i])$ for $i \in \{0, 1\}$, we have $F\mathsf{+\text{-}elim}_{C,f_0,f_1} = \mathsf{+\text{-}elim}_{FC,Ff_0,Ff_1}$.

□

Definition 10.16. A *universe* U for a cwf $\mathcal{C}$ consists of, for $\Gamma \in \mathcal{C}$,

- a type $\mathsf{U}_\Gamma \in \mathbf{Ty}(\Gamma)$,
- a type $\mathsf{El}_\Gamma \in \mathbf{Ty}(\Gamma.\mathsf{U})$

that are stable under substitution, i.e. given a substitution $\sigma : \Delta \to \Gamma$:

- we have $\mathsf{U}_\Gamma[\sigma] = \mathsf{U}_\Delta$,
- we have $\mathsf{El}_\Gamma[\sigma^+] = \mathsf{El}_\Delta$.

□

Definition 10.17. A *morphism of cwf's with universes* is a morphism $F : \mathcal{C} \to \mathcal{D}$ of underlying cwf's such that for $\Gamma \in \mathcal{C}$:

- we have $F\mathsf{U}_\Gamma = \mathsf{U}_{F\Gamma}$,
- we have $F\mathsf{El}_\Gamma = \mathsf{U}_{F\Gamma}$.

□

Definition 10.18. Given a cwf $\mathcal{C}$ with a universe U, the *cwf of* U*-types* $\mathcal{C}^{\mathsf{U}} =_{\text{def}} (\mathcal{C}, \mathbf{Ty}^{\mathsf{U}}, \mathbf{Tm}^{\mathsf{U}})$ has underlying category $\mathcal{C}$, types $\mathbf{Ty}^{\mathsf{U}}(\Gamma) =_{\text{def}} \mathbf{Tm}(U_\Gamma)$, and terms $\mathbf{Tm}^{\mathsf{U}}(A) = \mathbf{Tm}(\mathsf{El}(A))$, with substitution defined from substitution in the cwf $\mathcal{C}$. Given $\Gamma \in \mathcal{C}$ and $A \in \mathbf{Ty}^{\mathsf{U}}(\Gamma)$, context extension and projection is defined as $\Gamma.^{\mathsf{U}}A =_{\text{def}} \Gamma.\mathsf{El}(A)$ and $p^{\mathsf{U}}_A =_{\text{def}} p_{\mathsf{El}(A)}$, and the representing term is $q^{\mathsf{U}}_A =_{\text{def}} q_{\mathsf{El}(A)}$.

The *embedding morphism* $\mathcal{C}^{\mathsf{U}} \to \mathcal{C}$ is the identity on underlying categories, sends a type $A \in \mathbf{Ty}^{\mathsf{U}}(\Gamma)$ to $\mathsf{El}(A) \in \mathbf{Ty}(\Gamma)$, and is the identity on terms. □

Remark 10.19. The preceeding definition is much nicer in the discrete comprehension category style of presenting cwf's. There, one sees directly that any presheaf T on $\mathcal{C}$ with a natural transformation $u\colon T \to \mathbf{Ty}$ gives rise to a cwf C^T with the same underlying category and terms, but types "restricted" to T. Given a universe U, one defines $T(\Gamma)$ as the set of sections of $\chi(U_\Gamma)$. □

Definition 10.20. Let $\mathcal{C}$ be a cwf with a universe U.

- Let $\mathcal{C}$ have dependent sums. Then U-*dependent sums* consist of: given $A \in \mathbf{Tm}(\mathsf{U})$

□

Definition 10.21. Let T be a collection of type formers, excluding universes. Let α be an ordinal. An *α-indexed cumulative hierarchy with type formers T* is a diagram $\mathcal{C}\colon \alpha \to \mathbf{CwF}_T$ that is the identity on underlying categories together with, for $S\beta < \alpha$, a universe U_β in $\mathcal{C}_{S\beta}$ such that $\mathcal{C}_\beta$ and $\mathcal{C}^{\mathsf{U}_\beta}_{S\beta}$ are isomorphic over $\mathcal{C}_{S\beta}$; we require $\mathcal{C}^{\mathsf{U}_\beta}_{S\beta} \to \mathcal{C}_{S\beta}$ to be injective.[1] □

In the context of $\mathcal{C} \in \mathbf{CwF}^\alpha_T$, we write $\mathbf{Ty}_n(\Gamma)$ for $\mathbf{Ty}_{\mathcal{C}_n}(\Gamma)$.

Definition 10.22. Let $\mathcal{C}$ and $\mathcal{D}$ be α-indexed cumulative hierarchies with type formers T. A *morphism* $F\colon \mathcal{C} \to \mathcal{D}$ is a natural transformation such that, for $S\beta < \alpha$, $F_{S\beta}$ is a morphism of cwf's with universe. □

We denote $\mathbf{CwF}^\alpha_T$ the category of α-indexed cumulative hierarchy with type formers T. As a category of models of a generalized (or essentially) algebraic theory, it is cocomplete.

A *weak morphism* of objects in $\mathbf{CwF}^\alpha_T$ is a morphism of underlying objects in $\mathbf{CwF}^\alpha$. A *pseudomorphism* preserves context extension only up to canonical isomorphism. A *weak pseudomorphism* is like a pseudomorphism, but does not have to respect the type formers T.

11. Standard model

Let κ be an α-indexed hierarchy of regular cardinals. We have a standard model $\mathbf{Set} \in \mathbf{CwF}^\alpha_T$ that has as underlying category the category of sets $\mathbf{Set}$ and as types at level $n < \alpha$, the families that are valued in sets of cardinality bounded by κ_n.

Given $\mathcal{C} \in \mathbf{CwF}^\alpha_T$, we have a *global section* weak pseudomorphism $F\colon \mathcal{C} \to \mathbf{Set}$ that on underlying categories is given by $\mathcal{C}(1, -)$ and on types sends $A \in \mathbf{Ty}_n(\Gamma)$ to the family sending $\sigma\colon 1 \to \Gamma$ to $\mathbf{Tm}(A[\sigma])$.

Given a small category $\mathbb{C}$, we have a presheaf model $\widehat{\mathbb{C}} \in \mathbf{CwF}^\alpha_T$ that has as underlying category the presheaf category $\widehat{\mathbb{C}}$ and as types at level $n < \alpha$ the presheaves that are valued in sets of cardinality bounded by κ_n. This construction is contravariantly weakly functorial: given a functor $F\colon \mathcal{C} \to \mathcal{D}$, we obtain a weak morphism $\widehat{D} \to \widehat{C}$ induced by precomposition with F (it is a nice accident that we obtain a weak morphism instead of just a weak pseudomorphism).

Given $\mathcal{C} \in \mathbf{CwF}^\alpha_T$, we have a *Yoneda* weak pseudomorphism $F\colon \mathcal{C} \to \widehat{\mathcal{C}}$ that on underlying categories is given by the Yondeda functor y and on types sends $A \in \mathbf{Ty}_n(\Gamma)$ to the presheaf sending an element $\sigma\colon \Delta \to \Gamma$ of $y\Gamma$ to $\mathbf{Tm}(A[\sigma])$.

[1] Investigate whether this restriction can be lifted.

12. Manual specification of the category of renamings

For the remainder of the section, let us fix an object $\mathcal{C}$ of $\mathbf{CwF}^{\alpha}_{T}$. Given $n < \alpha$, we write $\mathbf{Ty}_n$ and $\mathbf{Tm}_n$ for $\mathbf{Ty}_{\mathcal{C}_n}$ and $\mathbf{Tm}_{\mathcal{C}_n}$, respectively.

Let $\mathcal{D}$ denote the colimit of $\mathcal{C}$. Explicitly, the underlying category of $\mathcal{D}$ is the same as for $\mathcal{C}$; a type in $\mathcal{D}$ is a pair (n, A) with $n < \alpha$ and $A \in \mathbf{Ty}(\mathcal{C}_n)$, subject to the quotient that identifies (n, A) and (n', A') if the images of A and A' are equal in $\mathbf{Ty}(\mathcal{C}_{n+1})$ (this description uses that $\mathcal{C}$ consists of monomorphisms).

We define category $\mathsf{Ren}(\mathcal{C})$ of renamings of $\mathcal{C}$ as that of $\mathcal{D}$, i.e. $\mathsf{Ren}(\mathcal{C}) =_{\text{def}} \mathsf{Ren}(\mathcal{D})$.

13. Manual specification of neutral terms and normal forms

We define normal forms and neutral terms. These are defined mutually inductive with natural interpretation maps to terms. That is, we define

$$\mathsf{NF} \in \widehat{\mathbf{Ty}_{\mathsf{Ren}(\mathcal{C})}}, \qquad \mathsf{tm}\colon \mathsf{NF}F \to \mathbf{Tm}_{\mathcal{C}},$$

$$\mathsf{NE} \in \widehat{\mathbf{Ty}_{\mathsf{Ren}(\mathcal{C})}}, \qquad \mathsf{tm}\colon \mathsf{NE}F \to \mathbf{Tm}_{\mathcal{C}},$$

all by mutual induction.

Given $n < \alpha$ and $A \in \mathbf{Ty}_n$, we write $\mathsf{NF}_n(\Gamma, A)$ for $\mathsf{NF}(\Gamma, (n, A)))$

Neutral to normal

$$\frac{t \in \mathsf{NE}_n(\Gamma, A)}{\mathsf{ne}(t) \in \mathsf{NF}_n(\Gamma, A) \qquad \mathsf{tm}(\mathsf{ne}(t)) = \mathsf{tm}(t)}$$

Variables

$$\frac{A \in \mathbf{Ty}_n(\Gamma) \qquad a \in \mathbf{Tm}_{\mathsf{Ren}(\mathcal{C})}(\Gamma, A)}{\mathsf{var}(a) \in \mathsf{NF}_n(\Gamma, a) \qquad \mathsf{tm}(\mathsf{var}(a)) = F(a)}$$

Dependent sums.

$$\frac{A \in \mathbf{Ty}_n(\Gamma) \qquad B \in \mathbf{Ty}_n(\Gamma.A) \qquad s \in \mathsf{NE}_n(\Gamma, \Sigma(A, B))}{\begin{array}{c}\mathsf{fst}(s) \in \mathsf{NE}_n(\Gamma, A) \qquad \mathsf{tm}(\mathsf{fst}(s)) = \mathsf{fst}(\mathsf{tm}(s)) \\ \mathsf{snd}(s) \in \mathsf{NE}_n(\Gamma, B(\mathsf{fst}(\mathsf{tm}(s)))) \qquad \mathsf{tm}(\mathsf{snd}(s)) = \mathsf{snd}(\mathsf{tm}(s))\end{array}}$$

$$\frac{A \in \mathbf{Ty}_n(\Gamma) \qquad B \in \mathbf{Ty}_n(\Gamma.A) \qquad a \in \mathsf{NF}_n(\Gamma, A) \qquad b \in \mathsf{NF}_n(\Gamma, B(\mathsf{tm}(a)))}{\mathsf{pair}(a, b) \in \mathsf{NF}_n(\Gamma, \Sigma(A, B))) \qquad \mathsf{pair}(a, b)) = \mathsf{pair}(\mathsf{tm}(a), \mathsf{tm}(b))}$$

Dependent products.

$$\frac{A \in \mathbf{Ty}_n(\Gamma) \qquad B \in \mathbf{Ty}_n(\Gamma.A) \qquad f \in \mathsf{NE}_n(\Gamma, \Pi(A, B)) \qquad a \in \mathsf{NF}_n(\Gamma, A)}{\mathsf{app}(f, a) \in \mathsf{NF}_n(\Gamma, B(\mathsf{tm}(a))) \qquad \mathsf{tm}(\mathsf{app}(f, a)) = \mathsf{app}(\mathsf{tm}(f), \mathsf{tm}(a))}$$

$$\frac{A \in \mathbf{Ty}_n(\Gamma) \qquad B \in \mathbf{Ty}_n(\Gamma.A) \qquad f \in \mathsf{NF}_n(\Gamma.A, B)}{\mathsf{lam}(b) \in \mathsf{NF}(\Gamma, \Pi(A, B)) \qquad \mathsf{tm}(\mathsf{lam}(b)) = \mathsf{lam}(\mathsf{tm}(b))}$$

Universes.

$$\frac{m < n \qquad A \in \mathsf{NF}_n(\Gamma, \mathsf{U}_m) \qquad B \in \mathsf{NF}_n(\Gamma.\mathsf{El}(A), \mathsf{U}_m)}{\Sigma(A, B) \in \mathsf{NF}_n(\Gamma, \mathsf{U}_m) \qquad \mathsf{tm}(\Sigma(A, B)) = \Sigma(\mathsf{tm}(A), \mathsf{tm}(B))}$$

$$\frac{m < n \qquad A \in \mathsf{NF}_n(\Gamma, \mathsf{U}_m) \qquad B \in \mathsf{NF}_n(\Gamma.\mathsf{El}(A), \mathsf{U}_m)}{\Pi(A, B) \in \mathsf{NF}_n(\Gamma, \mathsf{U}_m) \qquad \mathsf{tm}(\Sigma(A, B)) = \Pi(\mathsf{tm}(A), \mathsf{tm}(B))}$$

$$\frac{i < m < n}{\mathsf{U}_i \in \mathsf{NF}_n(\Gamma, \mathsf{U}_m) \qquad \mathsf{tm}(\mathsf{U}_i) = \mathsf{U}_i}$$

We have to check that these definitions respect the quotient defining the types of $\mathsf{Ren}(\mathcal{C})$.

14. Manual specification of the glueing

We will define an object $\mathcal{E}$ of $\mathbf{CwF}^{\alpha}_{T}$.

Write $i\colon \mathsf{Ren}(\mathcal{C}) \to \mathcal{C}$ for the canonical weak morphism. The underlying category is defined as the sconing of the functor

$$\mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{\hat{i}} \widehat{\mathsf{Ren}(\mathcal{C})},$$

i.e. $\mathcal{E} =_{\mathrm{def}} \widehat{\mathsf{Ren}(\mathcal{C})} \downarrow F$ where $F =_{\mathrm{def}} \hat{i} \circ y$.

Contexts and substitutions. The objects of $\mathcal{E}$ are triples $(\Gamma, |\Gamma|, \gamma)$ where $\Gamma \in \mathcal{C}$, $|\Gamma| \in \widehat{\mathsf{Ren}(\mathcal{C})}$, and $\gamma\colon |\Gamma| \to F\Gamma$. We write such an object simply as $|\Gamma| \xrightarrow{\gamma} F\Gamma$. A morphism from $(\Delta, |\Delta|, \delta)$ to $(\Gamma, |\Gamma|, \gamma)$ is written as a pair $(\sigma, |\sigma|)$ with morphisms $\sigma\colon \Delta \to \Gamma$ and $|\sigma|\colon |\Delta| \to |\Gamma|$ such that the following diagram commutes:

$$\begin{array}{ccc} |\Delta| & \xrightarrow{\delta} & F\Delta \\ \downarrow{\scriptstyle |\sigma|} & & \downarrow{\scriptstyle F\sigma} \\ \Gamma & \xrightarrow{\gamma} & F\Gamma. \end{array}$$

Types. Let $|\Gamma| \xrightarrow{\gamma} F\Gamma$ be an object of $\mathcal{E}$. We define $\mathbf{Ty}_n(|\Gamma| \xrightarrow{\gamma} F\Gamma)$ as the set of triples $(A, |A|, \alpha)$ with types $A \in \mathbf{Ty}_n(\Gamma)$, $|A| \in \mathbf{Ty}_n(|\Gamma|)$, i.e. $|A|$ is a κ_n-small presheaf over $\int |\Gamma|$, and a natural transformation $\alpha\colon |A| \to FA[\gamma]$ of presheaves over $\int |\Gamma|$. The action of substitutions on $\mathbf{Ty}_n$ is evident.

[Should we represent α as a map $|\Gamma|.|A| \to F(\Gamma.A)$ with a commuting diagram condition?]

Terms. Given a type $(A, |A|, \alpha) \in \mathbf{Ty}_n(|\Gamma| \xrightarrow{\gamma} F\Gamma)$, we define $\mathbf{Tm}(A, |A|, \alpha)$ as the set of pairs $(t, |t|)$ with terms $t \in \mathbf{Tm}(A)$ and $|t| \in \mathbf{Tm}(|A|)$ such that $\alpha(|t|) = Ft[\gamma]$. The action of substitutions on $\mathbf{Tm}_n$ is evident.

Context extension. Consider a type $(A, |A|, \alpha) \in \mathbf{Ty}_n(|\Gamma| \xrightarrow{\gamma} F\Gamma)$. The associated context extension is $(\Gamma.A, |\Gamma|.|A|, \gamma.\alpha)$ where $\gamma.a =_{\mathrm{def}} \langle \gamma \circ p_{|A|}, \alpha(q_{|A|}) \rangle$. Here, we take the liberty of using cwf notation also for the non-canonical context extension $F(\Gamma.A)$ (recall that F is a weak pseudomorphism and preserves context extension up to canonical isomorphism). The associated context projection is $p_{(A,|A|,\alpha)} =_{\mathrm{def}} (p_A, p_{|A|})$. The representing term is $q_{(A,|A|,\alpha)} =_{\mathrm{def}} (q_A, q_{|A|})$. One easily checks the universal property of context extension.

Hierarchy. Given $n \leq n' < \alpha$, we have a natural transformation $\mathbf{Ty}_n \to \mathbf{Ty}_{n'}$. These assemble into an α-indexed diagram of cwf's.

Dependent sums. Consider types $(A, |A|, \alpha) \in \mathbf{Ty}_n(|\Gamma| \xrightarrow{\gamma} F\Gamma)$ and $(B, |B|, \beta) \in \mathbf{Ty}_n(\Gamma.A, |\Gamma|.|A|, \gamma.\alpha)$. The associated dependent sums is $(\Sigma(A, B), \Sigma(|A|, |B|), \Sigma(\alpha, \beta))$ where

$$\Sigma(\alpha, \beta) \colon \Sigma(|A|, |B|) \to F(\Sigma(A, B))[\gamma]$$

comes from the map $\gamma.\alpha.\beta$ over γ.

|Γ|.|A|.|B| → F(Γ.A.B) → F(Γ.Σ(A,B)) → FΓ.F(Σ(A,B)) | |Γ| → FΓ

is the composition of

$$\Sigma(|A|, |B|) \longrightarrow \Sigma(|A|, FB[\gamma.\alpha]) \longrightarrow \Sigma(FA[\gamma], FB[\gamma])$$

FA   Ty(FΓ) FB   Ty(F(Γ.A))

|A| Ty(|Γ|) |B| Ty(|Γ|.|A|)

|Γ| : |A| → FA[ ] |Γ|.|A| : |B| → FB[.]

|Γ| ? : Σ(|A|, |B|) → F(Σ(A, B))[ ]

Tm(|Γ|.Σ(|A|, |B|), F(Σ(A, B))[ ][p]) — Tm(|Γ|.|A|.|B|, F(Σ(A, B))[ ][pp]) —

Γ.A.B → Γ with Tm(Γ.A.B, Σ(A, B)[pp]) has universal property of context extension

hence has F(Γ.A.B) → FΓ with Tm(F(Γ.A.B), F(Σ(A, B)[pp]))

goes via $\Sigma(FA, FB)$.

Universes.

Given a context $|\Gamma| \xrightarrow{\gamma} F\Gamma$, we define the universe

$$\begin{aligned} &\mathsf{U}_n \in \mathbf{Ty}_{n+1}(|\Gamma| \xrightarrow{\gamma} F\Gamma), \\ &\mathsf{U}_n =_{\text{def}} (\mathsf{U}_n, (X : F\mathsf{U}_n[\gamma]) \times (F\mathsf{El}_n[\gamma^+](X) \to \mathsf{U}_n), \pi_1) \end{aligned}$$

with

$$\begin{aligned} &\mathsf{El}_n \in \mathbf{Ty}_{n+1}(\Gamma.\mathsf{U}_n, |\Gamma|.((X : F\mathsf{U}_n[\gamma]) \times (P : F\mathsf{El}_n[\gamma^+](X) \to \mathsf{U}_n)), \gamma.\pi_1), \\ &\mathsf{El}_n = (\mathsf{El}_n, (x : F\mathsf{El}_n(X)) \times \mathsf{El}_n(P(x)), \pi_1). \end{aligned}$$

An element of $\mathbf{Tm}(\mathsf{U}_n, (X : F\mathsf{U}_n[\gamma]) \times (F\mathsf{El}_n(X) \to \mathsf{U}_n), \pi_1)$ consists of a term $t \in \mathbf{Tm}(\mathsf{U}_n)$ and a term $|t| \in \mathbf{Tm}((X : F\mathsf{U}_n[\gamma]) \times (F\mathsf{El}_n(X) \to \mathsf{U}_n))$ such that $\pi_1(|t|) = Ft$. This corresponds to a type $A \in \mathbf{Ty}_n(\Gamma)$ and a type $|A| \in \mathbf{Ty}_n(|\Gamma|.FA[\gamma])$.

[So this forces types to be defined fiberwise. Let us try to define universes differently.]

Given a context $|\Gamma| \xrightarrow{\gamma} F\Gamma$, we define the universe

$$\begin{aligned} &\mathsf{U}_n \in \mathbf{Ty}_{n+1}(|\Gamma| \xrightarrow{\gamma} F\Gamma), \\ &\mathsf{U}_n =_{\text{def}} (\mathsf{U}_n, (X : F\mathsf{U}_n[\gamma]) \times (|X| : \mathsf{U}_n) \times ((F\mathsf{El}_n[\gamma^+](X) \to \mathsf{El}_n(|X|))), \pi_1) \end{aligned}$$

with

$$\begin{aligned} &\mathsf{El}_n \in \mathbf{Ty}_{n+1}(\Gamma.\mathsf{U}_n, |\Gamma|.((X : F\mathsf{U}_n[\gamma]) \times (|X| : \mathsf{U}_n) \times ((F\mathsf{El}_n[\gamma^+](X) \to \mathsf{El}_n(|X|)))), \gamma.\pi_1), \\ &\mathsf{El}_n = (\mathsf{El}_n, \mathsf{El}_n(\pi_2), \pi_3). \end{aligned}$$

An element of $\mathbf{Tm}(\mathsf{U}_n, (X : F\mathsf{U}_n[\gamma]) \times (|X| : \mathsf{U}_n) \times ((F\mathsf{El}_n(X) \to \mathsf{El}_n(|X|))), \pi_1)$ consists of a term $t \in \mathbf{Tm}(\mathsf{U}_n)$ and a term $|t| \in \mathbf{Tm}(X : F\mathsf{U}_n[\gamma]) \times (|X| : \mathsf{U}_n) \times ((\mathsf{El}_n(|X|) \to F\mathsf{El}_n(X)))$ such that $\pi_1(|t|) = Ft$. This corresponds to a type $A \in \mathbf{Ty}_n(\Gamma)$ and a type $|A| \in \mathbf{Ty}_n(|\Gamma|)$ with a term $\mathbf{Tm}(|A| \to FA[\gamma])$.

[But that is really different from before. Here, the total space $|X|$ is small, before only the fibers of the map $|X| \to FX[\gamma]$ were small. I am curious to know whether both of these variants of glueings will work.]

In the twisted glueing, the types are now modified. A type over $|\Gamma| \xrightarrow{\alpha} \hat{i}y\Gamma$ additionally consists of maps

$$\begin{array}{ccccc} \mathsf{NE}_n(A)[\alpha] & \dashrightarrow & |A| & \dashrightarrow & \mathsf{NF}_n(A)[\alpha] \\ & \searrow^{\mathsf{tm}[\alpha]} & \downarrow & \swarrow^{\mathsf{tm}[\alpha]} & \\ & & \hat{i}yA[\alpha] & & \end{array}$$

15. Taking fibrations of cwf seriously

Given a cwf $\mathcal{C}$ and a context $\Gamma \in \mathcal{C}$, we write

$$\mathbf{TyCat}_{\mathbb{C}}(\Gamma) \longrightarrow \mathcal{C}/\Gamma$$

for the fully faithful functor sending $A \in \mathbf{Ty}(\Gamma)$ to $p_A \colon \Gamma.A \to \Gamma$. We call $\mathbf{TyCat}_{\mathcal{C}}(\Gamma)$ the *category of types* in context Γ. These assemble to a category-valued presheaf $\mathbf{TyCat}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \to$ **Cat**.

Definition 15.1. A *fibration* $P \colon \mathcal{E} \to \mathcal{C}$ of cwf's is a strict morphism P with the following structure.

- On underlying categories, it is a cloven isofibration.
- For $\Gamma \in \mathcal{E}$, $X \in \mathbf{Ty}_{\mathcal{E}}(\Gamma)$, $B \in \mathbf{Ty}_{\mathcal{C}}(P\Gamma)$, and $f \colon P\Gamma.B \to P\Gamma.PX$ over $P\Gamma$, we have $K_{X,B,f} \in \mathbf{Ty}_{\mathcal{E}}(\Gamma)$ and $k_{X,B,f} \colon \Gamma.K_{X,B,f} \to \Gamma.X$ over Γ such that $k_{X,B,f}$ is a cartesian lift of f with respect to the functor $P_\Gamma \colon \mathcal{E}/\Gamma \to \mathcal{C}/P\Gamma$:

$$\begin{array}{ccccc} K_{X,B,f} & \xrightarrow[\mathsf{cart}]{k_{X,B,f}} & \Gamma.X & \qquad & \mathcal{E}/\Gamma \\ & & & & \big\downarrow {\scriptstyle P_\Gamma} \\ P\Gamma.B & \longrightarrow & P\Gamma.PX & \qquad & \mathcal{C}/P\Gamma. \end{array}$$

  Furthermore, this data is stable under substitution, i.e. given $\sigma \colon \Delta \to \Gamma$, we have

$$K_{X,B,f}[\sigma] = K_{X[\sigma,B[P\sigma],f[P\sigma]]}$$
$$k_{X,B,f}[\sigma] = k_{X[\sigma,B[P\sigma],f[P\sigma]]}.$$

□

Let now $P \colon \mathcal{E} \to \mathcal{C}$ be a fibration of cwf's and $F \colon \mathcal{D} \to \mathcal{C}$ a pseudomorphism of cwf's. Construct the pullback

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{E} \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle Q} \\ \mathcal{D} & \xrightarrow{F} & \mathcal{C} \end{array}$$

in the category of cwf's without context extension.

References

[Awo16] Steve Awodey. Natural models of homotopy type theory. *Mathematical Structures in Computer Science*, pages 1–46, 2016.