EXPLANATION: HIGHER CWFS IN SEGAL TYPES

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The goal of this note is to explain the unfolding of the notion of category with family in the setting of (complete) Segal types. We start by choosing a suitably abstract definition of category with family in the 1-categorical setting. This definition then generalizes to the higher setting in a straightforward manner.

0.1. **Categories with families.** A functor $\mathcal{E} \to \mathcal{B}$ *in displayed form* is a category $\mathcal{E}$ displayed over $\mathcal{B}$, i.e. with all sorts and operations of $\mathcal{E}$ indexed by the corresponding sorts and operations of $\mathcal{B}$. This is what one naturally gets by taking the definition of functor into $\mathcal{B}$ and systematically applying the equivalence $\mathbf{Set}/I \simeq \mathbf{Set}^I$. In the notation $\mathcal{E} \to \mathcal{B}$, note that $\mathcal{E}$ refers not to the displayed category itself, but rather its dependent sum with $\mathcal{B}$ (in the theory of categories), i.e. the total category. We consider this overloading harmless.

Definition 0.1. A *category with family (cwf)* is a category $\mathcal{C}$ with a terminal object together with discrete Grothendieck fibrations $\mathsf{Ty} \to \mathcal{C}$ and $\mathsf{Tm} \to \mathsf{Ty}$ in displayed form with a right adjoint for $\mathsf{Tm} \to \mathsf{Ty}$. □

The higher analogue of a discrete Grothendieck fibration is a right fibration. This is a map $p\colon \mathcal{E} \to \mathcal{B}$ (in displayed form) between higher (pre-)categories such that for $f\colon A \to B$ and $Y \in \mathcal{E}$ with $p(Y) = B$, the type of $X \in \mathcal{E}$ and $g\colon X \to Y$ such that $p(X) = A$ and $p(g) = f$ is contractible. Seeing $(\mathcal{E}, p)$ as a displayed higher category over $\mathcal{B}$, this means that for $f\colon A \to B$ and $Y \in \mathcal{E}_0(B)$, the type of $X \in \mathcal{E}_0(A)$ and $g \in \mathcal{E}_1(f, X, Y)$ is contractible.

In the 1-categorical setting, note that a right fibration with base $\mathcal{B}$ in displayed form is nothing but a presheaf over $\mathcal{B}$. In higher categories, right fibrations with base $\mathcal{B}$ are one particular model of presheaves over $\mathcal{B}$.

0.2. **Reedy types.** Let D be a direct category with finite slices. We have a model of homotopy type theory in presheaves over D where types satisfy a so-called *Reedy fibrancy* condition. The contexts are just (strict) presheaves over D. They do not necessarily have to be fibrant (even levelwise can be "pretypes" or "non-fibrant types"), but we will not run into that situation.

Given such a presheaf X over D, let us make explicit what a Reedy type Y over X consists of. Inductively over $I \in D$, for $x \in X_I$ and

$$y_f \in Y_J(xf, (y_{fg})_{g\colon K \to J, g \neq \mathrm{id}_J})$$

for non-identities $f\colon J \to I$, we have a type $Y_I(x, y)$.

We have the usual type theoretic terminology for Reedy fibrant presheaves over D. For example, the above is a Reedy fibrant family Y over X. We could have also expressed it as a map $X \to U$ where U is a certain Reedy fibrant presheaf acting as a universe.

Let $i\colon A \to B$ be an inclusion of finite presheaves over D. Given a Reedy fibrant family Y over X in presheaves over D, the *Leibniz hom* $\widehat{\hom}(i, Y)$ of Y with i is the family indexed over $f\colon B \to X$ with $y_a : Y(I, f(i(a)), (y_{af})_{f\colon J \to I, f \neq \mathrm{id}_I})$ for $a \in A(I)$ given by the iterated dependent sum of $y_b : Y(I, f(b), (y_{bf})_{f\colon J \to I, f \neq \mathrm{id}_I})$

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for $b \in B(I)$ not in the image of i. We say that Y is *local* with respect to i if the Leibniz hom $\hom(i, Y)$ is a family of contractible types.

The Leibniz hom and the associated notion of locality generalizes to the case for i is not necessarily an inclusion. Instead of a family of types, which is then posited to be contractible, one gets a map between families of types, which is then posited to be an equivalence.

0.3. **(Marked) semisimplicial types.** Let Δ_+ denote the direct fragment of the simplex category. Explicitly, it is the category with objects $[n] = \{0 < \ldots < n\}$ for $n \geq 0$ and morphisms the order-preserving injections. Let $\Delta_{+,\mathsf{m}}$ be the extension of Δ_+ generated by an object $[1]_\mathsf{m}$ (thought of as the homotopical poset $\{0 < 1\}$ with the map $0 \to 1$ a weak equivalences) with a map $[1] \to [1]_\mathsf{m}$. Both Δ_+ and $\Delta_{+,\mathsf{m}}$ are a direct categories with finite slices. Presheaves over Δ_+ and $\Delta_{+,\mathsf{m}}$ are referred to as *semisimplicial* and *marked semisimplicial sets*, respectively. Fibrant families in the sense of the Reedy presheaf model are referred to as *Reedy fibrant (marked) semisimplicial families.*

0.4. **Segal types.** Let $I = \{\partial\Delta[n] \to \Delta[n]\}$ be the collection of boundary inclusions. If we are in the marked setting (presheaves over $\Delta_{+,\mathsf{m}}$), we also add the boundary inclusion of $[1]_\mathsf{m}$, i.e. the map $\Delta[1] \to y[1]_\mathsf{m}$ to I. A Reedy fibrant (marked) semisimplicial family is local with respect to I exactly if all its component families are contractible.

We do not recall what a horn inclusion $\Lambda_k[n] \to \Delta[n]$ is, but only remark that n always has to be positive. To such horn inclusion, we associate a set of *critical edges*:

- if $0 < k < n$, there is no critical edge,
- if $k = 0$, the edge from 0 to 1 is critical,
- if $k = n$, the edge from $n - 1$ to n is critical.

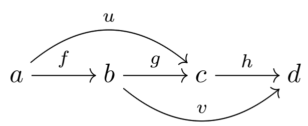
The critical edges are precisely the edges pointing in the wrong direction when one wants to categorically compose a horn to fill it.

In marked semisimplicial sets, let J_inner be the collection of horn inclusions where the marked edges are exactly the critical edges. A Reedy fibrant marked semisimplicial family that is local with respect to J_inner is called *inner fibrant.* We also use J_inner and inner fibrancy in semisimplicial sets, but then it just stands for the inner horn inclusions.

We obtain J_left and J_right be adding the horn inclusions $\Lambda_0[n] \to \Delta[n]$ and $\Lambda_n[n] \to \Delta[n]$ to J_inner. A Reedy fibrant marked semisimplicial family that is local with respect to J_left and J_right is called *left fibrant* and *right fibrant*, respectively. If it is both left and right fibrant, it is called *Kan fibrant.* Note that a Kan fibrant marked semisimplicial type is determined up to equivalence by its level zero component.

Let K be the collection of the following two inclusions of marked semisimplicial sets. Both maps are the identity on underlying semisimplicial sets and only differ in the marking.

- The underlying semisimplicial set is



with two triangles as indicated. The source has u and v marked, and the target has all edges marked. For an inner fibrant marked semisimplicial type, locality with respect to this map expresses that every equivalence edge is marked.

- The map is the codiagonal of the boundary inclusion for $[1]_{\mathsf{m}}$, i.e. the map $y[1]_{\mathsf{m}} +_{\Delta[1]} y[1]_{\mathsf{m}} \to y[1]_{\mathsf{m}}$. Explicitly, it goes from an edge with two markings to the edge with only one marking. Locality with respect to this map expresses that the family of markings is propositional.

An inner fibrant marked semisimplicial family that is local with respect to K is called *complete*. Note that any left or right fibrant marked semisimplicial family is automatically complete.

Definition 0.2. A *complete Segal type* is a complete inner fibrant marked semisimplicial type. □

The above setup has the benefit that it automatically yields the correct notion of "relative" complete Segal type, the analogue of a displayed category. This is now simply a complete inner fibrant semisimplicial family.

We also wish to discuss (non-complete) Segal types. Their definition is a bit more complicated in the semisimplicial rather than simplicial setting. We first discuss a definition given by Nicolai, but then use a different one.

Recall that an edge in an inner fibrant semisimplicial type is an *equivalence* if pre- and postcomposition with it is an equivalence, i.e. if two-dimensional (equivalently, arbitrary dimensional) outer horns have contractible fillers whenever this edge is the critical edge.

Definition 0.3 (Nicolai Kraus)**.** A *Segal type* is an inner fibrant semisimplicial type X such that for $x \in X_0$, there is an equivalence $i \in X_1(x, x)$ with $X_2(f, f, f)$. □

This definition is simple, but difficult to express using locality (which is useful for getting relative notions).

Definition 0.4. A *Segal type* is a Kan fibrant marked semisimplicial type $\overline{X}$ with an inner fibrant marked semisimplicial type X and a map $\overline{X} \to X$ that is an equivalence on points and marked edges. □

This definition can be seen as a variation of the definition of non-complete Segal types from complete Segal types. Instead of requiring that X is a complete Segal type and the map $\overline{X} \to X$ is surjective on points, we stipulate that $\overline{X} \to X$ is bijective on points. This makes $\overline{X}$ the core of X rather than a fully faithful cover of the core. To go from this definition to the usual one, one has to take the completion of X. The benefit of the current definition is that one does not require the completion operation to exhibit many important examples as instances. In a setting with colimits of Reedy fibrant semisimplicial types, the two definitions are equivalent.

It is possible to further strip the definition.

Definition 0.5 (Definition 0.4 reduced)**.** A *Segal type* is an inner fibrant marked semisimplicial type X together with a family $X_{2\mathsf{m}}$ on fully marked triangle boundaries such that the 2-restricted semisimplicial type $X_0, X_{1\mathsf{m}}, X_{2\mathsf{m}}$ is Kan fibrant. □

The last condition means that given two of the three marked edges $u : X_1(a, b)$, $v : X_1(b, c)$, and $w : X_1(a, c)$, the dependent sum of the third and $X_{2\mathsf{m}}(u, v, w)$ is contractible.

Note that Definition 0.5 is an instance of a Reedy fibrant diagram over some direct category satisfying a certain locality condition. As such, it immediately generalizes to give a relative notion.

0.5. **Cwfs in Segal types.** Let $f\colon X \to Y$ be a map of Segal types. We define what it means for f to have a right adjoint.

In 1-categories, we would form the comma category $f \downarrow y$ for $y : Y_0$ and demand a terminal object. This is exactly what we do here, using that we have a particularly nice model for this comma Segal type given by the semisimplicial structure.

For $y : Y_0$, the *over-Segal-type* Y/y is a displayed Segal type over Y. Its n-simplices are the $(n+1)$-simplices of Y whose last vertex is y. The comma Segal type $f \downarrow x$ is the base change (substitution) of X/x along $f\colon X \to Y$. It is a displayed Segal type over X.

Definition 0.6. A right adjoint to a map $f\colon X \to y$ of Segal types is a terminal object in $f \downarrow y$ for every $y : Y_0$. □

The first component of the terminal object in $f \downarrow y$ (a point in X) gives the action of the right adjoint on y. The second component gives the counit at y. Terminality witnesses the universal property of the counit. Since the definition is expressed in terms of universal properties, there are no further coherence conditions.

Now we have all the ingredients to state the definition of higher cwfs in Segal types.

Definition 0.7 (Higher cwf)**.** A (non-complete) *category with family (cwf)* is a Segal type X with a terminal vertex together with right fibrant Ty over X and right fibrant Tm over Ty such that the projection $\mathsf{Tm} \to \mathsf{Ty}$ has a right adjoint. □

Such a cwf is *complete* if the underlying Segal type is complete.