CYLINDRICAL MODEL STRUCTURES

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0.1. **Preliminaries.** Given categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathrm{Alg}(\mathcal{C}, \mathcal{D})$ for the category of adjunctions between $\mathcal{C}$ and $\mathcal{D}$. Briefly, this is the category of profunctors from $\mathcal{D}$ to $\mathcal{C}$ with a representation and corepresentation (these only matter for the objects, not the morphisms).

Explicitly, an object $\mathrm{Alg}(\mathcal{C}, \mathcal{D})$ is a tuple $(F_\mathsf{l}, F_\mathsf{r}, \alpha)$ of functors $F_\mathsf{l} \colon \mathcal{C} \to \mathcal{D}$ and $F_\mathsf{r} \colon \mathcal{D} \to \mathcal{C}$ with a natural isomorphism $\alpha \colon \mathcal{D}(F_\mathsf{l}(-), -) \simeq \mathcal{C}(-, F_\mathsf{r}(-))$. A morphism from $(F_\mathsf{l}, F_\mathsf{r}, \alpha)$ to $(G_\mathsf{l}, G_\mathsf{r}, \beta)$ consists of maps $u_\mathsf{l} \colon G_\mathsf{l} \to F_\mathsf{l}$ and $v \colon F_\mathsf{r} \to G_\mathsf{r}$ such that the diagram

$$\begin{array}{ccc}
\mathcal{D}(F_\mathsf{l}(-), -) & \xrightarrow{\alpha} & \mathcal{C}(-, F_\mathsf{r}(-)) \\
\Big\downarrow{\scriptstyle \mathcal{D}(u_\mathsf{l}, -)} & & \Big\downarrow{\scriptstyle \mathcal{C}(-, u_\mathsf{r})} \\
\mathcal{D}(G_\mathsf{l}(-), -) & \xrightarrow{\beta} & \mathcal{C}(-, G_\mathsf{r}(-))
\end{array}$$

commutes.

The forgetful functors $(-)_\mathsf{l} \colon \mathrm{Adj}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \to [\mathcal{C}, \mathcal{D}]$ and $(-)_\mathsf{r} \colon \mathrm{Adj}(\mathcal{C}, \mathcal{D}) \to [\mathcal{D}, \mathcal{C}]$ are fully faithful. We will treat their action on morphisms as implicit coercions. This allows us to denote a morphism of adjunctions by a morphism between the right adjoints or a morphism (in the opposite direction) between the left adjoints.

Given $M \in \mathrm{Alg}(\mathcal{C}, \mathcal{D})$, we write $M \odot_\mathsf{l} (-) \colon \mathcal{C} \to \mathcal{D}$ for the application of the left adjoint and $M \odot_\mathsf{r} (-) \colon \mathcal{D} \to \mathcal{C}$ for the application of the right adjoint. This is functorial in M. We choose an infix notation here so that we may easily denote the corresponding Leibniz constructions:

Given maps $f \colon A \to C$ and $g \colon B \to C$ in a category with binary coproducts, we write $[f, g] \colon A \amalg B \to C$ for the induced map from the coproduct.

1. Content

Let $\mathcal{E}$ be a category with classes of cofibrations and trivial cofibrations. Assume that $\mathcal{E}$ is tensored over finite sets. Then the functor $\mathrm{Id}_\mathcal{E} \amalg \mathrm{Id}_\mathcal{E}$ exists and is computed pointwise.

Definition 1.1. A *cylinder structure* on $\mathcal{E}$ is a tuple (C, δ_0, δ_1) where C is an endofunctor on $\mathcal{E}$ and $\delta_0, \delta_1 \colon \mathrm{Id}_\mathcal{E} \to C$ such that $[\delta_0, \delta_1] \colon \mathrm{Id}_\mathcal{E} \coprod \mathrm{Id}_\mathcal{E} \to C$ is a Quillen cofibration and δ_0, δ_1 are Quillen trivial cofibrations.

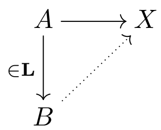
2. Weak factorization systems

We recall the notion of weak factorization system. We write $f \pitchfork g$ to indicate that a map f has the left lift property against a map g. We write ${}^{\pitchfork}(-)$ and $(-)^{\pitchfork}$ for the left and right lifting closure operations on classes of maps, respectively.

Definition 2.1. A *weak factorization system* $(\mathbf{L}, \mathbf{R})$ in a category $\mathcal{E}$ consists of classes of *left maps* $\mathbf{L}$ and *right maps* $\mathbf{R}$ such that $\mathbf{L} = {}^{\pitchfork}\mathbf{R}$, $\mathbf{R} = \mathbf{L}^{\pitchfork}$, and every map factors as a left map followed by a right map.

We assume a working knowledge of weak factorization systems. These include standard closure properties of left and right maps such as closure under retract, finitary composition, coproducts and pushout (for left maps), products and pullback (for right maps). Given a weak factorization system in $\mathcal{E}$, the slices and coslices of $\mathcal{E}$ inherit a weak factorization system, with classes of maps created by the forgetful functor. An object A is said to be *left* (or belong to $\mathbf{L}$) if $\emptyset \to A$ is a left map. Dually, an object X is said to be *right* (or belong to be $\mathbf{R}$) if $X \to 1$ is a right map.

Remark 2.2. The terminology of left and right objects makes sense even in the absence of initial and terminal objects by reading the given definition in copresheaves and presheaves over $\mathcal{E}$, respectively (for copresheaves, this uses the Yoneda embedding of $\mathcal{E}^{\mathrm{op}}$). For this, we extend the class $\mathbf{R}$ to presheaves over $\mathcal{E}$: a map $Y \to X$ in $[\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$ is called *right* if its pullback application $Y(B) \to Y(A) \times_{X(A)} X(B)$ to any left map $A \to B$ in $\mathcal{E}$ is split epi. Between representables, this agrees with the right maps in $\mathcal{E}$. The class $\mathbf{L}$ is extended dually, using pullback application in copresheaves over $\mathcal{E}$. With this convention, an object X is right just when $\mathcal{C}(B, X) \to \mathcal{C}(A, X)$ is split epi for every left map $A \to B$. This gives the standard lifting condition



avoiding a terminal object. If $\mathcal{E}$ has pushouts along left maps, X is right exactly if any left map in $\mathcal{E}$ with source X has a retraction.

Note that split epimorphisms are the right class of a weak factorization system in $\mathbf{Set}$. In particular, they enjoy standard closure properties such as under under pullback and composition. By Leibniz calculus, these closure properties transfer to the extended classes $\mathbf{L}$ and $\mathbf{R}$.

This convention makes sense in a broader context. It sometimes allows one to read arguments using certain (co)limits such as pullbacks of right maps in the context of a category $\mathcal{E}$ where these (co)limits do not exist. We keep it in mind for the remainder of this article, but will always accompany it with a reference to this remark when used. We warn that the extended classes $\mathbf{L}$ and $\mathbf{R}$ live in different categories and are not generally parts of weak factorization systems.

FiXme Note: We have an operation $[\mathcal{E}, \mathbf{Set}] \times [\mathcal{E}, \mathbf{Set}] \to Set$ that generalizes lifting. Figure out if its pullback operation sends left and right maps to split epis.

Definition 2.3. Let $\mathcal{E}$ be a category with a class of maps $\mathbf{S}$. We say that $\mathbf{S}$ is closed under:

- *left cancellation* if $f \in \mathbf{S}$ for composable f and g with $g, gf \in \mathbf{S}$,
- *right cancellation* if $g \in \mathbf{S}$ for composable f and g with $f, gf \in \mathbf{S}$,
- *2-out-of-3* if $\mathbf{S}$ is closed under left and right cancellation and composition.

We say that $\mathbf{S}$ has a closure property *among* a larger class $\mathbf{S}'$ if it has that closure property in the wide subcategory of maps in $\mathbf{S}'$.

Remark 2.4. The closure properties considered in Definition 2.3 descend to slices of $\mathcal{E}$. Here, we let the corresponding class of maps in the slice be created by the forgetful functor. We will use this fact implicitly in the rest of the article.

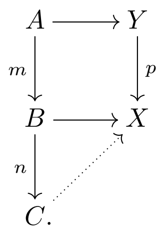
We have a converse to this observation. If one of the closure properties of Definition 2.3 holds in all slices of $\mathcal{E}$, then it holds in $\mathcal{E}$. To see this, note that a situation of maps $f \colon A \to B$ and $g \colon B \to C$ in $\mathcal{E}$ can be considered in the slice over C. This observation can sometimes be useful to reduce a claim of a cancellation property to the case where the ambient category has a terminal object.

A dual remark holds for coslices.

Lemma 2.5. *Let $\mathcal{E}$ be a category with a weak factorization system $(\mathbf{L}, \mathbf{R})$.*

(i) *Assume that the left maps are given by left lifting against right maps with right target. Then left maps are closed under left cancellation.*
(ii) *Assume that the right maps are given by right lifting against left maps with left source. Then right maps are closed under right cancellation.*

Proof. The assertions are dual, so it will suffice to prove part (i). Consider maps $m\colon A \to B$ and $n\colon B \to C$ such that n and nm are cofibrations. Consider a lifting problem of m against a right map $p\colon Y \to X$ with right target. We take a lift of n against X:



This reduces the lifting problem to one of nm against p, which we can solve since nm is a left map. This shows that m is a left map. □

Corollary 2.6. *Let $\mathcal{E}$ be a category with a weak factorization system* $(\mathbf{L}, \mathbf{R})$.
(i) *Assume that every object is right. Then left maps are closed under left cancellation.*
(ii) *Assume that every object is left. Then right maps are closed under right cancellation.* □

If the ambient category contains an initial and terminal objects, then a converse to Lemma 2.5 holds.

Lemma 2.7. *Let $\mathcal{E}$ be a category with a weak factorization system* $(\mathbf{L}, \mathbf{R})$.
(i) *Let $\mathcal{E}$ have a terminal objects. Assume that left maps are closed under left cancellation. Then left maps are given by the left lifting against right maps with right target.*
(ii) *Let $\mathcal{E}$ have an initial objects. Assume that right maps are closed under right cancellation. Then right maps are given by the right lifting against left maps with left target.*

Proof. The assertions are dual, so it will suffice to prove part (i). Let $A \to B$ be a map left lifting against right maps with right target. We will show that $A \to B$ is a left map.

Using that $\mathcal{E}$ has a terminal object, we factorize $B \to 1$ as a left map $B \to B'$ followed by a right map $B' \to 1$. Note that the composite $A \to B \to B'$ lifts against right maps with right target. By the cancellation assumption, it will suffice to show that the composite $A \to B'$ is a left map. This follows from the retract argument: the map $A \to B'$ lifts against its last factor of its (left, right)-factorization, hence is a retract of its first factor. □

3. Model categories and model setups

Definition 3.1. A *model setup* is a category $\mathcal{E}$ with the following data.
(i) We have a weak factorization system $(\mathbf{C}, \mathbf{TF})$ of *cofibrations* and *trivial fibrations.*
(ii) We have a weak factorization system $(\mathbf{TC}, \mathbf{F})$ of *trivial cofibrations* and *fibrations.*
(iii) Every trivial (co)fibration is a (co)fibration.
(iv) $\mathcal{E}$ has pushouts along cofibrations and pullbacks along fibrations.

Note that the notion of model setup is self-dual. The two variations in condition (iii) are equivalent. An object is called (trivially) cofibrant if it belongs to $\mathbf{C}$ ($\mathbf{TC}$) and (trivially) fibrant if it belongs to $\mathbf{F}$ ($\mathbf{TF}$). Given a model setup $\mathcal{E}$, there slices and coslices of $\mathcal{E}$ inherit the structure of a model setup with classes of maps created by the forgetful functor.

Definition 3.2. A *model structure* on a category $\mathcal{E}$ is given by the following.

(i) $\mathcal{E}$ has classes of maps: *weak equivalences* $\mathbf{W}$, *cofibrations* $\mathbf{C}$, *fibrations* $\mathbf{F}$.
(ii) $\mathcal{E}$ has pushouts along cofibrations and pullbacks along fibrations.
(iii) The pairs $(\mathbf{C}, \mathbf{F} \cap \mathbf{W})$ and $(\mathbf{C} \cap \mathbf{W}, \mathbf{F})$ form weak factorization systems.
(iv) The class $\mathbf{W}$ is closed under 2-out-of-3.

We refer to $\mathcal{E}$ as a *model category*.

The notion of model structure is self-dual. Given a model category $\mathcal{E}$, the slices and coslices of $\mathcal{E}$ inherit a model structure with classes of maps created by the forgetful functor.

Every model category has an underlying model setup given by $\mathbf{TC} = \mathbf{C} \cap \mathbf{W}$ and $\mathbf{TF} = \mathbf{F} \cap \mathbf{W}$. By 2-out-of-3, the weak equivalences are determined as those maps factoring as a trivial cofibration followed by a trivial fibration. Thus, every model setup extends to a model category in at most one way; we then say that it forms a model category (with the above choice of weak equivalences). This is characterized by the following standard recognition criterion.

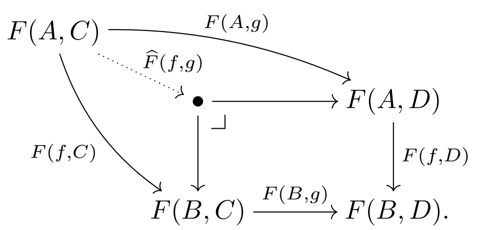
Lemma 3.3. *Let $\mathcal{E}$ be a model setup. Let $\mathbf{W}$ be the class of maps factoring as a trivial cofibration followed by a trivial fibration. Then $\mathbf{TC} = \mathbf{C} \cap \mathbf{W}$ and $\mathbf{TF} = \mathbf{F} \cap \mathbf{W}$. In particular, $\mathcal{E}$ forms to a model category exactly if $\mathbf{W}$ is closed under 2-out-of-3.*

Proof. By duality, it suffices to show $\mathbf{TC} = \mathbf{C} \cap \mathbf{W}$. The forward inclusion holds by definition of $\mathbf{W}$. The reverse inclusions follow from the retract argument: a cofibration that factors as a trivial cofibration followed by a trivial fibration lifts against its second factor, hence is a retract of its first factor. □

4. Leibniz constructions

We recall the theory of Leibniz constructions from [RV14]. We deviate slightly in our presentation from that source. For example, we do not use blanket assumptions of (co)continuity. Instead, we treat the Leibniz construction as a partial operation. We also focus on the pullback Leibniz construction and treat the pushout variant as its dual. This is due to the profunctorial point of view we adopt in the rest of our development: homming out of a pushout Leibniz construction (as well as homming into a pullback Leibniz construction) becomes a *pullback* of hom-sets.

Let $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a partial two-variable functor. Its *pullback Leibniz construction* $\widehat{F} : \mathcal{C}^{\to} \times \mathcal{D}^{\to} \to \mathcal{E}^{\to}$ is the partial functor sending a map $f : A \to B$ in $\mathcal{C}$ and a map $g : C \to D$ in $\mathcal{D}$ to the pullback corner map $\widehat{F}(f, g)$ in the diagram



(4.1)

The object $\widehat{F}(f, g)$ exists when all four applications of the functor F and the pullback in $\mathcal{E}$ in the above diagram exist. In particular, $\widehat{F}$ is total if F is total and $\mathcal{E}$ has pullbacks.

The *pushout Leibniz construction* is obtained by dualizing the above construction, treating F as a partial two-variable functor $F\colon \mathcal{C}^{\to} \times \mathcal{D}^{\to} \to \mathcal{E}^{\to}$. This uses a pushout in $\mathcal{E}$ instead of a pullback.

If F is a particular two-variable functor such as composition in a bicategory or functor application, we simply call $\widehat{F}$ the *pullback composition*, *pullback functor application*, and so on. The same convention applies to the pushout Leibniz construction. The namesake is the *pushout product* considered in a Cartesian monoidal category. Both pullback and pushout Leibniz construction will be denoted by putting a hat on the two-variable functor in question, often an infix operator. It will always be made clear which of the two we mean.

There are n-ary analogues of the Leibniz construction. In particular, the nullary pullback Leibniz construction of an object $X \in \mathcal{E}$ is the arrow $X \to 1$ in $\mathcal{E}$, assuming that the terminal object exists. The highest arity appearing in this development is $n = 3$.

The Leibniz construction enjoys many beautiful properties. Let us focus first on the pullback Leibniz construction $\widehat{F}$ defined in (4.1):

- The partial functor $\widehat{F}$, lying over F via codomain functors on both sides, preserves Cartesian morphisms.
- Argumentwise properties of F often lift to $\widehat{F}$. For example, this is the case for preservation of limits.
- We have that $\widehat{F}(f, g)$ is invertible if f or g are invertible.
- Composition in each argument is preserved up to pullback. For example, given $f\colon A \to B$ in $\mathcal{C}$ and $g\colon C \to D$ and $h\colon D \to E$ in $\mathcal{D}$, then $\widehat{F}(f, h \circ g)$ is the composite of $\widehat{F}(f, g)$ followed by a pullback of $\widehat{F}(f, h)$ (assuming that these Leibniz construction applications exist).

Leibniz constructions also interact nicely with each other. Natural isomorphisms between composition trees of multi-variable functors lift to natural isomorphisms between the corresponding composition trees of pullback Leibniz constructions, assuming that all involved functors preserve pullbacks in those argument positions functors of positive arity appear in and terminal objects in those argument positions functors of nullary arity appear in.

A prototypical example is the associativity isomorphism $(H \circ G) \circ F \simeq H \circ (G \circ F)$ in a bicategory $\mathcal{E}$ natural in $F \in \mathcal{E}(A, B)$, $G \in \mathcal{E}(B, C)$, and $H \in \mathcal{E}(C, D)$. Assuming that composition in $\mathcal{E}$ preserves pullbacks in each argument, this turns into an associativity isomorphism $(w \mathbin{\widehat{\circ}} v) \mathbin{\widehat{\circ}} u \simeq w \mathbin{\widehat{\circ}} (v \mathbin{\widehat{\circ}} u)$ for pullback composition natural in $u \in \mathcal{E}(A, B)^{\to}$, $v \in \mathcal{E}(B, C)^{\to}$, and $w \in \mathcal{E}(C, D)^{\to}$. More generally, we obtain a pullback Leibniz bicategory with the same objects as $\mathcal{E}$, but hom-categories replaced by arrow categories. We have a forgetful bifunctor back to $\mathcal{E}$ that is the identity on objects and the codomain functor on hom-categories.

These interaction properties are well-established in the mathematical practice of the Leibniz construction. Therefore, we do not prove them here. We will refer to specific properties whenever they are used.

5. Profunctors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The category $\mathbf{Prof}(\mathcal{C}, \mathcal{D})$ of *profunctors* from $\mathcal{C}$ to $\mathcal{D}$ is the functor category $[\mathcal{D}^{\mathrm{op}} \times \mathcal{C}, \mathbf{Set}]$.

Left and right evaluation. We introduce infix operators for evaluation of a profunctor $H \in \mathbf{Prof}(\mathcal{C}, \mathcal{D})$ on either side:

- given $X \in \mathcal{C}$, the *right evaluation* $H \circ_{\mathsf{r}} X \in \mathbf{Psh}(\mathcal{C})$ is $H(-, X)$,
- given $Y \in \mathcal{D}$, the *left evaluation* $Y \circ_{\mathsf{l}} H \in \mathbf{coPsh}(\mathcal{D})$ is $H(Y, -)$.

Left and right evaluation form two-variable functors

$$- \circ_{\mathsf{r}} -\colon \mathbf{Prof}(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \to \mathbf{Psh}(\mathcal{D}),$$
$$- \circ_{\mathsf{l}} -\colon \mathcal{D}^{\mathrm{op}} \times \mathbf{Prof}(\mathcal{C}, \mathcal{D}) \to \mathbf{coPsh}(\mathcal{C}).$$

We write $- \mathbin{\widehat{\circ}_{\mathsf{r}}} -$ and $- \mathbin{\widehat{\circ}_{\mathsf{l}}} -$ for pullback left and right evaluation, respectively.

We have isomorphisms $\mathbf{Psh}(\mathcal{E}) \simeq \mathbf{Prof}(1, \mathcal{E})$, $\mathbf{coPsh}(\mathcal{E}) \simeq \mathbf{Prof}(\mathcal{E}, 1)$, and $\mathbf{Set} \simeq \mathbf{Prof}(1, 1)$. We use these isomorphisms to implicitly treat presheaves as profunctors with source 1, copresheaves as profunctors with target 1, and sets as proendofunctors on 1. With this convention, it makes sense to use right evaluation also for evaluation of presheaves and left evaluation also for evaluation of copresheaves. For a profunctor $H \in \mathbf{Prof}(\mathcal{C}, \mathcal{D})$ with $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, this allows us to write $H(X, Y)$ as the *two-sided evaluation* $Y \circ_{\mathsf{l}} H \circ_{\mathsf{r}} X$. Since the order of left and right evaluation does not matter, we omit parentheses. This interchange law lifts to an interchange isomorphism for pullback left and right evaluation that factors via a ternary Leibniz construction we call *pullback two-sided evaluation*.

Bicategory of profunctors. We write **Prof** for the bicategory of categories and profunctors. Its objects are categories and the hom-category from $\mathcal{C}$ to $\mathcal{D}$ is given by $\mathbf{Prof}(\mathcal{C}, \mathcal{D})$. The identity on $\mathcal{C}$ is given by $\mathcal{C}(-, -) \in \mathbf{Prof}(\mathcal{C}, \mathcal{C})$. The composite of $H \in \mathbf{Prof}(\mathcal{C}, \mathcal{D})$ and $K \in \mathbf{Prof}(\mathcal{D}, \mathcal{E})$ is given by the coend

$$(K \circ H)(Z, X) = \int^{Y \in \mathcal{D}} H(Y, X) \times K(Z, Y).$$

The neutrality isomorphisms are given by the co-Yonedal lemma. The associativity isomorphism is given by the Fubini law for coends.

The pullback Leibniz construction turns **Prof** into a bicategory with the same objects, but hom-categories replaced by arrow categories. The identity on $\mathcal{C}$ is the map $\mathcal{C}(-, -) \to 1$. The two-variable functor of composition is denoted $- \widehat{\circ} -$, given by pullback composition of profunctors. Rarely, we will also use the bicategorical structure given instead by pushout Leibniz construction. Its identity on $\mathcal{C}$ is the map $0 \to \mathcal{C}(-, -)$. We write $- \widehat{\circ}' -$ for pushout composition of profunctors.

Remark 5.1. Left and right evaluation of profunctors may be regarded as special cases of profunctor composition. Indeed, by the co-Yoneda lemma, we have isomorphisms $H \circ_{\mathsf{r}} X \simeq H \circ \mathcal{C}(-, X)$ natural in $H \in \mathbf{Prof}(\mathcal{C}, \mathcal{D})$ and $X \in \mathcal{C}$ as well as $X \circ_{\mathsf{l}} H \simeq \mathcal{D}(Y, -) \circ H$ natural in $H \in \mathbf{Prof}(\mathcal{C}, \mathcal{D})$ and $Y \in \mathcal{D}$. This is the reason of our choice of notation for left and right evaluation, suggestive of composition. Note that the irrelevance of the order of left and right application in the two-sided evaluation $Y \circ_{\mathsf{l}} H \circ_{\mathsf{r}} X$ is reflected in the associativity isomorphism

$$(\mathcal{D}(Y, -) \circ H) \circ \mathcal{C}(-, X) \simeq \mathcal{D}(Y, -) \circ (H \circ \mathcal{C}(-, X)).$$

The above isomorphisms lift to pullback Leibniz constructions. For example, given by a map $u\colon H \to H'$ in $\mathbf{Prof}(\mathcal{C}, \mathcal{D})$ and $f\colon X \to Y$ in $\mathcal{C}$, we have $u \mathbin{\widehat{\circ}_{\mathsf{r}}} f \simeq u \mathbin{\widehat{\circ}} \mathcal{C}(-, f)$. Sometimes, it will be useful to analyze pullback left or right applications in the wider context of pullback composition.

Left and right application. Let $H \in \mathbf{Prof}(\mathcal{C}, \mathcal{D})$ be a profunctor. We introduce infix operator notation for (co)representing objects where they exist.

- Given $X \in \mathcal{C}$, if the presheaf $H \circ_{\mathsf{r}} X = H(-, X)$ is representable, the *right application* $H \odot_{\mathsf{r}} X \in \mathcal{D}$ is its representing object.

- Given $Y \in \mathcal{D}$, if copresheaf $Y \circ_{\mathsf{l}} H = H(Y, -)$ is corepresentable, the *left application* $H \odot_{\mathsf{l}} Y \in \mathcal{C}$ is its representing object.

If all right applications of H exist, we obtain a functor $H \odot_{\mathsf{r}} - \colon \mathcal{C} \to \mathcal{D}$ *representing* H. Dually, if all left applications of H exist, we obtain a functor $H \odot_{\mathsf{l}} - \colon \mathcal{D} \to \mathcal{C}$ *corepresenting* H. We call H *birepresentable* if it is both representable and corepresentable. In that case, H presents an adjunction

$$\mathcal{C} \underset{H \odot_{\mathsf{r}} -}{\overset{H \odot_{\mathsf{l}} -}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{D}.$$

More generally, we treat left and right application as partial two-variable functors

$$- \odot_{\mathsf{l}} - \colon \mathbf{Prof}(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \to \mathcal{D},$$
$$- \odot_{\mathsf{r}} - \colon \mathbf{Prof}(\mathcal{C}, \mathcal{D}) \times \mathcal{D} \to \mathcal{C}.$$

Remark 5.2. We can use left and right evaluation as stand ins for speaking about left and right application, which do not always exist. This is the spirit of Remark 2.2. By Yoneda, $\mathcal{D}$ embeds fully faithfully and continuously in $\mathbf{Psh}(\mathcal{D})$. This allows us to transfer functorial constructions for right evaluation to right application as long as we confirm that the action on objects is representable. Furthermore, limit constructions involving representables in $\mathbf{Psh}(\mathcal{D})$ agree with their corresponding constructions in $\mathcal{D}$.

A dual remark holds for left evaluation and left application, but note that there is a switch to opposite categories: left application corresponds to the opposite functor of left evaluation. Again by Yoneda, $\mathcal{C}^{\mathrm{op}}$ embeds fully faithfully and continuously in $\mathbf{coPsh}(\mathcal{D})$. In particular, limit constructions involving corepresentables in $\mathbf{coPsh}(\mathcal{D})$ agree with corresponding dual colimit constructions in $\mathcal{C}$.

Remark 5.3. Our choice of argument order of the left application infix operator is motivated as follows. We wish to mirror the argument order for the application of an actual functor from $\mathcal{D}$ to $\mathcal{C}$ rather than that of the left evaluation infix operator. Note that there is already a disconnect to left evaluation due to the switch to opposites.

Another principled option would have been to regard left application as going from $\mathcal{D}^{\mathrm{op}}$ to $\mathcal{C}^{\mathrm{op}}$ and swap the arguments of the operator $- \odot_{\mathsf{l}} -$. This would yield a non-standard notation for the left adjoint of an adjunction. The benefit of this convention would be that it presents all functors in an adjunction as right adjoints, so one would for example always speak about preservation of limits. This seems to fit nicely into the profunctorial point of view.

We write $- \widehat{\odot}_{\mathsf{l}} -$ for pushout left application and $- \widehat{\odot}_{\mathsf{r}} -$ for pullback right application, again partial two-variable functors.

Remark 5.4. By Remark 5.2, pullback right application, whenever it exists, agrees under the Yoneda embedding with pullback right evaluation. Dually, pushout left application agrees, whenever it exists, with pullback left evaluation. Again, note the switch of variance in the latter case.

Recall from Section 4 that we say that a pullback right application $u \mathbin{\widehat{\odot}_{\mathsf{r}}} g$ with $u \in \mathbf{Prof}(\mathcal{C}, \mathcal{D})^{\to}$ and $g \in \mathcal{C}^{\to}$ exists if four individual right applications and a pullback exist in $\mathcal{D}$. In this situation, since right application tracks right evaluation under Yoneda, we could more generally say that a pullback right application exists if the corresponding pullback right evaluation has representable source and target. A dual remark applies to pushout left application.

6. Functorial notion of homotopy

Definition 6.1. A *functorial notion of homotopy* I in a category $\mathcal{E}$ is a factorization

$$\begin{array}{ccc} \mathcal{E}(-,-) & \xrightarrow{\langle \mathrm{id},\mathrm{id}\rangle} & \mathcal{E}(-,-)^{\{0,1\}} \\ & {\scriptstyle \epsilon}\searrow \quad \nearrow{\scriptstyle \langle \delta_0,\delta_1\rangle} & \\ & I & \end{array}$$

of the diagonal of the hom-functor $\mathcal{E}(-,-)$ in proendofunctors on $\mathcal{E}$. We call δ_0, δ_1 the *endpoint maps*, $\langle \delta_0, \delta_1 \rangle$ the *boundary map*, and ϵ the *constant map*.

Throughout this paper, our notation for a functorial notion of homotopy and its associated data will always be the one established in Definition 6.1. This saves us from having to introduce local names.

Remark 6.2. The concept of functorial notion of homotopy is self-dual. That is, a functorial notion of homotopy on a category $\mathcal{E}$ is the same as a functorial notion of homotopy on $\mathcal{E}^{\mathrm{op}}$.

Remark 6.3. Let $\Delta^{\leq 1}$ denote the restriction of the simplex category to objects $[0]$ and $[1]$. It is generated by maps $d_1, d_0 \colon [0] \to [1]$ and $s \colon [1] \to [0]$ with $sd_1 = sd_0 = \mathrm{id}_{[0]}$. A functorial notion of homotopy in $\mathcal{E}$ is a functor $(\Delta^{\leq 1})^{\mathrm{op}} \to \mathbf{Prof}(\mathcal{E}, \mathcal{E})$ that sends the terminal object $[0]$ of $\Delta^{\leq 1}$ to the neutral object $\mathcal{E}(-,-)$.

The functorial notion of homotopy I in $\mathcal{E}$ is called *representable* (*corepresentable*, *birepresentable*) if the underlying profunctor I is so. A corepresentable functorial notion of homotopy is the same as a *functorial cylinder*, i.e. a functorial factorization of diagonals in $\mathcal{E}$. The functorial cylinder corresponding to I is given in our notation by

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & X \\ \downarrow{\scriptstyle \delta_0 \odot_{\mathsf{l}} A} & & \\ I \odot_{\mathsf{l}} A & \xrightarrow{\epsilon \odot_{\mathsf{l}} A} & X, \\ \uparrow{\scriptstyle \delta_1 \odot_{\mathsf{l}} A} & & \\ A & \xrightarrow{\mathrm{id}} & X \end{array} \tag{6.1}$$

functorially in $A \in \mathcal{E}$. Dually, a representable functorial notion of homotopy is the same as a *functorial cocylinder*, i.e. a functorial factorization of codiagonals in $\mathcal{E}$. The functorial cocylinder corresponding to I is given in our notation by

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ & & \uparrow{\scriptstyle \delta_0 \odot_{\mathsf{r}} X} \\ X & \xrightarrow{\epsilon \odot_{\mathsf{r}} X} & I \odot_{\mathsf{r}} X \\ & & \downarrow{\scriptstyle \delta_1 \odot_{\mathsf{r}} X} \\ X & \xrightarrow{\mathrm{id}} & X, \end{array} \tag{6.2}$$

functorially in $X \in \mathcal{E}$. A birepresentable functorial notion of homotopy is the same as a functorial cylinder left adjoint to a functorial cocylinder (with endpoint and constant maps corresponding to each other under transposition). A benefit of our setup is that it gives a uniform notation for the structure of this adjoint pair of functorial cylinder and cocylinder; we do not have to choose separate names for each side.

Remark 6.4. The notion of functorial cylinder makes sense in the absence of binary products in $\mathcal{E}$. This is clear from the diagram (6.1), which does not use products, but data indexed by $(\Delta_{\leq 1})^{\mathrm{op}}$ as in Remark 6.3. However, also the definition in terms of factorizations of diagonals in $\mathcal{E}$ can be made precise without limits by working in presheaves over $\mathcal{E}$ as in Remark 2.2: one asks for a functorial factorization of the diagonal of representables in presheaves (where binary products always exist) and requires the middle object to be representable. Indeed, without the representability condition this gives exactly a functorial notion of homotopy. A dual remark applies to the notion of functorial cocylinder.

Remark 6.5 (Stability under (co)slicing)**.** Let $\mathcal{E}$ be a category with a functorial notion of homotopy I, and let X be an object of $\mathcal{E}$. Then we have an induced functorial notion of homotopy $I_{\downarrow X}$ in $\mathcal{E} \downarrow X$. For this, recall the equivalence between $\mathbf{Psh}(\mathcal{E} \downarrow X)$ and $\mathbf{Psh}(\mathcal{E}) \downarrow \mathcal{E}(-, X)$. Functorially in an object (A, f) of $\mathcal{E} \downarrow X$, we have a commuting diagram

$$\begin{array}{ccccc} \mathcal{E}(-,A) & \xrightarrow{\epsilon} & I(-,A) & \xrightarrow{\langle \delta_0, \delta_1 \rangle} & \mathcal{E}(-,A)^{\{0,1\}} \\ \downarrow{\scriptstyle \mathcal{E}(-,f)} & & \downarrow{\scriptstyle I(-,f)} & & \downarrow{\scriptstyle \mathcal{E}(-,f)^{\{0,1\}}} \\ \mathcal{E}(-,X) & \xrightarrow{\epsilon} & I(-,X) & \xrightarrow{\langle \delta_0, \delta_1 \rangle} & \mathcal{E}(-,X)^{\{0,1\}}. \end{array}$$

Pulling back the top row to $\mathcal{E}(-, X)$ gives the factorization of the diagonal of the object $\mathcal{E}(-, A)$ in the slice over $\mathcal{E}(-, X)$. We see that if $\mathcal{E}$ has pullbacks and I is representable, then so is $I_{\downarrow X}$. Explicitly, $I_{\downarrow X}$ is given functorially in $(A, f) \in (\mathcal{E} \downarrow X)^{\mathrm{op}}$ and $(B, g) \in \mathcal{E} \downarrow X$ by the pullback

$$\begin{array}{ccccc} I_{\downarrow X}((A,f),(B,f)) & \longrightarrow & & & I(A,B) \\ \downarrow \;\lrcorner & & & & \downarrow{\scriptstyle I(A,g)} \\ 1 & \xrightarrow{f} & \mathcal{E}(A,X) & \xrightarrow{\epsilon} & I(A,X). \end{array}$$

If $I(A, -)$ is corepresented by $I \odot_{\mathrm{l}} A$, then $I_{\downarrow X}((A, f), -)$ is corepresented by $(I \odot_{\mathrm{l}} A, f \circ (\epsilon \odot_{\mathrm{l}} A))$. So if I is corepresentable, then so is $I_{\downarrow X}$, with no assumptions on colimits in $\mathcal{E}$. A dual remark holds for coslicing.

In terms of Remark 6.3, stability under (co)slicing make sense more generally for an arbitrary category $\mathcal{C}$ with a terminal object 1 and structure consisting of a functor $K : \mathcal{C}^{\mathrm{op}} \to \mathbf{Prof}(\mathcal{E}, \mathcal{E})$ sending 1 to the neutral object $\mathcal{E}(-, -)$. For the slice over X, we take $K_{\downarrow X}(S)((A, f), (B, g))$ as the pullback of $K(S)(A, B)$ along the composite $1 \to \mathcal{E}(A, X) \to K(A, X)$. If K is valued in (co)representable profunctors (and $\mathcal{E}$ has pullbacks for the case of representability), then so is $K_{\downarrow X}$.

Let $\mathcal{E}$ be a category with a functorial notion of homotopy I. A *homotopy* $u : f_0 \sim f_1$ between maps $f_0, f_1 : A \to B$ is $u \in I(A, B)$ with $\delta_0(u) = f_0$ and $\delta_1(u) = f_1$; the maps f_0 and f_1 are its *endpoints*. The *constant homotopy* on $f : A \to B$ is $\epsilon(f) : f \sim f$. Since I is functorial, homotopies can be whiskered strictly on both sides, interchangably. We overload composition to also denote this whiskering, for example writing $gu : gf_0 \sim gf_1$ for $I(A, g)(u)$ where u is as above and $g : B \to C$.

A map $f : A \to B$ is a 0*-sided homotopy equivalence* if we have $g : B \to A$ with $u : gf \sim \mathrm{id}_A$ and $v : gf \sim \mathrm{id}_A$; the 1-sided version uses homotopies $u : \mathrm{id}_A \sim gf$ and $v : \mathrm{id}_B \sim fg$. Such a homotopy equivalence is called *strong* if $fu = vf$. It is a *deformation retract* if $u = \epsilon_A$ and the *dual of a deformation retract* if $v = \epsilon_B$.

Remark 6.6. Under the self-duality of functorial notions of homotopy observed in Remark 6.2, all of the above notions are self-dual, except for the notion of deformation retracts, which dualizes as indicated.

We can equivalently talk about the above notions using left evaluation (in copresheaves over $\mathcal{E}$) or right evaluation (in presheaves over $\mathcal{E}$). If I is corepresentable, this specializes to the standard definitions in terms of the associated functorial cylinder (6.1). If I is representable, we obtain dual expressions in terms of the associated functorial cocylinder (6.2). We will use all of these versions interchangably.

We next give an abstract characterization of strong homotopy equivalences generalizing [GS17, Lemma 4.3]. This needs a little bit of setup. Consider the diagram

$$\begin{array}{ccc} I & \xrightarrow{\delta_0} & \mathcal{E}(-,-) \\ {\scriptstyle \delta_1}\big\downarrow & & \big\downarrow{\scriptstyle !} \\ \mathcal{E}(-,-) & \xrightarrow{!} & 1 \end{array}$$

in endoprofunctors on $\mathcal{E}$. Read horizontally or vertically, we obtain a map $\theta_i \colon \delta_i \to \delta_i \to !_{\mathcal{E}(-,-)\to 1}$ of arrows for $i = 0, 1$.

For $X \in \mathcal{X}$, we have an adjunction.

$$\mathcal{E} \underset{\mathcal{E}(X,-)}{\overset{\mathcal{E}(-,X)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} 1 \tag{6.3}$$

in the bicategory of profunctors. The unit map is

$$1 \xrightarrow{\mathrm{id}_X} \mathcal{E}(X,X) \xrightarrow{\simeq} \mathcal{E}(X,-) \circ \mathcal{E}(-,X)$$

and the counit map is

$$\mathcal{E}(-,X) \circ \mathcal{E}(X,-) \xrightarrow{\simeq} \mathcal{E}(X,-) \times \mathcal{E}(-,X) \xrightarrow{-\circ-} \mathcal{E}(-,-).$$

In the below statement, recall from Section 5 that $- \mathbin{\widehat{\circ}} -$ and $- \mathbin{\widehat{\circ}'} -$ denote pullback and pushout composition of profunctors, respectively. The bottom arrows in the diagrams (6.4) and (6.5) are induced by the counit and unit maps of the adjunction (6.3), respectively.

Lemma 6.7. *Let $\mathcal{E}$ be a category with a functorial notion of homotopy I. The following are equivalent for a map f and $i = 0, 1$.*

(i) *f is an i-sided strong homotopy equivalence,*

(ii) *in $\mathbf{Prof}(\mathcal{E}, \mathcal{E})^{\to}$, we have a lift*

$$\begin{array}{ccc} & & \delta_i \\ & \nearrow & \big\downarrow{\scriptstyle \theta_i} \\ \mathcal{E}(-,f) \mathbin{\widehat{\circ}'} \mathcal{E}(f,-) & \longrightarrow & !_{\mathcal{E}(-,-)\to 1}, \end{array} \tag{6.4}$$

(iii) *in $\mathbf{Set}^{\to}$, we have a lift*

$$\begin{array}{ccccc} & & & & f \mathbin{\widehat{\circ}_{\mathsf{l}}} \delta_i \mathbin{\widehat{\circ}_{\mathsf{r}}} f \\ & & & \nearrow & \big\downarrow{\scriptstyle f \mathbin{\widehat{\circ}_{\mathsf{l}}} \theta_i \mathbin{\widehat{\circ}_{\mathsf{r}}} f} \\ !_{\emptyset\to 1} & \longrightarrow & \mathcal{E}(f,-) \mathbin{\widehat{\circ}} \mathcal{E}(-,f) & \xrightarrow{\simeq} & f \mathbin{\widehat{\circ}_{\mathsf{l}}} !_{\mathcal{E}(-,-)\to 1} \mathbin{\widehat{\circ}_{\mathsf{r}}} f, \end{array} \tag{6.5}$$

(iv) *in $\mathbf{coPsh}(\mathcal{E})^{\to}$, the map $f \mathbin{\widehat{\circ}_{\mathsf{l}}} \theta_i \colon f \mathbin{\widehat{\circ}_{\mathsf{l}}} \delta_i \to \mathcal{E}(f,-)$ is split epi,*

(v) *in $\mathbf{Psh}(\mathcal{E})^{\to}$, the map $\theta_i \mathbin{\widehat{\circ}_{\mathsf{r}}} f \colon \delta_i \mathbin{\widehat{\circ}_{\mathsf{r}}} f \to \mathcal{E}(-,f)$ is split epi.*

Proof. We begin by showing that (i) and (iii) are equivalent. Let $f \colon X \to Y$. We evaluate the two-sided pullback evaluations on the right of (6.5).

- The target of $f \mathbin{\widehat{\circ}_{\mathsf{l}}} \delta_i \mathbin{\widehat{\circ}_{\mathsf{r}}} f$ computes to the set of triples (u, g, v) of $u \in I(X, X)$, $g \in \mathcal{E}(Y, X)$, and $v \in I(Y, Y)$ satisfying $\delta_i(u) = gf$, $I(X, f)(u) = I(Y, f)(v)$, and $\delta_i(v) = fg$.
- The target of $f \mathbin{\widehat{\circ}_{\mathsf{l}}} !_{\mathcal{E}(-,-)\to 1} \mathbin{\widehat{\circ}_{\mathsf{r}}} f$ computes to the set of pairs (x, y) of $x \in \mathcal{E}(X, X)$ and $y \in \mathcal{E}(Y, Y)$ satisfying $fx = yf$.

Under these identifications, the right map in (6.5) acts on codomains by sending a triple (u, g, v) to the pair $(\delta_{1-i}(u), \delta_{1-i}(v))$. The bottom map in (6.5) selects the pair (x, y) given by $x = \mathrm{id}_X$ and $y = \mathrm{id}_Y$. A preimage of this pair under the action of the right map in (6.5) on codomains is precisely the data making f a strong homotopy equivalence.

For the remaining conditions, we have a square of equivalences

$$\begin{array}{ccc} \text{(ii)} & \Longleftrightarrow & \text{(iv)} \\ \Updownarrow & & \Updownarrow \\ \text{(v)} & \Longleftrightarrow & \text{(iii)}. \end{array} \tag{6.6}$$

Each equivalence is explained by the adjunction (6.3).

In detail, let $L \in \mathbf{Prof}(\mathcal{A}, \mathcal{B})$ and $R \in \mathbf{Prof}(\mathcal{B}, \mathcal{A})$ be adjoint profunctors. Then for any category $\mathcal{C}$, postcomposition induces an adjunction

$$\mathbf{Prof}(\mathcal{C}, \mathcal{A}) \underset{R \circ -}{\overset{L \circ -}{\rightleftarrows}} \mathbf{Prof}(\mathcal{C}, \mathcal{B}), \quad \bot$$

Given such adjoint profunctors $L \dashv R$ and $L' \dashv R'$ and maps $l \colon L' \to L$, $r \colon R \to R'$ corresponding to each other under transposition, we obtain a lifted adjunction between the pushout and pullback Leibniz constructions of postcomposition:

$$\mathbf{Prof}(\mathcal{C}, \mathcal{A})^{\to} \underset{r \mathbin{\widehat{\circ}} -}{\overset{l \mathbin{\widehat{\circ}'} -}{\rightleftarrows}} \mathbf{Prof}(\mathcal{C}, \mathcal{B})^{\to}. \quad \bot$$

We instantiate this adjunction to

$$l = \mathcal{E}(-, f) \colon \mathcal{E}(-, X) \to \mathcal{E}(-, Y),$$
$$r = \mathcal{E}(f, -) \colon \mathcal{E}(Y, -) \to \mathcal{E}(X, -).$$

Transposition with respect to this adjunction explains the horizontal equivalences in (6.6). Here, we use Remark 5.1 to mediate between pullback composition and pullback (left and right) evaluation. The bottom maps in the diagrams (6.4) and (6.5) become invertible under transposition. Thus, the lifts in (ii) and (iii) become the sections in (iv) and (v), respectively.

Dually, transposition with respect to the Leibniz adjunction for precomposition explains the vertical equivalences in (6.6). □

Under (co)representability assumptions, the last two conditions of the above statement simplify as follows. This procedure is explained in Remark 5.2.

Corollary 6.8. *Let $\mathcal{E}$ be a category with a functorial notion of homotopy I. The following are equivalent for a map f and $i = 0, 1$:*

(i) *f is an i-sided strong homotopy equivalence,*

if the pushout left application $\delta_i \mathbin{\widehat{\odot}_{\mathsf{l}}} f$ exists:

(ii) *the map* $\theta_i \mathbin{\widehat{\odot_{\mathsf{l}}}} f \colon f \to \delta_i \mathbin{\widehat{\odot_{\mathsf{l}}}} f$ *in* $\mathcal{E}^{\to}$ *is split mono.*

if the pullback right application $\delta_i \mathbin{\widehat{\odot_{\mathsf{r}}}} f$ *exists:*

(iii) *the map* $\theta_i \mathbin{\widehat{\odot_{\mathsf{r}}}} f \colon \delta_i \mathbin{\widehat{\odot_{\mathsf{r}}}} f \to f$ *in* $\mathcal{E}^{\to}$ *is split epi.* □

Definition 6.9. Let $\mathcal{E}$ be a category with a functorial notion of homotopy I. We define the following partial functorial factorizations.

(i) The 0-*sided mapping cylinder factorization* of a map $f \colon A \to B$ is

$$A \xrightarrow{\iota_1 \circ (\delta_1 \odot_{\mathsf{l}} A)} B \textstyle\coprod_A I \odot_{\mathsf{l}} A \xrightarrow{[\mathrm{id}, f \circ (\epsilon \odot_{\mathsf{l}} A)]} B$$

if the left application and pushout exist.

(ii) The 0-*sided mapping cocylinder factorization* of a map $g \colon Y \to X$ is

$$Y \xrightarrow{\langle \mathrm{id}, (\epsilon \odot_{\mathsf{r}} X) \circ g \rangle} Y \times_Y I \odot_{\mathsf{r}} X \xrightarrow{(\delta_1 \odot_{\mathsf{r}} X) \circ \pi_1} X$$

if the right application and pullback exist.

The 1-*sided mapping (co)cylinder factorization* is defined dually, i.e. with indices 0 and 1 swapped.

Here and later, the order of the arguments of the pushout or pullback indicates with respect to which endpoint map δ_0, δ_1 it is taken with. For example, the pushout in part (i) of Definition 6.9 is taken with respect to $\delta_0 \colon A \to I \odot_{\mathsf{l}} A$ because A notationally appears on the left side of $I \odot_{\mathsf{l}} A$.

In many of our arguments, the sidedness of the mapping (co)cylinder factorization does not matter and can be chosen arbitrary. When this is the case, we will omit the sideness. This applies equally to other notions with a sidedness aspect.

Remark 6.10. Following Remark 2.2, the mapping cylinder factorization of a map $f \colon A \to B$ in $\mathcal{E}$ can also be considered when the relevant left application and pushout in $\mathcal{E}$ does not exist. In that case, the factorization in part (i) of Definition 6.9 has to be regarded as a factorization of $\mathcal{E}(f, -)$ in $\mathbf{coPsh}(\mathcal{E})$:

$$\mathcal{E}(B, -) \xrightarrow{\langle \mathrm{id}, \epsilon \circ \mathcal{E}(f, -) \rangle} \mathcal{E}(B, -) \times_{\mathcal{E}(A, -)} I(A, -) \xrightarrow{\delta_1 \circ \pi_1} \mathcal{E}(A, -). \tag{6.7}$$

If the mapping cylinder factorization exists, then it corepresents the above factorization. More generally, we may regard the mapping cylinder factorization to exist if the middle object above is representable.

A dual remark applies to the mapping cocylinder factorization in part (ii) of Definition 6.9. For $g \colon Y \to X$, the corresponding factorization of $\mathcal{E}(g, -)$ in $\mathbf{Psh}(\mathcal{E})$ is given by

$$\mathcal{E}(Y, -) \xrightarrow{\langle \mathrm{id}, \epsilon \circ \mathcal{E}(-, g) \rangle} \mathcal{E}(Y, -) \times_{\mathcal{E}(Y, -)} I(Y, -) \xrightarrow{\delta_1 \circ \pi_1} \mathcal{E}(X, -). \tag{6.8}$$

Remark 6.11. In the context of Definition 6.9, assume that all mapping cocylinder factorizations exist. As explained in [Gar09], any functorial factorization induces a pointed endofunctor on arrows (over the identity on codomains) and a copointed endofunctor on arrows (over the identity on domains). For $i = 0, 1$, a map $g \colon Y \to X$ is an i-sided strong deformation retract just when it is a coalgebra for the copointed endofunctor of the i-sided mapping cocylinder factorization. This means a lift as follows (depicting the case $i = 0$):

$$\begin{array}{ccc} Y & \longrightarrow & Y \times_X I \odot_{\mathsf{r}} X \\ {\scriptstyle g}\downarrow & \nearrow & \downarrow \\ X & \xrightarrow{\mathrm{id}} & X. \end{array}$$

In particular, for a map to be a strong deformation retract, it suffices for it to lift against the second factor of its mapping cocylinder factorization.

This characterization makes sense also without assuming the mapping cocylinder factorization to exist. Instead, we consider the corresponding (not necessarily representable) factorization (6.8) of $\mathcal{E}(-, g)$ in $\mathbf{Psh}(\mathcal{E})$ as per Remark 6.10. Then the structure making g an i-sided strong deformation retract is given by a lift

$$\begin{array}{ccc} \mathcal{E}(-,Y) & \longrightarrow & \mathcal{E}(-,Y) \times_{\mathcal{E}(-,X)} I(-,X) \\ {\scriptstyle \mathcal{E}(-,g)}\downarrow & \nearrow & \downarrow \\ \mathcal{E}(-,X) & \xrightarrow{\quad \mathrm{id} \quad} & \mathcal{E}(-,X). \end{array}$$

In particular, for g to be a strong deformation retract, by Yoneda and Leibniz calculus, it suffices for pullback left evaluation of the second factor of (6.8) to send g to a split surjection.

A dual remark applies to the mapping cylinder factorization.

7. Cylindrical weak factorization systems

Recall left and right evaluation of profunctors as introduced in Section 5. We write **SplitSur** for the class of split surjections of sets.

Definition 7.1. A map u in $\mathbf{Prof}(\mathcal{C}, \mathcal{D})$ is said to *relate* weak factorization systems $(\mathbf{L}_\mathcal{C}, \mathbf{R}_\mathcal{C})$ in $\mathcal{C}$ and $(\mathbf{L}_\mathcal{D}, \mathbf{R}_\mathcal{D})$ in $\mathcal{D}$ if $\mathbf{L}_\mathcal{D} \mathbin{\widehat{\circ}_{\mathsf{l}}} u \mathbin{\widehat{\circ}_{\mathsf{r}}} \mathbf{R}_\mathcal{C} \subseteq \mathbf{SplitSur}$.

In words, the pullback two-sided evaluation of u with a left map on the left and a right map on the right is a split surjection.

Remark 7.2. Definition 7.1 is self-dual. The map u in $\mathbf{Prof}(\mathcal{C}, \mathcal{D})$ respects the weak factorization systems in $\mathcal{C}$ and $\mathcal{D}$ precisely if when regarded as a map in $\mathbf{Prof}(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$, it respects the dual weak factorization systems in $\mathcal{D}^{\mathrm{op}}$ and $\mathcal{C}^{\mathrm{op}}$.

Recall from Remark 2.2 the extensions of the classes of left and right maps in a weak factorization system in $\mathcal{E}$ to $\mathbf{Psh}(\mathcal{E})$ and $\mathbf{coPsh}(\mathcal{E})$, respectively. The conditions (ii) and (iii) in the following statement target these extensions.

Lemma 7.3. *In the context of Definition 7.1, the following are equivalent:*

(i) *u relates the weak factorization systems $(\mathbf{L}_\mathcal{C}, \mathbf{R}_\mathcal{C})$ and $(\mathbf{L}_\mathcal{D}, \mathbf{R}_\mathcal{D})$,*
(ii) *pullback left evaluation $- \mathbin{\widehat{\circ}_{\mathsf{l}}} u$ preserves left maps,*
(iii) *pullback right evaluation $- \mathbin{\widehat{\circ}_{\mathsf{r}}} u$ preserves right maps,*
(iv) *in $\mathbf{Prof}(\mathcal{C}, \mathcal{D})$, we have $\mathcal{D}(-, \mathbf{L}_\mathcal{D}) \mathbin{\widehat{\circ}'} \mathcal{C}(\mathbf{R}_\mathcal{C}, -) \pitchfork u$.*

Here, the last condition says that u has the right lifting property against the pushout compositions of any map $\mathcal{E}(r, -)$ in $\mathbf{Prof}(\mathcal{C}, 1)$ with $r \in \mathbf{R}_\mathcal{C}$ followed by any map $\mathcal{E}(-, l)$ in $\mathbf{Prof}(1, \mathcal{D})$ with $l \in \mathbf{L}_\mathcal{D}$.

Proof of Lemma 7.3. The equivalence between the first three conditions is immediate from the definition of the extended classes $\mathbf{L}$ in $\mathbf{coPsh}(\mathcal{E})$ and $\mathbf{R}$ in $\mathbf{Psh}(\mathcal{E})$, respectively.

The equivalence between (i) and (iv) is an exercise in Leibniz calculus. By Remark 5.2, we may write (i) as a lifting condition $!_{0 \to 1} \pitchfork \mathcal{D}(\mathbf{L}_\mathcal{D}, -) \mathbin{\widehat{\circ}} u \mathbin{\widehat{\circ}} \mathcal{C}(-, \mathbf{R}_\mathcal{C})$ against a ternary pullback composition of profunctors. This is equivalent to (iv) using adjoint transposition as in the abstract part of the proof of Lemma 6.7. □

Under (co)representability assumptions, we obtain via Remark 5.2 the following equivalent standard conditions in terms of the associated (co)representing functor.

Corollary 7.4. *In the context of Definition 7.1, assume that u is birepresentable. Then the following are equivalent:*

(i) *u relates the weak factorization systems $(\mathbf{L}_\mathcal{C}, \mathbf{R}_\mathcal{C})$ and $(\mathbf{L}_\mathcal{D}, \mathbf{R}_\mathcal{D})$,*
(ii) *Pushout left application $u \mathbin{\widehat{\odot_\mathsf{l}}} -$ preserves left maps.*
(iii) *Pullback right application $u \mathbin{\widehat{\odot_\mathsf{r}}} -$ preserves right maps.* □

Remark 7.5. Relation of weak factorization systems is not generally closed under pullback composition of profunctors. However, given weak factorization systems on all relevant categories, we have that $v \mathbin{\widehat{\circ}} u$ relates weak factorization systems if u and v do and u is a map between representable profunctors or v is a map between corepresentable profunctors. This is a consequence of Corollary 7.4 and Lemma 7.3 and Leibniz calculus.

FiXme Note: Find counterexample.

Definition 7.6. Let $\mathcal{E}$ be a category with a functorial notion of homotopy I. A weak factorization system $(\mathbf{L}, \mathbf{R})$ in $\mathcal{E}$ is *cylindrical* if the boundary map $\langle \delta_0, \delta_1 \rangle$ respects the weak factorization system, i.e., relates it to itself.

Unfolded, this means the following. Let $l \colon A \to B$ be a left map and $r \colon Y \to X$ be a right map. Given $f_0, f_1 \colon B \to Y$ with $u \colon f_0 l \sim f_1 l$ and $v \colon r f_0 \sim r f_1$ such that $vl = ru$, there is $w \colon f_0 \sim f_1$ such that $wl = u$ and $rw = v$.

Remark 7.7 (Stability under (co)slicing)**.** Let $\mathcal{E}$ be a category with a functorial notion of homotopy I and a cylindrical weak factorization system $(\mathbf{L}, \mathbf{R})$. Given an object X, recall that the slice $\mathcal{E} \downarrow X$ inherits a functorial notion of homotopy $I_{\downarrow X}$ of Remark 6.5. The weak factorization system $(\mathbf{L}_{\downarrow X}, \mathbf{R}_{\downarrow X})$ in the slice remains cylindrical.

This can for example be seen as follows. By the description of $I_{\downarrow X}$ in Remark 6.5, For this, one checks that the pullback composition $\mathcal{E}(l, -) \mathbin{\widehat{\circ}} \langle \delta_0, \delta_1 \rangle \mathbin{\widehat{\circ}} \mathcal{E}(-, r)$ A dual remark applies to coslicing.

In a cylindrical weak factorization system, the mapping (co)cylinder factorization carries homotopical meaning. We record this in the following statement. Note that its assumptions do not guarantee the mapping (co)cylinder factorizations to exist in $\mathcal{E}$. In that case, the claims are to be interpreted as being about the corresponding factorizations in (co)presheaves as in Remark 6.10, using the extensions of the classes $\mathbf{L}$ and $\mathbf{R}$ introduced in Remark 2.2.

Lemma 7.8. *Let $\mathcal{E}$ be a category with a functorial notion of homotopy I and a cylindrical weak factorization system $(\mathbf{L}, \mathbf{R})$.*

(i) *Assume that $\mathcal{E}$ has an initial object. Given a map between left objects, the first factor of its mapping cylinder factorization is a left map.*
(ii) *Assume that $\mathcal{E}$ has a terminal object. Given a map between right objects, the last factor of its mapping cocylinder factorization is a right map.*

Proof. The assertions are dual, so it will suffice to prove part (i). We only do the case of the 0-sided mapping cylinder factorization of a map $f \colon A \to B$; the 1-sided case is dual. For clarity, we first assume that the mapping cylinder factorization exists in $\mathcal{E}$. Its first factor decomposes as follows:

$$A \longrightarrow B \amalg A \xrightarrow{\simeq} B \amalg_A (A \amalg A) \longrightarrow B \amalg_A I \odot_\mathsf{l} A.$$

The first factor is a pushout of the left map $\emptyset \to B$. The last factor is a pushout of the pushout left application of the boundary map $\langle \delta_0, \delta_1 \rangle$ to the left map $\emptyset \to A$. Hence, their composite is a left map.

In the general situation, we have to show the last factor of the factorization of $\mathcal{E}(f, -)$ in $\mathbf{coPsh}(\mathcal{E})$ from Remark 6.10 is a left map in the sense of Remark 2.2.

This factor decomposes as follows:

$$\begin{array}{ccc}
\mathcal{E}(B,-) \times_{\mathcal{E}(A,-)} (\mathcal{E}(A,-) \times \mathcal{E}(A,-)) & \longleftarrow & \mathcal{E}(B,-) \times_{\mathcal{E}(A,-)} I(A,-) \\
\big\downarrow \simeq & & \\
\mathcal{E}(B,-) \times \mathcal{E}(A,-) & \longrightarrow & \mathcal{E}(A,-).
\end{array}$$

The first factor is a pullback of the pullback left evaluation of $\langle \delta_0, \delta_1 \rangle$ at $\emptyset \to A$, hence a left map by Lemma 7.3. The last factor is a pullback of the image of $\emptyset \to B$ under the Yoneda embedding of $\mathcal{E}^{\mathrm{op}}$, hence a left map. □

Corollary 7.9. *Let $\mathcal{E}$ be a category with a functorial notion of homotopy I and a cylindrical weak factorization system* $(\mathbf{L}, \mathbf{R})$.

- (i) *Assume that $\mathcal{E}$ has a terminal object. Then left maps between right objects are strong deformation retracts.*
- (ii) *Assume that $\mathcal{E}$ has an initial object. Right maps between left objects are duals of strong deformation retracts.*

Proof. The assertions are dual, so it will suffice to prove part (i). Let f be a left map between right objects. For clarity, we first assume that its mapping cylinder factorization exists in $\mathcal{E}$. By part (ii) of Lemma 7.8, its last factor is a right map. Thus f lifts against it, making f a strong deformation retract by Remark 6.11.

In the general situation, part (ii) of Lemma 7.8 says that pullback left evaluation of the last factor of (6.8) sends left maps to split epis. In particular, it sends f to a split epi. This makes f a strong deformation retract by Remark 6.11. □

8. Cylindrical premodel categories

The following terminology is inspired by [Shu19, Definition 8.6 and Remark 8.9].

Definition 8.1. Let $u \in \mathbf{Prof}(\mathcal{C}, \mathcal{D})^{\to}$ be a map of profunctors between premodel categories. We call u a *Quillen fibration* if:

- u relates the weak factorization systems $(\mathbf{TC}_\mathcal{C}, \mathbf{F}_\mathcal{C})$ and $(\mathbf{TC}_\mathcal{D}, \mathbf{F}_\mathcal{D})$,
- u relates the weak factorization systems $(\mathbf{C}_\mathcal{C}, \mathbf{TF}_\mathcal{C})$ and $(\mathbf{C}_\mathcal{D}, \mathbf{TF}_\mathcal{D})$.

We call u a *Quillen trivial fibration* if:

- u relates the weak factorization systems $(\mathbf{TC}_\mathcal{C}, \mathbf{F}_\mathcal{C})$ and $(\mathbf{C}_\mathcal{D}, \mathbf{TF}_\mathcal{D})$.

In words, a map of profunctors is called a Quillen trivial fibration if pullback two-sided evaluation with a fibration on the right and cofibration on the left is a split surjection. For a Quillen fibration, we require this when one of the maps on the two sides is trivial. Note that a Quillen trivial fibration is in particular a Quillen fibration.

Remark 8.2. The notion of Quillen (trivial) fibration is self-dual. A map u in $\mathbf{Prof}(\mathcal{C}, \mathcal{D})^{\to}$ is a Quillen (trivial) fibration exactly if it is so when regarded as a map in $\mathbf{Prof}(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$ where $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{D}^{\mathrm{op}}$ carry the dual premodel category structures.

We have equivalent characterizations of Quillen (trivial) fibrations analogous to Lemma 7.3.

Lemma 8.3. *In the context of Definition 8.1, let u go between representable profunctors. Then we have the following alternative characterizations:*

- (i) *u is a Quillen fibration exactly if $u \mathbin{\widehat{\odot}_{\mathsf{r}}} - \colon \mathcal{C}^{\to} \to \mathcal{D}^{\to}$ preserves fibrations and trivial fibrations.*
- (ii) *u is a Quillen trivial fibration exactly if $u \mathbin{\widehat{\odot}_{\mathsf{r}}} - \colon \mathcal{C}^{\to} \to \mathcal{D}^{\to}$ sends fibrations to trivial fibrations.*

Remark 8.4. If the map u in Definition 8.1 goes between representable profunctors, we may equivalently describe Quillen (trivial) fibrations in terms of the map $u \odot_{\mathsf{r}} -$ between representing functors.

die note fuer thorsten hatte ich geschrieben, weil er mal eine zeit lang nicht aufgehoert hat, ueber higher cwfs zu reden und immer mit diesem D-zeug anfing. das war nachdem du die kurze non-univalent definition von non-complete higher categories gefunden hattest.

Definition 8.5. A *cylindrical premodel category* is a premodel category $\mathcal{E}$ with a birepresentable functorial notion of homotopy I such that the boundary map $\langle \delta_0, \delta_1 \rangle$ is a Quillen fibration and the endpoints maps δ_0, δ_1 are trivial Quillen fibrations.

Lemma 8.6. *Let $\mathcal{E}$ be a premodel category with a birepresentable functorial notion of homotopy I. Then the following are equivalent:*

(i) *The premodel category $\mathcal{E}$ is cylindrical,*
(ii)

The first two conditions say that the weak factorization systems of the premodel category are cylindrical. The last condition can be interpreted as

9. Cylindrical model setups

Definition 9.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories with a class of maps called cofibrations and a further subclass called trivial cofibrations. Let $u \colon F \to G$ be a map in $[\mathcal{C}, \mathcal{D}]$.

- We call u a *Quillen cofibration* if pushout application of u exists and preserves cofibrations and trivial cofibrations.
- We call u a *Quillen trivial cofibration* if pushout application of u exists and sends cofibrations to trivial cofibrations.

Remark 9.2. We have dual notions of *Quillen fibrations* and *Quillen trivial fibrations*, defined using pullback application. These are related to the notions of Definition 9.1 by adjointness. In detail, assume that $\mathcal{C}$ and $\mathcal{D}$ extend to model setups (minus part (iv)). Assume that F and G have respective right adjoints F' and G'. Then u transposes to a map $u' \colon G' \to F'$. Assume that pushout application of u and pullback application of u' exist. Then u is a Quillen (trivial) cofibration exactly if u' is a Quillen (trivial) fibration.

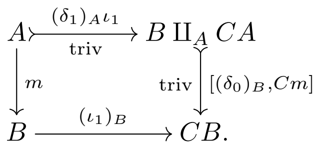
Definition 9.3. A *cylindrical model setup* is a model setup $\mathcal{E}$ equipped with a left adjoint functorial cylinder such that the boundary inclusion $[\delta_0, \delta_1]$ is a Quillen cofibration, and the endpoint inclusions δ_0, δ_1 are Quillen trivial cofibrations.

By the discussion in Section 6, we can equivalently describe a cylindrical model setup in terms of the associated right adjoint functorial cocylinder. From the discussion in Remark 9.2, the compatibility conditions become the following: the boundary projection $\langle p_0, p_1 \rangle$ is a Quillen fibration and the endpoint projections p_0, p_1 are Quillen trivial fibrations. This makes the notion of cylindrical model setup self-dual.

Lemma 9.4. *Let $\mathcal{E}$ be a cylindrical model setup.*

(i) *Cofibrations between trivially cofibrant objects are trivial cofibrations.* FiXme Note: (Co)completeness: needs initial object
(ii) *Fibrations between trivially fibrant objects are trivial fibrations.* FiXme Note: (Co)completeness: needs terminal object

Proof. The assertions are dual, so it will suffice to prove part (i). Let $m\colon A \to B$ be a cofibration with A and B trivially cofibrant. Consider the commuting diagram



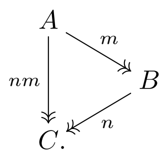
The top map is the first factor of the mapping cylinder factorization of m, a trivial cofibration by part (i) of Lemma 7.8 applied to the weak factorization system $(\mathbf{TC}, \mathbf{F})$. The right map is the pushout application of δ_0 to the cofibration m, also a trivial cofibration. The bottom map is split mono (with retraction ϵ_B). This makes m a codomain retract of a trivial cofibration. □

Corollary 9.5. *Let $\mathcal{E}$ be a cylindrical model setup.*

(i) *Assume that trivial cofibrations are closed under left cancellation among cofibrations. Then trivial cofibrations are closed under 2-out-of-3 in the wide subcategory of cofibrations.*
(ii) *Assume that trivial fibrations are closed under right cancellation among fibrations. Then trivial fibrations are closed under 2-out-of-3 in the wide subcategory of fibrations.*

FiXme Note: (Co)completeness: works always

Proof. The assertions are dual, so it will suffice to prove part (i). Consider a triangle of cofibrations



We then have the following.

- If m and n are trivial cofibrations, then so is nm.
- If m and nm are trivial cofibrations, then so is n by applying part (i) of Lemma 9.4 to n in the coslice under A.
- If n and nm are trivial cofibrations, then so is m by our left cancellation assumption. □

Lemma 9.6. *Let $\mathcal{E}$ be a cylindrical model setup. Let $f\colon A \to B$ be a cofibration between fibrant objects. Then the following are equivalent:*

(i) *f is a strong homotopy equivalence,*
(ii) *f is a strong deformation retract,*
(iii) *f is a trivial cofibration.*

We have dual equivalences for fibrations between cofibrant objects.

FiXme Note: (Co)completeness: works always

Proof. We produce a cycle of implications.

- Condition (iii) implies condition (ii) by part (i) of Corollary 7.9.
- Condition (ii) includes condition (i).
- Given condition (i), f is a retract of the trivial cofibration $\delta_i \mathbin{\widehat{\odot_{\mathsf{I}}}} f$ by the direction from (i) to (ii) of Lemma 6.7, so condition (iii) holds. □

Corollary 9.7. *Let $\mathcal{E}$ be a cylindrical model setup.*

(i) *Cofibrations between trivially fibrant objects are trivial cofibrations.*
(ii) *Fibrations between trivially cofibrant objects are trivial fibrations.*

FiXme Note: (Co)completeness: needs terminal object.

FiXme Note: (Co)completeness: needs initial object.

Proof. The assertions are dual, so it will suffice to prove part (i). The given cofibration is a strong deformation retract by part (i) of Corollary 7.9. Hence it is a trivial cofibration by Lemma 9.6. □

Lemma 9.8. *Let $\mathcal{E}$ be a cylindrical model setup. Assume that trivial cofibrations are closed under left cancellation among cofibrations. Let* $\mathbf{S}$ *denote the collection of cofibrant and trivially fibrant objects. Then the following are equivalent:*

(i) *there is trivially cofibrant $M \in \mathbf{S}$,*
(ii) *every $M \in \mathbf{S}$ is trivially cofibrant.*

We have a dual version of Lemma 9.8 that we do not bother to state.

FiXme Note: (Co)completeness: needs initial object, needs terminal object.

Proof of Lemma 9.8. Consider $\mathbf{S}$ as a category with cofibrations as morphisms. Given a finite collection $M_i \in \mathbf{S}$ with $i \in I$, we factor $\coprod_I M \to 1$ as a cofibration $\coprod_I M \to N$ followed by a trivial fibration $N \to 1$. Since all M_j are cofibrant, $\coprod_I M$ is cofibrant and the coprojections $M_i \to \coprod_I M$ are cofibrations. Thus, $N \in \mathbf{S}$ and the composites $M_i \to N$ are cofibrations. This shows that $\mathbf{S}$ is connected.

It remains to show: given a cofibration $M_0 \to M_1$ with $M_0, M_1 \in \mathbf{S}$, then M_0 is trivially cofibrant exactly if M_1 is. By part (i) of Corollary 9.7, $M_0 \to M_1$ is a trivial cofibration. So if M_0 is trivially cofibrant, so is M_1. The reverse direction is given by our left cancellation assumption. □

Remark 9.9. Consider the property of trivial cofibrations being given by left lifting against fibrations betwen fibrant objects. Is this property stable under slicing and/or coslicing? If not, we cannot really use Lemma 9.8 as stated for our application. Maybe instead use as assumption that trivial cofibrations are closed under left(?) cancellation.

Let $\mathcal{E}$ be a cylindrical model setup. We define the class of *weak equivalences* $\mathbf{W}$ as those maps factoring as a trivial cofibration followed by a trivial fibration. Recall from Lemma 3.3 that $\mathbf{TC} = \mathbf{C} \cap \mathbf{W}$ and $\mathbf{TF} = \mathbf{F} \cap \mathbf{W}$.

Lemma 9.10. *Let $\mathcal{E}$ be a cylindrical model setup. Let f be a map in $\mathcal{E}$.*

(i) *Assume that trivial cofibrations are closed under left cancellation among cofibrations. Then $f \in \mathbf{W}$ exactly if in every $(\mathbf{C}, \mathbf{TF})$-factorization of f, the first factor is a trivial cofibration.*
(ii) *Assume that trivial fibrations are closed under right cancellation among fibrations. Then $f \in \mathbf{W}$ exactly if in every $(\mathbf{TC}, \mathbf{F})$-factorization of f, the second factor is a trivial fibration.*

FiXme Note: (Co)completeness: works always.

Proof. The assertions are dual, so it will suffice to prove part (i). This is Lemma 9.8 applied in the category of factorizations of f. □

FiXme Note: Add remark discussing this cylindrical model setup.

Definition 9.11. Let $\mathcal{E}$ be a category with classes of maps $\mathbf{A}$ and $\mathbf{B}$. We say that $\mathbf{A}$ *exchanges with* $\mathbf{B}$ if for $X \to M$ in $\mathbf{A}$ and $M \to Y$ in $\mathbf{B}$, there is $X \to N$ in $\mathbf{B}$ and $N \to Y$ in $\mathbf{A}$ such that the following square commutes:

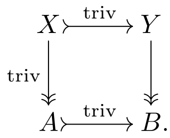
$$\begin{array}{ccc} X & \xrightarrow{\in \mathbf{B}} & Y \\ {\scriptstyle \in\mathbf{A}}\downarrow & & \downarrow{\scriptstyle \in\mathbf{A}} \\ M & \xrightarrow[\in \mathbf{B}]{} & N. \end{array}$$

Lemma 9.12. *Let $\mathcal{E}$ be a cylindrical model setup. Assume either of the following:*

(i) *Every object is cofibrant.*
(ii) *Every object is fibrant.*

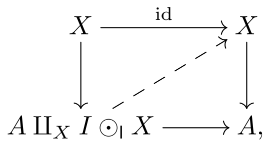
Then trivial fibrations exchange with trivial cofibrations.

Proof. By duality, it suffices to show the claim under condition (i). Given a trivial fibration $X \to A$ and a trivial cofibration $A \to B$, we use the weak factorization system $(\mathbf{TC}, \mathbf{F})$ to produce a square

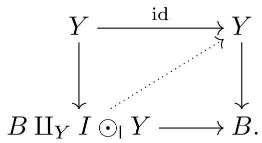


It remains to show that $Y \to B$ is a trivial fibration.

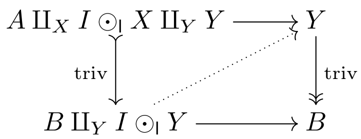
Using the assumption that all objects are cofibrant, trivial fibrations coincide with fibrations that are duals of a strong deformation retract. Recall their description from Remark 6.11. Given a diagonal filler



it thus remains to find a diagonal filler



For this, we fill the lifting problem



where the top map is induced by the dashed map above. The left map is a trivial cofibration as it writes as a pushout of the pushout application of $[\delta_0, \delta_1]$ to $X \to Y$ followed by a pushout of $A \to B$. □

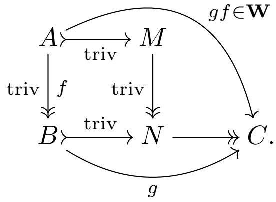
Theorem 9.13. *Let $\mathcal{E}$ be a cylindrical model setup. Then $\mathcal{E}$ extends to a cylindrical model category exactly if:*

(i) *trivial cofibrations are closed under left cancellation among cofibrations,*
(ii) *trivial fibrations are closed under right cancellation among fibrations,*
(iii) *trivial fibrations exchange with trivial cofibrations.*

Proof. If $\mathcal{E}$ is a model category, then both conditions hold by closure of weak equivalences under 2-out-of-3. Thus, the main work is in the forward implication, for which we use Lemma 3.3. Closure of $\mathbf{W}$ under composition is an immediate consequence of condition (iii). It remains to show closure under left and right cancellation. By duality, we may restrict our attention to the case of left cancellation. Consider maps $f \colon A \to B$ and $g \colon B \to C$ with $f, fg \in \mathbf{W}$. We have to show $g \in \mathbf{W}$. Since f factors as a trivial cofibration followed by a trivial fibration, it suffices to consider separately the cases where f is a trivial cofibration or a trivial fibration.

Assume that $f \colon A \to B$ is a trivial cofibration. Factor g as a cofibration $B \to M$ followed by a trivial fibration $M \to C$. Then $A \to M$ is a trivial cofibration by part (i) of Lemma 9.10 since $fg \in \mathbf{W}$. Applying part (i) of Lemma 9.4 in the slice under A, we see that $B \to M$ is a trivial cofibration. This shows $g \in \mathbf{W}$.

Assume that $f\colon A \to B$ is a trivial fibration. Factor g as a trivial cofibration $B \to N$ followed by a fibration $N \to C$. By condition (iii), the composite $A \to N$ factors as a trivial cofibration $A \to M$ followed by a trivial fibration $M \to N$:

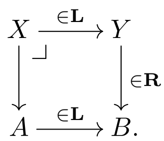


By part (ii) of Lemma 9.10, the map $M \to C$ is a trivial fibration. Then $N \to C$ is a trivial fibration by condition (ii). This shows $g \in \mathbf{W}$. □

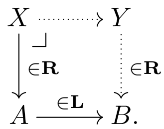
10. Application

In this section, let $\mathcal{E}$ is a category with a weak factorization system $(\mathbf{L}, \mathbf{R})$. We assume that pullbacks along maps in $\mathbf{R}$ exists.

Definition 10.1. We say that $(\mathbf{L}, \mathbf{R})$ has the *Frobenius property* if left maps are stable under pullback along right maps:



Definition 10.2. We say that $(\mathbf{L}, \mathbf{R})$ has the *extension property* if right maps descend along left maps in the following sense. For any left map $A \to B$ and right map $X \to A$, there is a right map $Y \to B$ pulling back to $X \to A$:

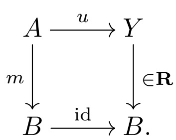


The Frobenius property and the extension property are both inherited by the induced weak factorization system on a slice or coslice of $\mathcal{E}$.

We record the following standard fact.

Lemma 10.3. *A map $m\colon A \to B$ is left exactly if pullback along m lifts sections of right objects. That is, given Y over B right and a section y' of its pullback m^*Y over A, there is a section y of Y pulling back to y'.*

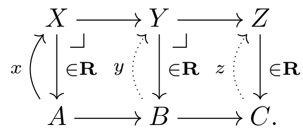
Proof. Let Y over B be right. A section y' of m^*Y corresponds to a map u making the below square commute:



An extension of y' to a section of Y corresponds to a lift in this square. This takes care of the forward direction. For the reverse direction, note that any left lifting problem of m against a right map reduces by taking a pullback to one where the bottom map is the identity on B. □

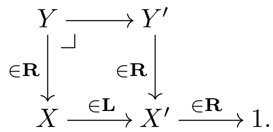
Lemma 10.4. *If $(\mathbf{L}, \mathbf{R})$ has the extension property, then left maps are closed under left cancellation.*

First proof. Given maps $m\colon A \to B$ and $n\colon B \to C$ with n, nm left, we show that m is left. We work with the characterization of left maps given by Lemma 10.3. Consider right Y over B with a section x of its pullback X over A. By the extension property, there is right Z over C pulling back to Y:



By pullback pasting, the composite square is a pullback. Since $A \to C$ is left, the section x extends to a section z of Z over C. This pulls back to a section y of Y over B, which further pulls back to x over A as required. □

Second proof. Using Remark 2.4, we reduce the claim to the situation where the ambient category has a terminal object. There, every right map is a pullback of a right map with right target: given a right map $Y \to X$, take a right replacement $X \to X'$ (using the terminal object) and extend $Y \to X$ along the left map $X \to X'$:



This shows that right maps and right maps with right target have the same left lifting closure. The claim follows by part (i) of Lemma 2.5. □

We now come to one of our main points. The extension and Frobenius properties suffice to extend a cylindrical model setup to a proper model category.

Theorem 10.5. *Let $\mathcal{E}$ be a cylindrical model setup. If:*

(i) the weak factorization system $(\mathbf{TC}, \mathbf{F})$ has the extension property,
(ii) every object is cofibrant,

then $\mathcal{E}$ extends to a cylindrical model category. If furthermore:

(iii) the weak factorization system $(\mathbf{TC}, \mathbf{F})$ has the Frobenius property,

this model category is proper.

Proof. This is a corollary of Theorem 9.13. Condition (i) is satisfied by part (i) of Corollary 2.6 (using assumption (ii)). Condition (ii) is satisfied by Lemma 10.4 (using assumption (i)). Condition (iii) is satisfied by Lemma 9.12 (using assumption (ii)).

The model structure is left proper since every object is cofibrant. For right properness, we need that pullback along fibrations preserves weak equivalences. This reduces to preservation of trivial cofibrations and trivial fibrations. The former is the Frobenius property (assumption (iii)) and the latter always hold since trivial fibrations are defined by right lifting. □

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