FAILURE OF THE ALGEBRAIC UNIVERSE OF TYPES WITH PRISM-BASED KAN COMPOSITION IN SIMPLICIAL SETS

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Extensional layer. We will call $\mathbf{SSet} = \mathrm{Presheaf}(\Delta)$ the category of context. Given a context Γ, we will call $\mathrm{Presheaf}(\int \Gamma)$ the category of types over Γ, with a type A over Γ written $\Gamma \vdash A$. Under the equivalence $\mathrm{Presheaf}(\int \Gamma) \simeq \mathrm{Presheaf}(\Delta)/\Gamma$, such a type corresponds to a morphism $\Gamma.A \to \Gamma$, called the context extension of Γ with A.

Let V be the Hofmann-Streicher universe in simplicial sets, i.e. V_n is the set of (small) types over Δ^n (with the presheaf structure given by precomposition of functors). Let $\tilde{V}$ be its pointed variant, i.e. $\tilde{V}_n$ is the set of (small) types $\Delta^n \vdash A$ together with a point $1 \to A$. Viewing $\tilde{V}$ as a type over V, it forms a classifier of (small) types. (The projection $\tilde{V} \to V$ forms a weak classifier (in the sense of Cisinski) of all (small) maps in $\mathbf{SSet}$.)

Homotopical layer. We will now introduce the homotopical layer following the newer cubical model of Coquand et al. Given a context Γ and a type $\Gamma \vdash A$, we have a set of composition structures $\mathrm{Comp}(\Gamma, A)$ where we use all monomorphisms as cofibrations and Δ^1 as the interval (with evident endpoints). This gives a functor $\mathrm{Comp}\colon (\int \mathrm{Type})^{\mathrm{op}} \to \mathbf{Set}$. Observe that Comp is furthermore contravariantly functorial in $\Gamma \vdash A$ as an arrow $\Gamma.A \to A$ in $\mathbf{SSet}$ (though this may not be used in this note).

Our goal for this note is to examine the algebraic universe $U \in \mathrm{Presheaf}(\Delta)$ of (small) types with composition. It is is defined by letting U_n be the set of (small) types $\Delta^n \vdash A$ together with a composition structure $\mathrm{Comp}(\Delta^n, A)$. This gives an evident projection $U \to V$, inducing the classifier $U \vdash \tilde{U}$ via pullback of $V \vdash \tilde{V}$.

Let Γ be a context and $\Gamma \vdash A$ a (small) type classified by a map $\ulcorner A \urcorner\colon \Gamma \to V$. Lifts

$$\begin{array}{ccc} & & U \\ & \nearrow & \downarrow \\ \Gamma & \xrightarrow[\ulcorner A \urcorner]{} & V \end{array} \tag{0.1}$$

correspond to a coherent family of composition structures $\mathrm{Comp}(\Delta^n, \sigma^* A)$ for $\sigma\colon \Delta^n \to \Gamma$. We denote the set of such coherent families by $\mathrm{Comp}'(\Gamma, A)$.

Functoriality of $\mathrm{Comp}\colon (\int \mathrm{Type})^{\mathrm{op}} \to \mathbf{Set}$ induces a map $\mathrm{Comp}(\Gamma, A) \to \mathrm{Comp}'(\Gamma, A)$. For an algebraic universe, we would want lifts (0.1) to be in canonical bijection with composition structures on $\Gamma \vdash A$, so would want the map $\mathrm{Comp}(\Gamma, A) \to \mathrm{Comp}'(\Gamma, A)$ to be a bijection.

A recursive description of compositions. Here, we restrict to decidable simplicial sets. In our counterexamples, all simplicial sets will be decidable.

Let $\mathrm{CompProb}(\Gamma, A)$ and $\mathrm{CompProb}_\bullet(\Gamma, A)$ denote the category of composition problems and solved composition problems with right-hand side $\Gamma.A \to A$, respectively. Forgetting the solution yields a discrete Grothendieck fibration $\mathrm{CompProb}_\bullet(\Gamma, A) \to \mathrm{CompProb}(\Gamma, A)$. The projection to the left-hand side gives a discrete Grothendieck fibration $\mathrm{CompProb}(\Gamma, A) \to \{0, 1\} \times I$ where

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$I \to \mathbf{SSet}^{\to}$ is the generating category of cofibrations. We have a further discrete Grothendieck fibration $I \to \Delta$ given by the codomain functor. Thus, all of the involved categories inherit an elegant Reedy category structure from Δ.

An element of $\mathrm{Comp}(\Gamma, A)$ is a section F to the functor $\mathrm{CompProb}_{\bullet}(\Gamma, A) \to \mathrm{CompProb}(\Gamma, A)$. Such a section can be specified recursively as follows. Let $X \in \mathrm{CompProb}(\Gamma, A)$.

If X is degenerate, there is a unique non-identity degeneracy $d\colon X \to Y$ with Y non-degenerate. Since Y has lower degree than X, we already have the value $F(Y)$. We then let $F(X)$ be the base change of $F(Y)$ along d.

Otherwise, we consider the faces of X. Since they have lower degree, we already have their values under F. Letting $U \hookrightarrow \Delta^n$ denote the left-hand side of X. The information from the faces of X specified the composition filler on the restriction on the boundary of Δ^n, so it only remains to choose an n-simplex in A over the given n-simplex in Γ that has the boundary prescribed by the value of F on the faces of X.

It is easy to verify that this indeed yields a section F as required and that every section F arises in this way.

Lemma 0.1. *The map* $\mathrm{Comp}(\Gamma, A) \to \mathrm{Comp}'(\Gamma, A)$ *is not always injective.*

Proof. Let $\Gamma =_{\mathrm{def}} \Delta^1 \times \Delta^1$ and $A =_{\mathrm{def}} \mathrm{Cosk}^1(\Gamma \cup \{a, b\})$ where a, b are two additional copies of the edge $e\colon (1,0) \to (1,1)$. The map $A \to \Gamma$ is the unique extension of the identity on Γ. Viewing A as a type in context Γ, we have

$$A(\sigma) = \begin{cases} \{\bullet, a, b\} & \text{for } \sigma = e, \\ \{\bullet\} & \text{else} \end{cases}$$

for $\sigma\colon \Delta^n \to \Gamma$. We define composition structures $\alpha, \beta \in \mathrm{Comp}(\Gamma, A)$ as follows. Since $A(\sigma)$ is a singleton except for $\sigma = e$, we only need to specify these solutions over the edge e. We use the description of solutions to composition problems from before. So we only need to consider a 1-dimensional problem $X \in \mathrm{CompProb}(\mathrm{Comp}, A)$ where the missing edge is over e. The solution to potentially missing vertices is given by recursion. If the lifting problem lifts to a lifting problem in $\mathrm{Comp}(\Delta^2, u^* A)$ where $u\colon \Delta^2 \to \Gamma$ is the unique inclusion whose image includes e, we let the solution over e be the canonical element $\bullet \in A(e)$. Otherwise, we let the solution over e be a in case of α and b in case of β.

We claim that α and β pull back to the same element of $\mathrm{Comp}(\Delta^n, \sigma^* A)$ for any $\sigma\colon \Delta^n \to \Gamma$. This is clear when the edge e is not in the image of σ. Otherwise, there is $\sigma'\colon \Delta^2 \to \Delta^n$ such that $\sigma = \sigma' u$. Thus, it suffices to check that α and β pull back to the same element of $\mathrm{Comp}(\Delta^2, u^* A)$. But this is also clear by construction.

Thus, the distinct elements $\alpha, \beta \in \mathrm{Comp}(\Gamma, A)$ get send to the same element of $\mathrm{Comp}'(\Gamma, A)$ under the map $\mathrm{Comp}(\Gamma, A) \to \mathrm{Comp}'(\Gamma, A)$. □

We will now look at the dual question: is the map $\mathrm{Comp}(\Gamma, A) \to \mathrm{Comp}'(\Gamma, A)$ always surjective? Note that it suffices to examine the universal case of $\Gamma = U$ and $A = \mathrm{El}$, i.e. $\Gamma.A = \tilde{U}$.

For a particular $\Gamma \vdash A$, the answer is positive if for every the bottom map $\Delta^1 \times \Delta^n \to \Gamma$, the category of factorizations of this map through a representable is connected. This suggests a way to build a counterexample.

Lemma 0.2. *The map* $\mathrm{Comp}(\Gamma, A) \to \mathrm{Comp}'(\Gamma, A)$ *is not always surjective.*

Proof. Let S and T be 3-simplices glued together along inclusions $\Delta^1 \times \Delta^1 \to S$ and $\Delta^1 \times \Delta^1 \to T$ to yield a simplicial set Γ. We name an edge $e\colon (1,0) \to (1,1)$.

In the presheaf category over $\int \Gamma$, obtain a type A by starting with the terminal object that is $\{\bullet\}$ over any element of Γ, adding two edges s, t over e, and then taking the 1-coskeleton. Let A_S and A_T be the restriction of A to S and T, respectively.

Just like in Lemma 0.1, we build an element of $\mathrm{Comp}(S, A_S)$ by using the edge s in the solution whenever possible, but only if the lifting problem does not factor through a triangle of $\Delta^1 \times \Delta^1$ (otherwise we use the canonical element $\bullet$ over e). We build an element of $\mathrm{Comp}(T, A_T)$ just like that, only using the edge t instead. It can then be verified that these composition structures give rise to a coherent family as required for an element of $\mathrm{Comp}'(\Gamma, A)$.

There cannot be a preimage of this element in $\mathrm{Comp}(\Gamma, A)$, for the composition problem with base the identity on $\Delta^1 \times \Delta^1$ and only an edge missing over e would have to be solved using both a and b, but $a \neq b$. □

Let us consider Presheaf(**SSet**) (with a larger universe of sets in the outer presheaf layer). We have a Hofmann-Streicher universe Type and Comp can be viewed as a presheaf on $\int \mathrm{Type}$.

Corollary 0.3. Comp *is not continuous.*

Proof. Using the fact that presheaves are the free cocompletion of the base category, Comp is continuous precisely if the map $\mathrm{Comp}(\Gamma, A) \to \mathrm{Comp}'(\Gamma, A)$ is an isomorphism for every element (Γ, A) of Type. But by Lemmata 0.1 and 0.2, it is neither a monomorphism nor an epimorphism. □

Unrelatedly, we can still say the following.

Lemma 0.4. *The map $\tilde{U} \to U$, i.e. the type $U \vdash \mathrm{El}$, is a Kan fibration.*

Proof. So it suffices to solve lifting problems

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \tilde{U} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \xrightarrow[\ulcorner A \urcorner]{} & U. \end{array}$$

This pulls back to a lifting problem

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \xrightarrow[\mathrm{id}]{} & \Delta^n. \end{array}$$

But $\Delta^n \vdash A$ had composition, so $A \to \Delta^n$ is a Kan fibration. □