# GLUING ALONG A PROFUNCTOR

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## 1. Preliminaries on profunctors

Given categories $\mathcal{C}$ and $\mathcal{D}$, a *profunctor* $H\colon \mathcal{C} \nrightarrow \mathcal{D}$ is a functor $H\colon \mathcal{D}^{\mathrm{op}} \times C \to \mathbf{Set}$. The elements of $H(Y, X)$ are called *heteromorphisms* and written $Y \rightsquigarrow_H X$. We may omit the subscript if it is clear from the context. The action of $H$ on morphisms of $\mathcal{C}$ and $\mathcal{D}$ allows us to to precompose heteromorphisms with morphisms in $\mathcal{D}$ and postcome them with morphisms in $\mathcal{C}$, in the usual associative manner.

Categories and profunctors form a bicategory **Prof** (in fact, they form a main example). The identity profunctor on $\mathcal{C}$ is given by its hom-sets $\mathcal{C}(-,-)$. The composition of $H\colon \mathcal{C} \nrightarrow \mathcal{D}$ and $K\colon \mathcal{D} \nrightarrow \mathcal{E}$ is given by

$$(K \circ H)(Z, X) = \int^{Y \in \mathcal{D}} H(Y, X) \times K(Z, Y).$$

Neutrality and associativity are witnessed by evident natural isomorphisms.

**Remark 1.1** (Duality of profunctors)**.** Profunctors admit an evident duality. This is pseudo-equivalence $i\colon \mathbf{Prof} \simeq \mathbf{Prof}^{\mathsf{co,op}}$ satisfying $i^{\mathsf{co,op}} \simeq i^{-1}$. On objects, it is given by taking opposite categories. On categories of morphisms, it sends a profunctor $H\colon \mathcal{C} \nrightarrow \mathcal{D}$ to the profunctor $H'\colon \mathcal{D}^{\mathrm{op}} \nrightarrow \mathcal{C}^{\mathrm{op}}$ defined by $H'(X, Y) = H(Y, X)$. □

**Remark 1.2** (Representable profunctors)**.** The 2-category **Cat** of categories and functors embeds into **Prof** via an identity-on-objects pseudofunctor. The action on morphisms sends $R\colon \mathcal{C} \to \mathcal{D}$ to $\mathcal{D}(-, R(-))\colon \mathcal{C} \nrightarrow \mathcal{D}$ and is fully faithful. The profunctors $H$ in its essential image are called *representable*. This happens exactly if $H(Y, -) \in \mathrm{Psh}(\mathcal{C}^{\mathrm{op}})$ is representable for all $Y \in \mathcal{D}$.

Under the duality of Remark 1.1, we similarly have a pseudofunctor from $\mathbf{Cat}^{\mathsf{co,op}}$ to **Prof**. On objects, it sends each category to its opposite. On morphisms, it sends $L\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}(L(-), -)\colon \mathcal{C} \nrightarrow \mathcal{D}$. The profunctors $H$ in its essential image are called *corepresentable*. This happens exactly if $H(-, X) \in \mathrm{Psh}(\mathcal{D})$ is representable for all $X \in \mathcal{C}$.

Birepresentable profunctors corresponds to adjunctions. The right adjoint is given by the representing functor and the left adjoint is given by the corepresenting functor. □

### 1.1. Cographs.

Consider a profunctor $H\colon \mathcal{C} \nrightarrow \mathcal{D}$. Its *cograph* $\mathsf{Cograph}(H)$ is the following category over $[1] = \{0 \leq 1\}$. On objects, we have

$$\mathsf{Cograph}(H)_0 = \mathcal{D},$$
$$\mathsf{Cograph}(H)_1 = \mathcal{C}.$$

On morphisms, we have

$$\mathsf{Cograph}(H)_{0 \leq 0} = \mathcal{D}(-,-),$$
$$\mathsf{Cograph}(H)_{0 \leq 1} = H,$$
$$\mathsf{Cograph}(H)_{1 \leq 1} = \mathcal{C}(-,-).$$

Identities are given by identities in $\mathcal{C}$ and $\mathcal{D}$. Composition is given by composition in $\mathcal{C}$ and $\mathcal{D}$ as well as functoriality of $H$ in each argument.

**Remark 1.3.** We have inclusions $\iota_0\colon \mathcal{D} \to \mathsf{Cograph}(H)$ and $\iota_1\colon \mathcal{C} \to \mathsf{Cograph}(H)$. This makes $\mathsf{Cograph}(H)$ into a discrete two-sided cofibration from $\mathcal{C}$ to $\mathcal{D}$. In fact, this is a characterization of the category of profunctors from $\mathcal{C}$ to $\mathcal{D}$. □

**Remark 1.4.** The profunctor $H$ is representable exactly if $\mathsf{Cograph}(H) \to [1]$ is a cartesian fibration. Dually, $H$ corepresentable if that projection is a cocartesian fibration. □

**Remark 1.5.** Heteromorphisms $Y \rightsquigarrow X$ correspond to morphisms $\iota_0(Y) \to \iota_1(X)$ in $\mathsf{Cograph}(H)$. Composition of heteromorphisms with morphisms in $\mathcal{C}$ and $\mathcal{D}$ corresponds to ordinary composition in $\mathsf{Cograph}(H)$. We may therefore understand $\mathsf{Cograph}(H)$ as an explanation of heteromorphisms and their comosition in terms of ordinary morphisms and their composition. □

1.2. **Graphs.** Consider a profunctor $H\colon \mathcal{C} \nrightarrow \mathcal{D}$. Its *graph* $\mathsf{Graph}(H)$ is the category of heteromorphisms of $H$. The objects are triples $(X, Y, h)$ of $X \in C$ and $Y \in \mathcal{D}$ and a heteromorphism $h\colon Y \rightsquigarrow X$. The morphisms from $(X_0, Y_0, h_0)$ to $(X_1, Y_1, h_1)$ are pairs $(f, g)$ of $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ forming a square as follows:

$$\begin{array}{ccc} Y_0 & \overset{h_0}{\rightsquigarrow} & X_0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y_1 & \overset{h_1}{\rightsquigarrow} & X_1, \end{array}$$

i.e., $H(Y_0, f)(h_0) = H(g, X_1)(h_1)$. Composition is defined componentwise.

**Remark 1.6.** We have projections $\rho_0\colon \mathsf{Cograph}(H) \to \mathcal{D}$ and $\rho_1\colon \mathsf{Cograph}(H) \to \mathcal{C}$. Note that $\rho_0$ is a cartesian fibration and $\rho_1$ is a cocartesian fibration. Jointly, this forms a discrete two-sided fibration from $\mathcal{C}$ and $\mathcal{D}$. In fact, this is a characterization of the category of profunctors from $\mathcal{C}$ to $\mathcal{D}$. □

**Remark 1.7.** Recall the adjunction between two-sided cofibrations and fibrations from $\mathcal{C}$ and $\mathcal{D}$ given by forming comma and cocomma categories. This restricts to an equivalence between discrete two-sided cofibrations and fibrations. This equivalence relates the cograph of a profunctor with its graph. In particular, $\mathsf{Graph}(H)$ is the comma category of $\mathsf{Cograph}(H)$ and $\mathsf{Cograph}(H)$ is the cocomma category of $\mathsf{Graph}(H)$. □

1.3. **Heteropullbacks.** Consider a profunctor $H\colon \mathcal{C} \nrightarrow \mathcal{D}$.

**Lemma 1.8.** *Consider a heterogeneous square*

$$\begin{array}{ccc} Y_0 & \overset{h_0}{\rightsquigarrow} & X_0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y_1 & \overset{h_1}{\rightsquigarrow} & X_1. \end{array}$$

*The following are equivalent:*

(1) *The corresponding morphism in* $\mathsf{Graph}(H)$ *is cartesian for* $\rho_1\colon \mathsf{Graph}(H) \to \mathcal{C}$.

(2) *The corresponding morphism in* $\mathsf{Graph}(H)$ *is locally cartesian for* $\rho_1$.

(3) *The corresponding square in* $\mathsf{Cograph}(H)$ *is a pullback,*

*Proof.* Conditions (1) and (2) are equivalent because $\rho_1$ is a cocartesian fibration. Conditions (2) and (3) unfold to the same thing. □

If the conditions of the lemma are satisfied, we call the given square a *(hetero)pullback*. This is displayed with the usual pullback corner symbol:

$$\begin{array}{ccc} Y_0 & \overset{h_0}{\rightsquigarrow} & X_0 \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y_1 & \overset{h_1}{\rightsquigarrow} & X_1. \end{array}$$

## 2. Promorphisms of cwfs

Consider cwfs $\mathcal{C}$ and $\mathcal{D}$. A *promorphism of cwfs* from $\mathcal{C}$ to $\mathcal{D}$ is a profunctor $H\colon \mathcal{C} \nrightarrow \mathcal{D}$ together with a cwf structure on $\mathsf{Cograph}(H)$ such that the inclusions $\iota_0\colon \mathcal{D} \to \mathsf{Cograph}(H)$ and $\iota_1\colon \mathcal{C} \to \mathsf{Cograph}(H)$ form strict cwf morphisms that are identities on types.

Let us unfold this definition.

Concerning types, we have $\mathsf{Ty} \circ \iota_1 = \mathsf{Ty}_{\mathcal{C}}$ and $\mathsf{Ty} \circ \iota_0 = \mathsf{Ty}_{\mathcal{D}}$. Since $\iota_0$ and $\iota_1$ are cwf morphisms, the substitution squares for types in $\mathcal{D}$ and $\mathcal{C}$ remain pullbacks in $\mathsf{Cograph}(H)$. In addition, we have a heterogeneous substitution operation:

$$\begin{array}{ccc} \Gamma'.\gamma^* A & \overset{\gamma.A}{\rightsquigarrow} & \Gamma.A \\ {\scriptstyle p_{HA}}\downarrow & \lrcorner & \downarrow{\scriptstyle p_A} \\ \Gamma' & \overset{\gamma}{\rightsquigarrow} & \Gamma, \end{array} \tag{2.1}$$

compatible with substitution in $\mathcal{C}$ and $\mathcal{D}$.

**Remark 2.1.** Promorphisms of cwfs generalize the notion of (weak) morphism of cwfs. Concretely, consider a functor $R\colon \mathcal{C} \to \mathcal{D}$. Extensions of $R$ to a cwf morphism correspond to extensions of the representable profunctor $\mathcal{D}(-, R(-))\colon \mathcal{C} \nrightarrow \mathcal{D}$ to a promorphism of cwfs. The heterogeneous substitution operation 2.1 is given by $\gamma^*(RA)$ when seeing $\gamma\colon \Gamma' \rightsquigarrow \Gamma$ as $\gamma\colon \Gamma' \to R\Gamma$. Reversely, we can recover the action of $R$ on types and their extensions from heterogeneous substitution along $\mathrm{id}_{R\Gamma}\colon R\Gamma \rightsquigarrow \Gamma$. □

**Remark 2.2** (Right adjoint on types)**.** Consider cwfs $\mathcal{C}$ and $\mathcal{D}$. Recall that a functor $L\colon \mathcal{D} \to \mathcal{C}$ has a right adjoint on types (also called dependent right adjoints) if the action $L_\Gamma\colon \mathcal{D} \downarrow \Gamma' \to \mathcal{C} \downarrow L\Gamma'$ of $L$ on slices has a partial right adjoint defined on and valued in extensions of types, all strictly natural. Equivalently, this is an operation $R_{\Gamma'}\colon \mathsf{Ty}_{\mathcal{C}}(L\Gamma') \to \mathsf{Ty}_{\mathcal{D}}(\Gamma')$ such that sections of $R_{\Gamma'}A$ are isomorphic to sections of $A$, all natural in $\Gamma'$. □

**Remark 2.3.** Promorphisms of cwfs generalize functors with right adjoint on types (dependent right adjoints). Concretely, consider a functor $L\colon \mathcal{C} \to \mathcal{D}$. A right adjoint on types for $L$ corresponds to an extension of the corepresentable profunctor $\mathcal{C}(L(-), -)\colon \mathcal{C} \nrightarrow \mathcal{D}$ to a promorphism of cwfs. The heterogeneous substitution operation 2.1 is given by $R_{\Gamma'}(\gamma^* A)$ when seeing $\gamma\colon \Gamma' \rightsquigarrow \Gamma$ as $\gamma\colon L\Gamma' \to \Gamma$. Reversely, we can recover the action of $R_{\Gamma'}$ on types and their extensions from heterogeneous substitution along $\mathrm{id}_{L\Gamma'}\colon \Gamma' \rightsquigarrow L\Gamma'$. □

**Remark 2.4.** Consider cwfs with terminal objects $\mathcal{C}$ and $\mathcal{D}$. We say that a promorphism of cwfs $H\colon \mathcal{C} \nrightarrow \mathcal{D}$ respects the terminal object if $\iota_1\colon \mathcal{C} \to \mathsf{Cograph}(H)$ preserves the terminal object. This means that $H(Y, 1)$ is terminal for $Y \in \mathcal{D}$.

Assume further that every object of $\mathcal{C}$ is isomorphic to the extension of a global type. Then every promorphisms of cwfs $H\colon \mathcal{C} \nrightarrow \mathcal{D}$ that respects the terminal object is representable. For $X \in \mathcal{C}$, the representation $RX$ with $\mathrm{y}(RX) \simeq H(-, X)$ is given by writing $X \simeq 1.A$ with $A \in \mathsf{Ty}_C(1)$, substituting

$$\begin{array}{ccc} 1_{\mathcal{D}}.\gamma^* A & \overset{\gamma.A}{\rightsquigarrow} & 1_{\mathcal{C}}.A \\ \downarrow{\scriptstyle p_{HA}} & \lrcorner & \downarrow{\scriptstyle p_A} \\ 1_{\mathcal{D}} & \rightsquigarrow & 1_{\mathcal{C}}, \end{array}$$

and taking $RX = 1_{\mathcal{D}}.\gamma^* A$. □

## 3. Gluing along a profunctor

The gluing along a profunctor $H : \mathcal{C} \to \mathcal{D}$ is simply the graph $\mathsf{Graph}(H)$. Gluing of cwfs can now be replayed at the level of a promorphism $H$ of cwfs. If $H$ is represented by $F : \mathcal{C} \to \mathcal{D}$, we have $\mathsf{Graph}(H) \simeq \mathcal{D} \downarrow F$ and the construction reduces to the standard gluing construction for cwfs.

The extra generality is useful because it allows us to construct presheaves over a direct category with finite slices in a cwf $\mathcal{C}$ via iterated gluing. Here, the matching object does not generally exist because the category of contexts lacks finite limits. This is one reason why some people prefer to work with contextual or democratic cwfs, and semantic notions such as tribes and type-theoretic fibration categories ask that every object be fibrant, which ensures that the matching object exists. Instead, we can work with the profunctor that would correspond to mapping into the matching objects, if it existed.

This example is noteworthy for another reason: heteropullbacks preserve anodyne maps (maps lifting against extensions of types). If this holds, we can construct (indexed) inductive types in the gluing in a more direct way. For example, the identity type in the gluing would normally be an indexed (dependent) identity type in $\mathcal{D}$ over the heteropullback of the identity type in $\mathcal{C}$. If anodyne maps are preserved, this heteropullback retains the behaviour of an identity type, so we can use it to transport between the fibers of $\mathcal{D}$ and use the ordinary identity type in $\mathcal{D}$ to implement the indexed one.