THOUGTS ON THE GLUEING CONSTRUCTION FOR CATEGORIES WITH FAMILIES

Abstract. Categories with families (cwf's) are traditionally identified with full split comprehension categories. Peter Lumsdaine has advocated that they should better be identified with discrete comprehension categories. We agree with his sentiment.

While the equivalence of these concepts is relatively easy to verify, the bookkeeping is quite tremendous when building actual equivalences between categories (with strict morphisms, pseudomorphisms, or lax morphisms) of categories with families on one hand and discrete comprehension categories on the other. Here, we try to give a conceptual account using the theory of comonadicity. Our argument is suitable for formalization.

Contents

1. Recollections on comonads and adjunctions

1.1. **Functors.** We write $\mathbf{Func}_{\mathrm{strict}}$ for the category of functors and strict morphisms, i.e. the functor category from the walking arrow to $\mathbf{Cat}$. We write $\mathbf{Func}_{\mathrm{pseudo}}, \mathbf{Func}_{\mathrm{lax}}, \mathbf{Func}_{\mathrm{oplax}}$ for the category of 2-functors from the walking arrow to $\mathbf{Cat}$ with pseudo-, oplax, and lax natural transformations as morphisms. In all cases, given functors $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ and $F_2 : \mathcal{C}_2 \to \mathcal{D}_2$, a morphism from F_1 to F_2 is a tuple (U, V, ϕ) with functors $U : \mathcal{C}_1 \to \mathcal{C}_2$ and $V : \mathcal{D}_1 \to \mathcal{D}_2$, but the signature of the natural transformation ϕ differs. In the lax case, we have $\phi : F_2U \to VF_1$. In the oplax case, we have $\phi : VF_1 \to F_2U$. In the pseudo case, we have an isomorphism $\phi : VF_1 \to F_2U$. In the strict case, we require that $\phi : VF_1 \to F_2U$ is an identity. In summary, we have a diagram of inclusions of wide subcategories as follows:

$$\begin{tikzcd}
 & & \mathbf{Func}_{\mathrm{oplax}} \\
\mathbf{Func}_{\mathrm{strict}} \arrow[r, hook] & \mathbf{Func}_{\mathrm{pseudo}} \arrow[ur, hook] \arrow[dr, hook] & \\
 & & \mathbf{Func}_{\mathrm{lax}}.
\end{tikzcd}$$

In all cases, we have projection functors

$$\begin{tikzcd}
 & \mathbf{Cat} \\
\mathbf{Func}_{\bullet} \arrow[ur, "\mathrm{Dom}"] \arrow[dr, "\mathrm{Cod}"'] & \\
 & \mathbf{Cat}
\end{tikzcd}$$

returning the domain and codomain category of the given functor.

1.2. **Adjunctions.** Let us describe the category $\mathbf{Adj}$ of adjunctions.

- An object of $\mathbf{Adj}$ is a tuple $(\mathcal{C}, \mathcal{D}, L, R, \eta, \epsilon)$ consisting adjoint functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ with unit η and counit ϵ.
- A morphism
$$(\mathcal{C}_1, \mathcal{D}_1, L_1, R_1, \eta_1, \epsilon_1) \to (\mathcal{C}_2, \mathcal{D}_2, L_2, R_2, \eta_2, \epsilon_2)$$
is a tuple (U, V, l, r) with functors $U : \mathcal{C}_1 \to \mathcal{C}_2$ and $V : \mathcal{D}_1 \to \mathcal{D}_2$ and natural transformations $l : L_2U \to VL_1$ and $r : UR_1 \to R_2V$ such that
$$rL_1 \circ U\eta_1 = R_2l \circ \eta_2U$$
and
$$V\epsilon_1 \circ lR_1 = \epsilon_2V \circ L_2r.$$
- The identity on $(\mathcal{C}, \mathcal{D}, L, R, \eta, \epsilon)$ is $(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}}, \mathrm{id}, \mathrm{id})$. The composition of
$$(U_1, V_1, l_1, r_1) : (\mathcal{C}_1, \mathcal{D}_1, L_1, R_1, \eta_1, \epsilon_1) \to (\mathcal{C}_2, \mathcal{D}_2, L_2, R_2, \eta_2, \epsilon_2)$$
and
$$(U_2, V_2, l_2, r_2) : (\mathcal{C}_2, \mathcal{D}_2, L_2, R_2, \eta_2, \epsilon_2) \to (\mathcal{C}_3, \mathcal{D}_3, L_3, R_3, \eta_3, \epsilon_3)$$
is
$$(U_2U_1, V_2V_1, V_2L_1 \circ L_3U_1, r_2U_1 \circ U_2r_1).$$
- Neutrality and associativity laws clearly hold.

Note that the last two conditions on a morphism in **Adj** above are equivalent and both specify that l and r are adjoint transposes, i.e.

$$R_2 V\epsilon_1 \circ R_2 l R_1 \circ \eta_2 U R_1 = r$$

or equivalently

$$\epsilon_2 V L_1 \circ \circ L_2 r L_1 \circ L_2 U \eta_1 = l.$$

Thus, one may remove one of l and r from the data of a morphism and omit these equations altogether. One obtains fully faithful functors

$$\begin{tikzcd} & \mathbf{Func}_{\mathrm{oplax}} \\ \mathbf{Adj} \arrow[ur, "\mathrm{Left}"] \arrow[dr, "\mathrm{Right}"'] & \\ & \mathbf{Func}_{\mathrm{lax}} \end{tikzcd}$$

projecting to the left and right adjoint. One may alternatively define **Adj** in this way, letting the morphism structure be created via these maps.

We add subscripts left-pseudo, left-strict, right-pseudo, right-strict, pseudo, strict to **Adj** to indicate that we are taking the wide subcategory of morphisms as above for which l or r or both are invertible or an identity, respectively. These subscripts maybe combined.

1.3. **Comonads.** Let us describe the category **Cmd**.

- An object is a tuple $(\mathcal{D}, N, \epsilon, \nu)$ with a category $\mathcal{D}$ and a comonad (N, ϵ, ν) on $\mathcal{D}$.
- A morphism from $(\mathcal{D}_1, N_1, \epsilon_1, \nu_1)$ to $(\mathcal{D}_2, N_2, \epsilon_2, \nu_2)$ is a functor $V\colon \mathcal{D}_1 \to \mathcal{D}_2$ and a natural transformation $v\colon VN_1 \to N_2 V$ such that $V\epsilon_1 = \epsilon_2 V \circ v$ and $N_1 v \circ v N_1 \circ V\nu_1 = \nu_2 V \circ v$.
- The identity on $(\mathcal{C}, T, \eta, \mu)$ is $(\mathrm{Id}_{\mathcal{C}}, \mathrm{id})$. The composition of
$$(V_1, v_1)\colon (\mathcal{D}_1, N_1, \epsilon_1, \nu_1) \to (\mathcal{C}_2, T_2, \epsilon_2, \nu_2)$$
and
$$(V_2, v_2)\colon (\mathcal{D}_2, N_2, \epsilon_2, \nu_2) \to (\mathcal{C}_3, T_3, \epsilon_3, \nu_3)$$
is $(V_2 V_1, v_2 V_1 \circ V_2 v_1)$.
- Neutrality and associativity laws clearly hold.

We write $\mathbf{Cmd}_{\mathrm{pseudo}}$ and $\mathbf{Cmd}_{\mathrm{strict}}$ for the wide subcategory of **Cmd** of morphisms as above for which v is invertible or an identity, respectively. Analogous categories are defined for copointed endofunctors, in which case the comultiplication component is omitted. This gives a diagram of categories as follows:

$$\begin{tikzcd} \mathbf{Cmd}_{\mathrm{strict}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cmd}_{\mathrm{pseudo}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cmd} \arrow[d] \\ \mathbf{Endo}_{\mathrm{copt,strict}} \arrow[r, hook] & \mathbf{Endo}_{\mathrm{copt,pseudo}} \arrow[r, hook] & \mathbf{Endo}_{\mathrm{copt}}. \end{tikzcd}$$

The horizontal arrows are wide subcategory inclusions.

1.4. **Comonads and adjunctions.** There is a functor $S\colon \mathbf{Adj}_{\text{left-strict}} \to \mathbf{Cmd}$ sending an adjunction $(\mathcal{C}, \mathcal{D}, L, R, \eta, \epsilon)$ to the comonad $(\mathcal{D}, LR, \epsilon, L\eta R)$ and a morphism

$$(U_1, V_1, l_1, r_1)\colon (\mathcal{C}_1, \mathcal{D}_1, L_1, R_1, \eta_1, \epsilon_1) \to (\mathcal{C}_2, \mathcal{D}_2, L_2, R_2, \eta_2, \epsilon_2)$$

to the morphism

$$(V_1, L_2 r)\colon (\mathcal{D}_1, L_1 R_1, \epsilon_1, L_1 \eta_1 R_1) \to (\mathcal{D}_2, L_2 R_2, \epsilon_2, L_2 \eta_2 R_2).$$

The functor S has a right adjoint $T\colon \mathbf{Cmd} \to \mathbf{Adj}_{\text{left-strict}}$ sending a comonad $(\mathcal{D}, N, \epsilon, \nu)$ to the adjunction $(\mathrm{Coalg}(N), \mathcal{D}, L, R, \eta, \epsilon)$ where $L\colon \mathrm{Coalg}(N) \to \mathcal{D}$ is the forgetful functor sending a coalgebra (A, f) to A and R is the cofree coalgebra functor sending A to the coalgebra (TA, ν_A); the component $\eta_{(A,f)}\colon (A, f) \to (TA, \nu_A)$ of the unit is given by f. The morphism

$$(V, v)\colon (\mathcal{D}_1, N_1, \epsilon_1, \nu_1) \to (\mathcal{C}_2, T_2, \epsilon_2, \nu_2)$$

is send to the morphism

$$(U, V, \mathrm{id}, r)\colon (\mathrm{Coalg}(N)_2, \mathcal{D}_2, U_2, F_2, \eta_2, \epsilon_2) \to (\mathrm{Coalg}(N_2), \mathcal{D}_2, U_2, F_2, \eta_2, \epsilon_2)$$

where U sends a coalgebra (A, f) for N_1 to the coalgebra $(VA, v_A \circ Vf)$ for N_2 and we do not to describe r or check any more laws since the projection $\mathbf{Adj}_{\text{left-strict}} \to \mathbf{Func}_{\text{strict}}$ to the left adjoint is fully faithful.

The adjunction

$$\begin{tikzcd}
\mathbf{Adj}_{\text{left-strict}} \arrow[rr, bend left, "S"] \arrow[rr, phantom, "\perp"] \arrow[dr] & & \mathbf{Cmd} \arrow[ll, bend left, "T"] \arrow[dl] \\
& \mathbf{Cat} &
\end{tikzcd}$$

lives strictly over $\mathbf{Cat}$ as indicated where the left functor returns the target of the left adjoint and the right functor returns the underlying category. It is furthermore a reflection, i.e. the right adjoint T is fully faithful. The adjunctions in the essential image of T, i.e. those adjunctions X for which the unit $X \to TSX$ is an isomorphism, are called *strictly comonadic.*

Note that the above reflection restricts to reflections

$$\begin{tikzcd}
\mathbf{Adj}_{\text{pseudo,left-strict}} \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathbf{Cmd}_{\text{pseudo}} \arrow[l, bend left, "T"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{Adj}_{\text{strict}} \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathbf{Cmd}_{\text{strict}} \arrow[l, bend left, "T"]
\end{tikzcd}.$$

2. Stuff on cartesian copointings

Recall that a functor is *cartesian* if it preserves pullbacks. Recall that a natural transformation is *cartesian* if its naturality squares are pullbacks. Recall that a (co)monad is *cartesian* if it is cartesian as a functor and its (co)unit and (co)multiplication are cartesian natural transformations.

Lemma 2.1. *Let (N, ϵ) be a copointed endofunctor. If the copointing ϵ is cartesian, then so is the whole copointed endofunctor (N, ϵ).*

Proof. The functor N is cartesian by a pullback pasting argument in a cube, using that its copointing ϵ is cartesian. □

Lemma 2.2. *Let (N, ϵ, ν) be a comonad. If the counit ϵ is cartesian, then so is the whole comonad (N, ϵ, ν).*

Proof. The functor N is cartesian by Lemma 2.1. By pullback pasting cancellation in the neutrality law $\epsilon N \circ \nu$, we see that also ν is cartesian. □

Lemma 2.3. *Let (N, ϵ, ν) be a comonad with cartesian counit on a category $\mathcal{D}$. Then the forgetful functor*

$$F \colon \operatorname{Coalg}(N, \epsilon, \nu) \to \operatorname{Coalg}(N, \epsilon)$$

is an isomorphism.

Proof. Clearly F is injective on objects and fully faithful. It remains to show that it is surjective on objects. Let (A, u) with $u \colon A \to NA$ be an object of $\operatorname{Coalg}(N, \epsilon)$. To show that (A, u) lifts through F, we need to show that u respects the comultiplication, i.e. that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "u"] & NA \arrow[d, "Nu"] \\ NA \arrow[r, "\nu_A"] & N^2 A. \end{tikzcd} \tag{2.1}$$

Since ϵ is cartesian, we have the following pullback:

$$\begin{tikzcd} N^2 A \arrow[r, "\epsilon_{NA}"] \arrow[d, "N\epsilon_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & NA \arrow[d, "\epsilon_A"] \\ NA \arrow[r, "\epsilon_A"] & A. \end{tikzcd}$$

It thus suffices to verify that the two composites in (2.1) are equal when postcomposed with ϵ_{NA} and $N\epsilon_A$. The composites of v_A with these two maps are identities, so it remains to show that the postcompositions of $Nu \circ u$ with ϵ_{NA} and $N\epsilon_A$ are equal to u. In the former case, we have

$$\begin{aligned} \epsilon_{NA} \circ Nu \circ u &= u \circ \epsilon_{NA} \circ u \\ &= u \circ \mathrm{id} \\ &= u. \end{aligned}$$

In the latter case, we have

$$\begin{aligned} N\epsilon_A \circ Nu \circ u &= N(\epsilon_A \circ u) \circ u \\ &= N(\mathrm{id}) \circ u \\ &= u. \end{aligned}$$

□

Lemma 2.4. *The vertical forgetful functors in the diagram*

$$\begin{tikzcd} \mathbf{Cmd}_{\text{strict}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cmd}_{\text{pseudo}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cmd} \arrow[d] \\ \mathbf{Endo}_{\text{copt,strict}} \arrow[r, hook] & \mathbf{Endo}_{\text{copt,pseudo}} \arrow[r, hook] & \mathbf{Endo}_{\text{copt}} \end{tikzcd} \tag{2.2}$$

are isomorphisms on copointed endofunctors with cartesian copointing.

Proof. All vertical functors have the same action on objects. Let us show that is it bijective. Let $(\mathcal{C}, N, \epsilon)$ be a copointed endofunctor with ϵ cartesian. We wish to show that there is a unique comultiplication μ satisfying the left and right neutrality and associativity laws.

Since ϵ is cartesian, we have the following pullback:

$$\begin{tikzcd} N^2 \arrow[r, "\epsilon N"] \arrow[d, "N\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, "\epsilon"] \\ N \arrow[r, "\epsilon"'] & \mathrm{Id}\,. \end{tikzcd} \tag{2.3}$$

This allows us to define comultiplication ν follows:

$$
\begin{tikzcd}
N \arrow[dr, dotted, "\nu"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& N^2 \arrow[r, "\epsilon N"] \arrow[d, "N\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, "\epsilon"] \\
& N \arrow[r, "\epsilon"'] & \mathrm{Id}\,.
\end{tikzcd}
\tag{2.4}
$$

Since ϵ is natural transformation, so is ν. The left and upper triangles in (2.4) witness the neutrality laws for a comonad. In fact, as seen in (2.4), ν is determined uniquely by these requirements.

Let us check the associativity law:

$$
\begin{tikzcd}
N \arrow[r, "\nu"] \arrow[d, "\nu"'] & N^2 \arrow[d, "N\nu"] \\
N^2 \arrow[r, "\nu N"] & N^3.
\end{tikzcd}
\tag{2.5}
$$

From (2.3), we get the following pullback:

$$
\begin{tikzcd}
N^3 \arrow[r, "\epsilon N^2"] \arrow[d, "N\epsilon N"'] \arrow[dr, phantom, "\lrcorner", very near start] & N^2 \arrow[d, "\epsilon N"] \\
N^2 \arrow[r, "\epsilon N"'] & N.
\end{tikzcd}
$$

It thus suffices to check that the two composites in (2.5) are equal when postcomposed with ϵN^2 and $N\epsilon N$. In both cases, this is easily verified using the neutrality laws.

Thus, we have shown that (N, ϵ, ν) is a comonad. Together with the unicity of ν observed earlier, this concludes the verification that the action on objects of the vertical maps in (2.2) are bijections.

It remains to show that the vertical maps in (2.2) are fully faithful. Since this property is preserved under pullback, it suffices to check this for the rightmost vertical map $\mathbf{Cmd} \to \mathbf{Endo}_{\mathrm{copt}}$. It is clearly faithful. To show that is full, consider objects $(\mathcal{D}_1, N_1, \epsilon_1, \nu_1)$ and $(\mathcal{D}_2, N_2, \epsilon_2, \nu_2)$ of $\mathbf{Cmd}$ with cartesian counits and a morphism

$$(V, v)\colon (N_1, \epsilon_1) \to (N_2, \epsilon_2)$$

in $\mathbf{Endo}_{\mathrm{copt}}$. To lift this morphism to $\mathbf{Cmd}$, we need to check that the following diagram commutes:

$$
\begin{tikzcd}
VN_1 \arrow[rr, "v"] \arrow[d, "V\nu_1"'] & & N_2V \arrow[d, "\nu_2 V"] \\
VN_1^2 \arrow[r, "vN_1"] & N_2VN_1 \arrow[r, "N_2 v"] & N_2^2V.
\end{tikzcd}
\tag{2.6}
$$

From (2.3) for the counit ϵ_2, we get the following pullback:

$$
\begin{tikzcd}
N_2^2V \arrow[r, "\epsilon_2 N_2 V"] \arrow[d, "N_2\epsilon_2 V"'] \arrow[dr, phantom, "\lrcorner", very near start] & N_2V \arrow[d, "\epsilon_2 V"] \\
N_2V \arrow[r, "\epsilon_2 V"'] & V.
\end{tikzcd}
$$

It thus suffices to check that the two composites in (2.6) are equal when postcomposed with $\epsilon_2 N_2 V$ and $N_2\epsilon_2 V$. For the upper right composite, we obtain v using the neutrality laws for the comonad (N_2, ϵ_2, ν_2). For the lower left composite, we calculate

$$\begin{aligned}\epsilon_2 N_2 V \circ N_2 v \circ v N_1 \circ V\nu_1 &= v \circ \epsilon_2 V N_1 \circ v N_1 \circ V\nu_1\\ &= v \circ V\epsilon_1 N_1 \circ V\nu_1\\ &= v\end{aligned}$$

and

$$\begin{aligned}N_2\epsilon_2 V \circ N_2 v \circ v N_1 \circ V\nu_1 &= N_2 V\epsilon_1 \circ v N_1 \circ V\nu_1\\ &= v \circ V N_1\epsilon_1 \circ V\nu_1\\ &= v.\end{aligned}$$

□

Lemma 2.5. *Consider an endofunctor N on a category $\mathcal{B}$ with a cartesian copointing ϵ. Then the forgetful functor $\mathrm{Coalg}(N,\epsilon) \to \mathcal{B}$ is a discrete fibration.*

Proof. Let $f\colon A \to B$ be a map in $\mathcal{B}$ and consider a coalgebra structure $v\colon B \to NB$ on B. Our goal is to show that there is a unique coalgebra structure $u\colon A \to NA$ on A such that f forms a coalgebra morphism from (A,u) to (B,v). That is, we must show that there is a unique dotted map in the diagram

$$\begin{tikzcd} A \arrow[r, dotted] \arrow[d, "f"'] & NA \arrow[r, "\epsilon_A"] \arrow[d, "Nf"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"'] \\ B \arrow[r, "v"'] & NB \arrow[r, "\epsilon_B"'] & B \end{tikzcd}$$

commutes, where we ommited drawing the horizontal composite identities. This follows from the universal property of the pullback given by the naturality square of ϵ at f. □

3. Stuff on discrete fibrations

3.1. **Connected limits in discrete fibrations.**

Lemma 3.1. *A discrete fibration that is bijective on objects is an isomorphism.*

Proof. Trivial. □

Lemma 3.2. *Let $P\colon \mathcal{E} \to \mathcal{B}$ be a discrete fibration and $F\colon \mathcal{I} \to \mathcal{E}$ be a diagram with $\mathcal{I}$ connected. Then the induced functor*

$$Q\colon\ \mathrm{const} \downarrow_{[\mathcal{I},\mathcal{E}]} F \to \mathrm{const} \downarrow_{[\mathcal{I},\mathcal{B}]} PF$$

is bijective on objects.

Proof. Since Q is a discrete fibration, so is $Q^{\mathcal{I}}\colon [\mathcal{I},\mathcal{E}] \to [\mathcal{I},\mathcal{B}]$, hence the functor

$$Q'\colon [\mathcal{I},\mathcal{E}]/F \to [\mathcal{I},\mathcal{B}]/PF$$

is an isomorphism of categories. By functoriality of the comma category construction, we have a commuting diagram of categories

$$\begin{tikzcd} \mathrm{const} \downarrow_{[\mathcal{I},\mathcal{E}]} F \arrow[r, "Q"] \arrow[d, "L"] & \mathrm{const} \downarrow_{[\mathcal{I},\mathcal{B}]} PF \arrow[d, "R"] \\ {[\mathcal{I},\mathcal{E}]/F} \arrow[r, "Q'"] & {[\mathcal{I},\mathcal{B}]/PF.} \end{tikzcd} \tag{3.1}$$

Note that L and R are injective on objects since $\mathcal{I}$ is inhabited. Since L and Q' are injective on objects, so is Q.

Let us show that Q is is surjective on objects. Consider an object (B, u), where $u\colon \operatorname{const}(B) \to PF$, of the codomain of Q. Let (Y', v) the (unique) lift of $R(B, u) = (\operatorname{const} B, u)$ through Q'. Since R is injective on objects, it remains to lift (Y', v) through L.

Since P reflects identities, we have that $Y'K = Y'K'$ and $Y'j = \mathrm{id}$ for any map $j\colon K \to K'$ in $\mathcal{I}$. Since $\mathcal{I}$ is connected, there hence is an object $Y \in \mathcal{E}$ such that $Y' = \operatorname{const} Y$. Then (Y, v) is a lift of (Y', v) through L. □

Lemma 3.3. *Let $P\colon \mathcal{E} \to \mathcal{B}$ be a discrete fibration and $F\colon \mathcal{I} \to \mathcal{E}$ be a diagram with $\mathcal{I}$ connected. Then the induced functor*

$$Q\colon \operatorname{const} \downarrow_{[\mathcal{I},\mathcal{E}]} F \to \operatorname{const} \downarrow_{[\mathcal{I},\mathcal{B}]} PF$$

is an isomorphism.

Proof. We know by Lemma 3.2 that Q is bijective on objects. Using Lemma 3.1, it remains to show that Q is a discrete fibration. So let (Y, v) be an object of its domain and $(A, s) \to (PY, Pv)$ a map in its codomain; we want to show that it has a unique lift $(X, u) \to (Y, v)$ in its codomain.

We extend F using the cone (Y, v), obtaining a new functor $G\colon 1 \star \mathcal{I} \to \mathcal{E}$. Now consider the functor

$$R\colon \operatorname{const} \downarrow_{[1\star\mathcal{I},\mathcal{E}]} G \to \operatorname{const} \downarrow_{[1\star\mathcal{I},\mathcal{B}]} PG.$$

The map $(A, s) \to (PY, Pv)$ forms an object of its codomain. Lifts $(X, u) \to (Y, v)$ correspond to lifts of this object to its codomain. By Lemma 3.2, lifts of objects through R are unique. □

Alternative to our setup, one could show directly that the diagram (3.2) in the proof of Lemma 3.2 is a pullback.

Corollary 3.4. *Discrete fibrations create connected limits.*

Proof. Using the characterization of limits as terminal objects in appropriate comma categories, this is just Lemma 3.3. □

Corollary 3.5. *Discrete fibrations create pullbacks.* □

Corollary 3.6. *Discrete fibrations create coequalizers.* □

3.2. **Discrete fibrations with right adjoints.**

Lemma 3.7. *Consider a discrete fibration $P\colon \mathcal{E} \to \mathcal{B}$ with a right adjoint (R, ϵ). Then ϵ is a cartesian natural transformation.*

Proof. Let $g\colon B \to C$ be a map in $\mathcal{B}$. We want to show that the square

$$\begin{tikzcd}
PRB \arrow[r, "PRg"] \arrow[d, "\epsilon_B"] & PRC \arrow[d, "\epsilon_C"] \\
B \arrow[r, "g"] & C
\end{tikzcd}$$

is a pullback. Let us consider a cone

$$\begin{tikzcd}
A \arrow[dr, dotted, "f"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "f'"'] & & \\
& PRB \arrow[r, "PRg"] \arrow[d, "\epsilon_B"] & PRC \arrow[d, "\epsilon_C"] \\
& B \arrow[r, "g"] & C;
\end{tikzcd} \tag{3.2}$$

our goal is to show that there is a unique map f as indicated making the diagram commute.

Let $w\colon X \to RC$ be the unique lift of h to $\mathcal{E}$, using that P is a discrete fibration. In particular, we have $A = PX$. The outer square in (3.2) states commutativity of the triangle

$$\begin{tikzcd} PX \arrow[dr, "f'"'] \arrow[drr, bend left, "\epsilon_C \circ Pw"] & & \\ & B \arrow[r, "g"] & C. \end{tikzcd} \tag{3.3}$$

We will now work with transposes with respect to the adjunction $P \dashv R$. Let us write $u\colon X \to RB$ be the transpose of f'. Using naturality of transposition in the codomain, we obtain from (3.3) the commuting triangle

$$\begin{tikzcd} X \arrow[dr, "u"'] \arrow[drr, bend left, "w"] & & \\ & RB \arrow[r, "Rg"] & RC. \end{tikzcd} \tag{3.4}$$

For existence of f in (3.2), we can now put $f =_{\mathrm{def}} Pu$. The left triangle commutes because f' is the transpose of u. The upper triangle commutes because it is the image of (3.4) under P.

For uniqueness of f, we argue as follows. Since P is a discrete fibration, we must have $f = P\overline{u}$ for some map $\overline{u}$ in $\mathcal{E}$ with codomain RB. Using the left triangle in (3.2), we have that $\overline{u}$ is the transpose of f'. This determines $\overline{u}$, and hence f, uniquely. ☐

We call an adjunction *cartesian* if the left and right adjoints are cartesian functors and the unit and counit are cartesian natural transformations. Note that this implies that the induced monad and comonad are cartesian.

Proposition 3.8. *Consider a discrete fibration $P\colon \mathcal{E} \to \mathcal{B}$ with a right adjoint R. Then the adjunction $P \dashv R$ is cartesian.*

Proof. We know that P preserves pullbacks by Corollary 3.5. Note that R preserves pullbacks as it is a right adjoint. The counit ϵ is cartesian by Lemma 3.7.

The unit η is cartesian by the following reasoning. Recall the triangle law $P\eta \circ \epsilon P = \mathrm{id}_P$. Since ϵ is cartesian, so is ϵP, and hence $P\eta$ by pullback pasting cancellation. Then η is cartesian as P reflects pullbacks by Corollary 3.5. ☐

Let us recall a few items from the theory of comonadic adjunctions. Consider a functor $P\colon \mathcal{E} \to \mathcal{B}$ with right adjoint R, denoting the unit η and counit ϵ. The adjunction $P \dashv R$ is called *comonadic* if the canonical comparison functor

$$\begin{tikzcd} \mathcal{E} \arrow[rr] \arrow[dr, "P"'] & & \mathrm{Coalg}(PR) \arrow[dl] \\ & \mathcal{B} & \end{tikzcd}$$

over $\mathcal{B}$ sending X to the comonad algebra $(PX, P\eta_X)$ is an equivalence. It is *strictly comonadic* if this comparison functor is an isomorphism. In case P is an amnestic isofibration, for example because it is a discrete fibration, comonadicity is equivalent to strict comonadicity.

There are various versions of the comonadicity theorem, which gives sufficient (and sometimes equivalent) conditions for detecting comonadicity. In our case, we will only need the following. Given an adjunction $P \dashv R$ as above, if P creates coequalizers, then $P \dashv R$ is comonadic.

Proposition 3.9. *Consider a discrete fibration $P\colon \mathcal{E} \to \mathcal{B}$ with a right adjoint R. Then the adjunction $P \dashv R$ is strictly comonadic.*

Proof. It is comonadic because P creates coequalizers by Corollary 3.6. It is then strictly comonadic as P is a discrete fibration. □

We add the subscript left-discfib to the category **Adj** to indicate that we restrict to objects where the left adjoint is a discrete fibration.

4. Variants of categories with families

4.1. Standard definition.

Definition 4.1. A *category with families (cwf, standard variation)* $\mathcal{C} = (\mathcal{C}, \mathbf{Ty}^{\mathcal{C}}, \mathbf{Tm}^{\mathcal{C}}, p^{\mathcal{C}}, q^{\mathcal{C}})$ consists of a category $\mathcal{C}$, a presheaf $\mathbf{Ty}^{\mathcal{C}}$ on $\mathcal{C}$, a presheaf $\mathbf{Tm}^{\mathcal{C}}$ on $\int \mathbf{Ty}^{\mathcal{C}}$, and for each $\Gamma \in \mathcal{C}$ and $A \in \mathbf{Ty}^{\mathcal{C}}(\Gamma)$, a universal element

$$(p^{\mathcal{C}}_{(\Gamma,A)} \colon \Gamma.A \to \Gamma, q^{\mathcal{C}}_{(\Gamma,A)} \in \mathbf{Tm}^{\mathcal{C}}(\mathbf{Ty}^{\mathcal{C}}(p^{\mathcal{C}}_{(\Gamma,A)})(A)))$$

of the presheaf

$$\mathrm{WkTm}^{\mathcal{C}}_{(\Gamma,A)} \colon (\mathcal{C}/\Gamma)^{\mathrm{op}} \longrightarrow (\textstyle\int \mathbf{Ty}^{\mathcal{C}})^{\mathrm{op}} \xrightarrow{\mathbf{Tm}^{\mathcal{C}}} \mathbf{Set}$$

where the first map is the functor sending $\sigma \colon \Delta \to \Gamma$ to $(\Delta, \mathbf{Ty}^{\mathcal{C}}(\sigma)(A))$.

We introduce some shorthand notation. We omit superscripts if they are evident from the context. The category $\mathcal{C}$ is also referred to as the *category of contexts and substitutions*. Its objects are usually denoted with uppercase greek letters $\Gamma, \Delta, \Xi, \ldots$. The elements of **Ty** and **Tm** are called *types* and *terms*, respectively.

Let $\Gamma \in \mathcal{C}$ be a context and $A \in \mathbf{Ty}(\Gamma)$ be a type over it. The context $\Gamma.A$ is called the *context extension* (of Γ) with A. The morphism $p_A \colon \Gamma.A \to \Gamma$ is called the *context projection* of A. Given also a substitution $\sigma \colon \Delta \to \Gamma$, we write $A[\sigma] =_{\mathrm{def}} \mathbf{Ty}(\sigma)(A)$ for the *substitution* of A by σ. Given additionally a term $t \in \mathbf{Tm}(\Gamma, A)$, we write $t[\sigma] =_{\mathrm{def}} \mathbf{Tm}(\sigma, A)(t)$ for the *substitution* of t by σ. Note that this notation is ambiguous if the values of the presheaf **Ty** respectively **Tm** are not disjoint.

Definition 4.2. A *lax morphism* of cwf's from $\mathcal{C}$ to $\mathcal{D}$ is a tuple (F, u, v) with a functor $F \colon \mathcal{C} \to \mathcal{D}$, a natural transformation $u \colon \mathbf{Ty}_{\mathcal{C}} \to \mathbf{Ty}_{\mathcal{D}}F$, and a natural transformation $v \colon \mathbf{Tm}_{\mathcal{C}} \to \mathbf{Tm}_{\mathcal{D}}(\int u)$.

Given a lax morphism of cwf's as in Definition 4.2 and $\Gamma \in \mathcal{C}$ with $A \in \mathbf{Ty}^{\mathcal{C}}(\Gamma)$, note that $(Fp^{\mathcal{C}}_{(\Gamma,A)}, v_{(\Gamma.A,A[p_A])}q^{\mathcal{C}}_{(\Gamma,A)})$ is an element of the presheaf $\mathrm{WkTm}^{\mathcal{D}}_{(F\Gamma,u_\Gamma A)}$. By universality of $(p^{\mathcal{D}}_{(F\Gamma,u_\Gamma A)}, q^{\mathcal{D}}_{(F\Gamma,u_\Gamma A)})$, there is thus a unique *coercion substitution*

$$\begin{tikzcd}
F(\Gamma.A) \arrow[rr, dotted, "\tau_{(\Gamma,A)}"] \arrow[dr, "F(p^{\mathcal{C}}_{(\Gamma,A)})"'] & & F\Gamma.u_\Gamma A \arrow[dl, "p^{\mathcal{D}}(F\Gamma,u_\Gamma A)"] \\
& \Gamma &
\end{tikzcd}$$

over Γ as indicated such that $v_{(\Gamma.A,A[p_A])}q^{\mathcal{C}}_{(\Gamma,A)} = q^{\mathcal{D}}_{(F\Gamma,u_\Gamma A)}[\tau_{(\Gamma,A)}]$. The lax morphism (F, u, v) is a *pseudomorphism* if all coercion substitutions are isomorphisms and a *strict morphism* if they are identities.

Lax morphisms of cwf's compose in the evident way. The identity on $\mathcal{C}$ is given by $(\mathrm{Id}_{\mathcal{C}}, \mathrm{id}, \mathrm{id})$. The composition of morphisms

$$\mathcal{C} \xrightarrow{(F,u,v)} \mathcal{D} \xrightarrow{(F',u',v')} \mathcal{E}$$

is given by $(F'F, u'F \circ u, v'(\int u) \circ v)$. Neutrality and associativity laws are easily verified.

Observe that identities in $\mathbf{CwF}_{\mathrm{lax}}$ are strict morphisms and that pseudomorphisms and strict morphisms are closed under composition. This justifies the following definition.

Definition 4.3. We have categories of cwf's (standard variation) with strict morphisms $\mathbf{CwF}_{\mathrm{strict}}$, pseudomorphisms $\mathbf{CwF}_{\mathrm{pseudo}}$, and lax morphisms $\mathbf{CwF}_{\mathrm{lax}}$.

In summary, we have a sequence of wide subcategory inclusions

$$\mathbf{CwF}_{\mathrm{strict}} \hookrightarrow \mathbf{CwF}_{\mathrm{pseudo}} \hookrightarrow \mathbf{CwF}_{\mathrm{lax}}. \tag{4.1}$$

4.2. **Slickering.** Let us look for a more categorical way of defining cwf's. The first change is to present types and terms in terms of discrete fibrations instead of presheaves. This cuts down on instances of the category of elements construction, which passes from presheaves to discrete fibrations and appears already inside Definition 4.1.

Replacing all presheaves by discrete fibrations, and overloading names in the process, we obtain we the following. A category with families $\mathcal{C}$ is equivalently given by a tower of discrete fibrations

$$\begin{tikzcd}
\mathbf{Tm}^{\mathcal{C}} \arrow[d, two heads, "Q", "\mathrm{disc}"'] \\
\mathbf{Ty}^{\mathcal{C}} \arrow[d, two heads, "P", "\mathrm{disc}"'] \\
\mathcal{C}
\end{tikzcd}$$

together with, for each $A \in \mathbf{Ty}$, a universal element of the presheaf

$$(\mathcal{C}/PA)^{\mathrm{op}} \xrightarrow{\simeq} (\mathbf{Ty}^{\mathcal{C}}/A)^{\mathrm{op}} \xrightarrow{\mathrm{dom}} (\mathbf{Ty}^{\mathcal{C}})^{\mathrm{op}} \xrightarrow{Q^{-1}} \mathbf{Set} \tag{4.2}$$

where the first map is given by the unique lifting property of discrete fibrations.

Note that the first functor in (4.2) is fact an isomorphism as indicated. Thus, we may simplify the condition without changing it by omitting it, requiring instead for each $A \in \mathbf{Ty}$ a universal element of the presheaf

$$(\mathbf{Ty}^{\mathcal{C}}/A)^{\mathrm{op}} \xrightarrow{\mathrm{dom}} (\mathbf{Ty}^{\mathcal{C}})^{\mathrm{op}} \xrightarrow{Q^{-1}} \mathbf{Set}. \tag{4.3}$$

Note that this condition does not make use of the discrete fibration $P \colon \mathbf{Ty}^{\mathcal{C}} \to \mathcal{C}$ at all anymore, only its total space.

A final simplification is achieved by noting that the category of elements of the presheaf (4.3) is just the comma category $Q \downarrow A$. A universal element of the presheaf is a terminal object in there. Thus, a choice for each $A \in \mathbf{Ty}^{\mathcal{C}}$ of a universal element of the presheaf (4.3) is nothing but a right adjoint for Q. Therefore, we obtain the following categorical definition.

Definition 4.4. A *category with families (cwf, slick variation)* $\mathcal{C}$ is a tower of discrete fibrations

$$\begin{tikzcd}
\mathbf{Tm}^{\mathcal{C}} \arrow[d, two heads, "Q^{\mathcal{C}}" near start, "\mathrm{disc}"' near end] \arrow[d, phantom, shift left=3, "\dashv"] \\
\mathbf{Ty}^{\mathcal{C}} \arrow[u, bend right, "R^{\mathcal{C}}"'] \arrow[d, two heads, "\mathrm{disc}"', "P^{\mathcal{C}}"] \\
\mathcal{C}
\end{tikzcd}$$

with a right adjoint $R^{\mathcal{C}}$ to Q as indicated.

This categorical phrasing makes the definition of categories of cwf's much easier. Let us write **DiscFib** for the full subcategory of $\mathbf{Func}_{\mathrm{strict}}$ of functors that are discrete fibrations.

Definition 4.5. The category of cwf's (slick variation) with lax morphisms is defined by the pullback

$$\begin{tikzcd}
\mathbf{CwF}_{\mathrm{lax}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Adj}_{\text{left-discfib,left-strict}} \arrow[d, "\mathrm{Cod} \circ \mathrm{Left}"] \\
\mathbf{DiscFib} \arrow[r, "\mathrm{Dom}"] & \mathbf{Cat}.
\end{tikzcd}$$

The categories of cwf's with pseudomorphisms and strict morphisms are defined by an analogous pullback, but with $\mathbf{Adj}_{\text{left-discfib,left-strict}}$ replaced by

$$\mathbf{Adj}_{\text{left-discfib,left-strict,right-pseudo}}$$

and

$$\mathbf{Adj}_{\text{left-discfib,strict}},$$

respectively.

Unfolding the components of morphisms, identities, and composition in this definition, one finds that the categories defined are indeed equivalent to Definition 4.3 and that these equivalences respect the wide subcategory inclusions of (4.1).

5. Variants of discrete comprehension categories

5.1. Standard definition.

Definition 5.1. A *discrete comprehension category (standard variation)* $\mathcal{C}$ is a commuting diagram of categories

$$\begin{tikzcd}
\mathbf{Ty}^{\mathcal{C}} \arrow[rr, "\chi^{\mathcal{C}}"] \arrow[dr, "P^{\mathcal{C}}"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{C} &
\end{tikzcd}$$

where the left map is a discrete fibration and $\chi^{\mathcal{C}}$ maps morphisms to pullback squares.

We copy some usage and terminology from cwf's. We omit superscripts if they are evident from the context. The category $\mathcal{C}$ is again referred to as the *category of contexts and substitutions*. The objects of **Ty** are called *types*. Given $\Gamma \in \mathcal{C}$, we write $\mathbf{Ty}(\Gamma) =_{\mathrm{def}} P^{-1}(\Gamma)$ for the types over Γ.

Let $A \in \mathbf{Ty}$ be a type in context $\Gamma =_{\mathrm{def}} PA$. The map $\chi(A) \colon \Gamma.A \to \Gamma$ is called the *comprehension* or *context projection* of A. Its domain is the *context extension* with A. Given also a substitution $\sigma \colon \Delta \to \Gamma$, we write $A[\sigma] \to A$ for the unique lift of σ, using that P is a discrete fibration. We call $A[\sigma]$ the *substitution* of A by σ. Observe that, in contrast to the case of categories with families, this notation is never ambiguous.

Definition 5.2. A *lax morphism* of discrete comprehension categories from $\mathcal{C}$ to $\mathcal{D}$ consists of functors F, G and a natural transformation ζ fitting into a diagram

$$\begin{tikzcd}
\mathbf{Ty}^{\mathcal{C}} \arrow[rr, "\chi^{\mathcal{C}}"] \arrow[d, "G"'] \arrow[ddr] \arrow[drr, phantom, "\Downarrow \zeta"] & & \mathcal{C}^{\to} \arrow[d, "F^{\to}"] \arrow[ddl] \\
\mathbf{Ty}^{\mathcal{D}} \arrow[rr, "\chi^{\mathcal{D}}"] \arrow[ddr] & & \mathcal{D}^{\to} \arrow[ddl] \\
& \mathcal{C} \arrow[d, "F"] & \\
& \mathcal{D} &
\end{tikzcd}$$

such that the bottom left square commutes and ζ lies over the identity, i.e. $\operatorname{cod}_{\mathcal{D}} \zeta = \mathrm{id}$.

In Definition 5.2, note that we do not require ζ to be valued in pullback squares. If this is the case, equivalently if ζ is an isomorphism, we have a *pseudomorphism*. If ζ is an identity, we have a *strict morphism*.

Lax morphisms of comprehension categories compose in the evident way. Observe that the identity lax morphism is strict and that strict morphisms and pseudomorphisms are closed composition. This justifies the following definition.

Definition 5.3. We have categories of comprehension categories (standard variation) with strict morphisms $\mathbf{CompCat}_{\mathrm{strict}}$, pseudomorphisms $\mathbf{CompCat}_{\mathrm{pseudo}}$, and lax morphisms $\mathbf{CompCat}_{\mathrm{lax}}$.

In summary, we have a sequence of wide subcategory inclusions

$$\mathbf{CompCat}_{\mathrm{strict}} \hookrightarrow \mathbf{CompCat}_{\mathrm{pseudo}} \hookrightarrow \mathbf{CompCat}_{\mathrm{lax}}. \tag{5.1}$$

5.2. **Slickering.** Suppose we are given a comprehension category

$$\begin{tikzcd}
\mathbf{Ty}^{\mathcal{C}} \arrow[rr, "\chi^{\mathcal{C}}"] \arrow[dr, "P^{\mathcal{C}}"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{C}. &
\end{tikzcd} \tag{5.2}$$

Since $P^{\mathcal{C}}$ is a discrete fibration, we have a pullback square

$$\begin{tikzcd}
(\mathbf{Ty}^{\mathcal{C}})^{\to} \arrow[r, two heads, "\mathrm{disc}"] \arrow[d, "\mathrm{cod}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\to} \arrow[d, "\mathrm{cod}"] \\
\mathbf{Ty}^{\mathcal{C}} \arrow[r, two heads, "\mathrm{disc}"] & \mathcal{C}.
\end{tikzcd} \tag{5.3}$$

Note that the top arrow in (5.3) is again a discrete fibration as indicated, for example because discrete fibrations are stable under pullback. From the pullback (5.3), we see that giving $\chi^{\mathcal{C}}$ making the diagram (5.2) commute is the same thing as giving a section to

$$\operatorname{cod}_{\mathbf{Ty}^{\mathcal{C}}} \colon (\mathbf{Ty}^{\mathcal{C}})^{\to} \to \mathbf{Ty}^{\mathcal{C}}.$$

Furthermore, since discrete fibrations create pullbacks by Corollary 3.5, we have that $\chi^{\mathcal{C}}$ maps morphisms to pullback squares exactly if the corresponding section to $\operatorname{cod}_{\mathbf{Ty}^{\mathcal{C}}} \colon (\mathbf{Ty}^{\mathcal{C}})^{\to} \to \mathbf{Ty}^{\mathcal{C}}$ has that property.

Finally, observe that a section to $\mathrm{cod}_{\mathbf{Ty}^{\mathcal{C}}}$ that maps morphisms to pullback squares is the same thing as a copointed endofunctor on $\mathbf{Ty}^{\mathcal{C}}$ with cartesian copointing. Recall from Lemmata 2.1 and 2.4 that a copointed endofunctor with cartesian copointing is the same thing as a cartesian copointed endofunctor and a cartesian comonad. Thus, we obtain the following isomorphic definition of discrete comprehension categories.

Definition 5.4. A *discrete comprehension category (slick variation)* $\mathcal{C}$ is a discrete fibration $\mathbf{Ty}^{\mathcal{C}} \to \mathcal{C}$ together with a cartesian copointed endofunctor on $\mathbf{Ty}^{\mathcal{C}}$.

This definition makes it slightly easier to specify categories of comprehension categories. Let us write $\mathbf{Endo}_{\mathrm{copt},\mathsf{cart}}$ for the full subcategory of copointed endofunctor that are cartesian and similarly for its variants.

Definition 5.5. The category of comprehension categories (slick variation) with lax morphisms is defined by the pullback

$$\begin{tikzcd} \mathbf{CompCat}_{\mathrm{lax}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Endo}_{\mathrm{copt},\mathsf{cart}} \arrow[d] \\ \mathbf{DiscFib} \arrow[r, "\mathrm{Dom}"] & \mathbf{Cat} \end{tikzcd}$$

where the right functor returns the underlying category of an endofunctor. The categories of comprehension categories with pseudomorphisms and strict morphisms are defined by an analogous pullback, but with $\mathbf{Endo}_{\mathrm{copt},\mathsf{cart}}$ replaced by

$$\mathbf{Endo}_{\mathrm{copt},\mathrm{pseudo},\mathsf{cart}}$$

and

$$\mathbf{Endo}_{\mathrm{copt},\mathrm{strict},\mathsf{cart}},$$

respectively.

Unfolding the components of morphisms, identities, and composition in this definition, one finds that the categories defined are indeed equivalent to Definition 5.3 and that these equivalences respect the wide subcategory inclusions of (5.1).

6. Comparing categories with families and discrete comprehension categories

With the framework we have built up, comparing not just categories with families and discrete comprehension categories but also the different categories thereof is now possible in a very conceptual manner.

Theorem 6.1. *Categories with families and discrete comprehension categories are equivalent in the sense of vertical equivalences making the following diagram commute:*

$$\begin{tikzcd} \mathbf{CwF}_{\mathrm{strict}} \arrow[r, hook] \arrow[d, dotted, "\approx"] & \mathbf{CwF}_{\mathrm{pseudo}} \arrow[r, hook] \arrow[d, dotted, "\approx"] & \mathbf{CwF}_{\mathrm{lax}} \arrow[d, dotted, "\approx"] \\ \mathbf{CompCat}_{\mathrm{strict}} \arrow[r, hook] & \mathbf{CompCat}_{\mathrm{pseudo}} \arrow[r, hook] & \mathbf{CompCat}_{\mathrm{lax}}. \end{tikzcd} \tag{6.1}$$

Proof. All six categories are defined in Definitions 4.5 and 5.5 via a pullback along the same functor

$$\mathbf{DiscFib} \xrightarrow{\mathrm{Dom}} \mathbf{Cat}.$$

Instead of constructing the equivalences in (6.1) directly, we may thus instead construct equivalences as in

$$\begin{tikzcd}
\mathbf{Adj}_{\text{left-discfib,strict}} \arrow[r, hook] \arrow[d, dotted, "\approx"] & \mathbf{Adj}_{\text{left-discfib,left-strict,right-pseudo}} \arrow[r, hook] \arrow[d, dotted, "\approx"] & \mathbf{Adj}_{\text{left-discfib,left-strict}} \arrow[d, dotted, "\approx"] \\
\mathbf{Endo}_{\text{copt,strict},\mathsf{cart}} \arrow[r, hook] & \mathbf{Endo}_{\text{copt,pseudo},\mathsf{cart}} \arrow[r, hook] & \mathbf{Endo}_{\text{copt},\mathsf{cart}}
\end{tikzcd} \tag{6.2}$$

where each vertical arrow must live over **Cat** via the right vertical functors in Definitions 4.5 and 5.5.

Let us first construct the rightmost equivalence in (6.2). We have a commuting diagram

$$\begin{tikzcd}
\mathbf{Adj}_{\text{left-discfib,left-strict}} \arrow[dr, "\mathrm{Cod}\circ\mathrm{Left}"'] & \mathbf{Cmd}_{\mathsf{cart}} \arrow[l] \arrow[r] \arrow[d] & \mathbf{Endo}_{\text{copt},\mathsf{cart}} \arrow[dl] \\
& \mathbf{Cat} &
\end{tikzcd} \tag{6.3}$$

where the left horizontal arrow is a restriction of the fully faithful right adjoint of the adjunction of adjunctions and comonads of Subsection 1.4, using that the associated forgetful functor from the category of coalgebras for a cartesian comonad is a discrete fibration by Lemma 2.5 and the right horizontal arrow is the functor forgetting the comultiplication.

Recall from Proposition 3.9 that the adjunctions in $\mathbf{Adj}_{\text{left-discfib,left-strict}}$ are strictly comonadic. Since the associated comonad of an adjunction in $\mathbf{Adj}_{\text{left-discfib,left-strict}}$ is cartesian by Proposition 3.8, we have that the left horizontal functor in is essentially surjective (via an explicit functor) and hence an equivalence. The second horizontal arrow is an isomorphism by (2.4).

This gives the rightmost equivalence in (6.2). The middle and leftmost equivalence are constructed using the same argument, but with modified subscripts as in the diagrams

$$\begin{tikzcd}
\mathbf{Adj}_{\text{left-discfib,left-strict,right-pseudo}} \arrow[dr, "\mathrm{Cod}\circ\mathrm{Left}"'] & \mathbf{Cmd}_{\text{pseudo},\mathsf{cart}} \arrow[l] \arrow[r] \arrow[d] & \mathbf{Endo}_{\text{copt,pseudo},\mathsf{cart}} \arrow[dl] \\
& \mathbf{Cat} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{Adj}_{\text{left-discfib,strict}} \arrow[dr, "\mathrm{Cod}\circ\mathrm{Left}"'] & \mathbf{Cmd}_{\text{strict},\mathsf{cart}} \arrow[l] \arrow[r] \arrow[d] & \mathbf{Endo}_{\text{copt,strict},\mathsf{cart}} \arrow[dl] \\
& \mathbf{Cat}. &
\end{tikzcd}$$

Furthermore, all these diagrams fit together naturally. □

7. Some tools

For this section, fix a discrete comprehension category

$$\begin{tikzcd}
\mathbf{Ty}^{\mathcal{C}} \arrow[rr, "\chi^{\mathcal{C}}"] \arrow[dr, "P^{\mathcal{C}}"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{C}. &
\end{tikzcd}$$

7.1. Telescopes. For $n \in \mathbb{N}$, we recursively define category $\mathbf{Ty}^{(n)}$ of *telescopes of length* n together with a comprehension functor $\chi^{(n)} \colon \mathbf{Ty}^{(n)} \to \mathcal{C}^{[n]^{\mathrm{op}}}$ as follows.

- At the initial stage, we set $\mathbf{Ty}^{(0)} =_{\mathrm{def}} \mathcal{C}$ and let $\chi^{(0)} \colon \mathcal{C} \to \mathcal{C}^{[0]}$ be the canonical isomorphism.
- At the successor stage, we define $\mathbf{Ty}^{(n+1)}$ via the pullback

$$\begin{tikzcd}
\mathbf{Ty}^{(n+1)} \arrow[rr] \arrow[d, two heads, "\mathrm{disc}"', "P^{(n+1)}_{(n)}"] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathbf{Ty} \arrow[d, two heads, "\mathrm{disc}"', "P"] \\
\mathbf{Ty}^{(n)} \arrow[r, "\chi^{(n)}"] & \mathcal{C}^{[n]^{\mathrm{op}}} \arrow[r, "\mathcal{C}^{(0)}"] & \mathcal{C}
\end{tikzcd}$$

and $\chi^{(n+1)} \colon \mathbf{Ty}^{(n+1)} \to \mathcal{C}^{[n+1]^{\mathrm{op}}}$ via the induced map between pullbacks as indicated below:

$$\begin{tikzcd}
\mathbf{Ty}^{(n+1)} \arrow[rr] \arrow[dr] \arrow[dd, dotted, "\chi^{(n+1)}"'] & & \mathbf{Ty} \arrow[dr] \arrow[dd, "\chi"' near start] & \\
& \mathbf{Ty}^{(n)} \arrow[rr, crossing over] & & \mathcal{C} \arrow[dd, equal] \\
\mathcal{C}^{[n+1]^{\mathrm{op}}} \arrow[rr] \arrow[dr] & & \mathcal{C}^{\to} \arrow[dr, "\mathrm{cod}"] & \\
& \mathcal{C}^{[n]^{\mathrm{op}}} \arrow[rr, "\mathcal{C}^{\{n\}}"] \arrow[from=uu, crossing over, "\chi^{(n)}"'] & & \mathcal{C}.
\end{tikzcd}$$

We write $P^{(n)}$ for the following composite of functors: Consider the diagram

$$\begin{tikzcd}
\mathbf{Ty}^{(n)} \arrow[rr, "\chi^{(n)}"] \arrow[dr, "P^{(n)}"'] & & \mathcal{C}^{[n]^{\mathrm{op}}} \arrow[dl, "\mathcal{C}^{\{0\}}"] \\
& \mathcal{C}. &
\end{tikzcd}$$

By induction on n, we see that $P^{(n)}$ is a discrete fibration and that $\chi^{(n)}$ as a morphism of functors over $\mathcal{C}$ is cartesian, i.e. in this situation sends morphisms to cartesian natural transformations.

We denote an object of $\mathbf{Ty}^{(n)}$ as a tuple $(\Gamma; A_1, \ldots, A_n)$ with $\Gamma \in \mathcal{C}$ and $A_i \in \mathbf{Ty}(\Gamma_{i-1})$ for $i \in \{1, \ldots, n\}$ where $\Gamma_0 =_{\mathrm{def}} \Gamma$ and $\Gamma_i =_{\mathrm{def}} \Gamma_{i-1}.A_i$. A morphism from $(\Delta; B_1, \ldots, B_n)$ to $(\Gamma; A_1, \ldots, A_n)$ then is a substitution $\sigma \colon \Delta \to \Gamma$ such that $B_i = A_i[\sigma_{i-1}]$ where $\sigma_0 =_{\mathrm{def}} \sigma$ and $\sigma_i =_{\mathrm{def}} \sigma_{i-1}.A_i$ for $i \in \{1, \ldots, n\}$.

7.2. Pullbacks along types. Consider a context $\Gamma \in \mathcal{C}$, a substitution $\sigma \colon \Delta \to \Gamma$, and a type $A \in \mathbf{Ty}(\Gamma)$. The substitution σ lifts to a map $A[\sigma] \to A$ between types. Applying comprehension, we obtain a pullback square

$$\begin{tikzcd}
\Delta.A[\sigma] \arrow[r, "\sigma.A"] \arrow[d, "p_{A[\sigma]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "p_A"] \\
\Delta \arrow[r, "\sigma"] & \Gamma.
\end{tikzcd}$$

Ordinarily, we view this situation as the base change $A[\sigma]$ of the type A along the substitution σ. Dually, however, we may also view it as the pullback $\sigma.A$ of the substitution σ along a type A. This operation lifts to a functor, which we write

$$(-).(-) \colon \mathcal{C}^{\to} \times_{\mathcal{C}} \mathbf{Ty} \to \mathcal{C}^{\to}$$

where the pullback is taken with respect to cod and P.

8. Dependent sums and products

Consider a discrete comprehension category

$$\begin{tikzcd} \mathbf{Ty}^{\mathcal{C}} \arrow[rr, "\chi^{\mathcal{C}}"] \arrow[dr, "P^{\mathcal{C}}"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C}. & \end{tikzcd}$$

We have a pullback functor

$$\mathbf{Ty} \times_{\mathcal{C}} \mathcal{C}^{\to} \to \mathcal{C}^{\to}$$

$$\mathbf{Ty}^{(2)} \times_{\mathcal{C}} \mathbf{Ty} \to \mathcal{C}^{\to}$$

Definition 8.1. The *category of dependent products* $\mathcal{C}_\Pi$ is defined with a functor $\mathcal{C}_\Pi \to \mathbf{Ty}^{(2)}$ as follows.

- An object over $(\Gamma; A, B)$ consists of a type $\Pi_A B \in \mathbf{Ty}(\Gamma)$, and a bijection eval from $\mathcal{C}(\Gamma, \Gamma.\Pi_A B)$ to $\mathcal{C}(\Gamma.A, \Gamma.A.B)$.
- A morphism over
$$(\sigma, m, n) \colon (\Gamma, A, B) \to (\Gamma', A', B')$$
from $(\Pi_{A'} B', \mathrm{eval}')$ to $(\Pi_A B, \mathrm{eval})$ consists of a map $o \colon \Delta.\Pi_{A'} B' \to \Gamma.\Pi_A B$ over σ such that $m^* \circ \mathrm{eval} = \mathrm{eval} \circ o^*$.

Let I be a set.

Definition 8.2. The *category of weak I-coproducts* $\mathcal{C}_+$ is defined as follows.

- An object consists of a context $\Gamma \in \mathcal{C}$, types $A_i \in \mathbf{Ty}(\Gamma)$ for $i \in I$, a type $X \in \mathbf{Ty}(\Gamma)$, and maps $f_i \colon \Gamma.A_i \to \Gamma.X$ over Γ for $i \in I$.
- A morphism from (Δ, A', X', f') to (Γ, A, X, f) consists of a substitution $\sigma \colon \Delta \to \Gamma$, maps $m_i \colon A'_i \to A_i$ over σ for $i \in I$, and a map $u \colon \Delta.X' \to \Gamma.X$ over σ such that the diagram
$$\begin{tikzcd} \Delta.A'_i \arrow[r, "f'_i"] \arrow[d, "\chi(m_i)"] & \Delta.X' \arrow[d, "u"] \\ \Gamma.A_i \arrow[r, "f_i"] & \Gamma.X \end{tikzcd}$$
commutes for each $i \in I$.

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d, two heads, "\cong"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, two heads, "\cong"', "P"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{A} \end{tikzcd}$$

Given a context $(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}})$ in $\mathcal{D}$. Given types $(C_i \in \mathbf{Ty}(\Gamma_{\mathcal{C}}), B_i \in \mathbf{Ty}(\Gamma_{\mathcal{B}}))$ over it. We want to take their coproduct. So we take $C_1 + C_2 \in \mathbf{Ty}(\Gamma_{\mathcal{C}})$ and $B_1 + B_2 \in \mathbf{Ty}(\Gamma_{\mathcal{B}})$. But these do not map to the same thing in $\mathcal{A}$: $F(C_1 + C_2)$ vs $P(B_1 + B_2)$ in $\mathbf{Ty}(\Gamma_{\mathcal{A}})$.

Well, P is a strict map. So $P(B_1 + B_2) = P(B_1) + P(B_2) = F(C_1) + F(C_2)$. Have a canonical map $F(C_1) + F(C_2) \to F(C_1 + C_2)$.

$\mathcal{C}(\Delta, \Delta.\Pi_{A'} B')$ $\mathcal{C}(\Gamma, \Gamma.\Pi_A B)$

Definition 8.3. The *category of dependent sums* $\mathcal{C}_\Sigma$ is defined with a functor $\mathcal{C}_\Sigma \to \mathbf{Ty}^{(2)}$ as follows.

- An object over $(\Gamma; A, B)$ is a pair $(\Sigma_A B, \mathrm{snd})$ with $\Sigma_A B \in \mathbf{Ty}(\Gamma)$ and a map
$$\mathrm{snd}\colon \chi_{\Gamma.A}(B) \to \chi_\Gamma(\Sigma_A B).A$$
in $\mathcal{C}/\Gamma.A$ such that $(\chi_\Gamma(\Sigma_A B), \mathrm{snd})$ is an initial object in $\chi_{\Gamma.A}(B) \downarrow (-).A$.
- A morphism over
$$(\sigma, m, n)\colon (\Gamma, A, B) \to (\Gamma', A', B')$$
from $(\Sigma_A B, \eta)$ to $(\Sigma_{A'} B', \eta')$ is a map $o\colon \Sigma_{A'} B' \to \Sigma_A B$ in $\mathbf{Ty}$ over σ such that
$$\begin{tikzcd} \chi(B') \arrow[r, "\chi(n)"] \arrow[d, "\mathrm{snd}'"] & \chi(B) \arrow[d, "\mathrm{snd}"] \\ \chi(\Sigma_{A'} B').A' \arrow[r, "\chi(o).m"] & \chi(\Sigma_A B).A \end{tikzcd}$$
commutes.
- Identities and compositions are defined using that in $\mathbf{Ty}$.

Formally, we may write $\mathcal{C}_\Sigma$ as a full subcategory of the inserter
$$\begin{tikzcd} \mathbf{Ty}^{(2)} \times_{\mathcal{C}} \mathbf{Ty} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathcal{C}^{\to} \end{tikzcd}$$
where the pullback is taken with respect to $P^{(2)}$ and P, the functor F is given by
$$\mathbf{Ty}^{(2)} \times_{\mathcal{C}} \mathbf{Ty} \xrightarrow{\pi_0} \mathbf{Ty}^{(2)} \xrightarrow{\chi^{(2)}} \mathcal{C}^{[2]^{\mathrm{op}}} \xrightarrow{\mathcal{C}^{\{1,2\}}} \mathcal{C}^{\to}$$
and the functor G is given by
$$\mathbf{Ty}^{(2)} \times_{\mathcal{C}} \mathbf{Ty} \xrightarrow{P^{(2)}_{(1)} \times_{\mathcal{C}} \chi} \mathbf{Ty} \times_{\mathcal{C}} \mathcal{C}^{\to} \xrightarrow{(-).(-)} \mathcal{C}^{\to}.$$
We restrict to those objects whose inserter map lies over the identity at the functor ....

$$\begin{tikzcd} & (\Gamma : \mathcal{C}, A : \mathbf{Ty}(\Gamma), b : \mathcal{C}/\Gamma.A, x : \mathcal{C}/A, \mathcal{C}/\Gamma.A(b, x.A))c \arrow[d] \\ \mathbf{Ty}^{(2)} \arrow[r] \arrow[d] & \mathbf{Ty} \times_{\mathcal{C}} \mathcal{C}^{\to} \\ \mathcal{C} & \end{tikzcd}$$

Let $A \in \mathbf{Ty}$, with comprehension $p_A\colon \Gamma.A \to \Gamma$. We denote $(-).A$ the induced functor in the following commuting diagram:
$$\begin{tikzcd} \mathbf{Ty}/A \arrow[r, "N"] \arrow[d, "\simeq"] & \mathbf{Ty}/NA \arrow[d, "\simeq"] \\ \mathcal{C}/\Gamma \arrow[r, dotted, "(-).A"] & \mathcal{C}/\Gamma.A. \end{tikzcd}$$

Note that the vertical arrows are isomorphism since P is a discrete fibration. Since the counit ϵ of N is cartesian, observe that the top functor in the above diagram is a pullback functor in $\mathbf{Ty}$ along $\epsilon_A \colon NA \to A$ and hence so $(-).A$ in $\mathcal{C}$ along p_A.

Definition 8.4. Given $A \in \mathbf{Ty}(\Gamma)$ and $B \in \mathbf{Ty}(\Gamma.A)$, a *dependent sum* of A and B is a type $\Sigma_A B \in \mathbf{Ty}(\Gamma)$ and a map

$$\eta \colon \chi_{\Gamma.A}(B) \to \chi_\Gamma(\Sigma_A B).A$$

in $\mathcal{C}/\Gamma.A$ such that $(\chi_\Gamma(\Sigma_A B), \eta)$ is an initial object in $\chi_{\Gamma.A}(B) \downarrow (-).A$.

Consider a substitution $\sigma \colon \Gamma' \to \Gamma$, a map $m \colon A' \to A$ in $\mathbf{Ty}$ over Γ, and a map $n \colon B' \to B$ in $\mathbf{Ty}$ over $\sigma.m$. Consider dependent sums $(\Sigma_A B, \eta)$ and $(\Sigma_{A'} B', \eta')$. The dependent sums are called coherent if there is a morphism $o \colon \Sigma_A B \to \Sigma_{A'} B'$ over σ

Definition 8.5. The *category of dependent sums* in $\mathcal{C}$ is defined as follows.

- The objects are tuples $(\Gamma, A, B, \Sigma_A B, \eta)$ with a context Γ, types $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma.A)$, and $\Sigma_A B \in \mathbf{Ty}(\Gamma)$, and a map
$$\eta \colon \chi_{\Gamma.A}(B) \to \chi_\Gamma(\Sigma_A B).A$$
$$\eta \colon \chi_{\Gamma.A}(B) \to \chi_{\Gamma.A}((\Sigma_A B)[\sigma])$$
in $\mathcal{C}/\Gamma.A$ such that $(\chi_\Gamma(\Sigma_A B), \eta)$ is an initial object in $\chi_{\Gamma.A}(B) \downarrow (-).A$.
- The morphisms
$$(\Gamma, A, B, \Sigma_A B, \eta) \to (\Gamma', A', B', \Sigma_{A'} B', \eta')$$
are tuples (σ, m, n, o) with a substitution $\sigma \colon \Gamma' \to \Gamma$, a map $m \colon A' \to A$ in $\mathbf{Ty}$ over Γ, a map $n \colon B' \to B$ in $\mathbf{Ty}$ over $\sigma.m$, and a map $o \colon \Sigma_{A'} B' \to \Sigma_A B$ in $\mathbf{Ty}$ over σ such that
$$\begin{tikzcd} \chi(B') \arrow[r, "\chi(n)"] \arrow[d, "\eta'"] & \chi(B) \arrow[d, "\eta"] \\ \chi((\Sigma_{A'} B')[p_{A'}]) \arrow[r, "\chi(o[p_m])"] & \chi((\Sigma_A B)[p_A]) \end{tikzcd}$$
commutes.
- Identities and compositions are defined componentwise.

Note that
$$\eta \colon \chi_{\Gamma.A}(B) \to \chi_\Gamma(\Sigma_A B).A$$
may also be written
$$\eta \colon \chi_{\Gamma.A}(B) \to \chi_{\Gamma.A}((\Sigma_A B)[p_A])$$
Fix a substitution $\sigma \colon \Delta \to \Gamma$. Consider types $A \in \mathbf{Ty}(\Gamma)$ and $B \in \mathbf{Ty}(\Gamma.A)$ with a dependent sum $(\Sigma_A B, \eta)$. We have morphisms in $\mathbf{Ty}$ of
$$A[\sigma] \to A$$
over σ and
$$B[\sigma.A] \to B$$
over $\sigma.A$ and
$$(\Sigma_A B)[\sigma] \to \Sigma_A B$$
over σ. Applying χ, we get pullback squares
$$\begin{tikzcd} \Delta.A[\sigma] \arrow[r] \arrow[d] & \Gamma.A \arrow[d] \\ \Delta \arrow[r] & \Gamma \end{tikzcd}$$

and

$$\begin{tikzcd}
\Delta.A[\sigma].B[\sigma.A] \arrow[r] \arrow[d] & \Gamma.A.B \arrow[d] \\
\Delta.A[\sigma] \arrow[r] & \Gamma.A
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Delta.(\Sigma_A B)[\sigma] \arrow[r] \arrow[d] & \Gamma.\Sigma_A B \arrow[d] \\
\Delta \arrow[r] & \Gamma
\end{tikzcd}$$

$$\eta\colon \chi_{\Delta.A[\sigma]}(B[\sigma.A]) \to \chi_{\Delta}((\Sigma_A B)[\sigma]).A[\sigma]$$

Definition 8.6. Given $A \in \mathbf{Ty}(\Gamma)$ and $B \in \mathbf{Ty}(\Gamma.A)$, a *dependent product* of A and B is a type $\Pi_A B \in \mathbf{Ty}(\Gamma)$ and a map

$$\epsilon\colon \chi_\Gamma(\Sigma_A B).A \to \chi_{\Gamma.A}(B)$$

in $\mathcal{C}/\Gamma.A$ such that $(\chi_\Gamma(\Sigma_A B), \epsilon)$ is a terminal object in $(-).A \downarrow \chi_{\Gamma.A}(B)$.

Definition 8.7. Let $A \in \mathbf{Ty}(\Gamma)$, $B \in \mathbf{Ty}(\Gamma.A)$, and $\sigma\colon \Delta \to \Gamma$.

Remark 8.8. Fix $A \in \mathbf{Ty}(\Gamma)$. A choice of dependent sums for A and any $B \in \mathbf{Ty}(\Gamma.A)$ corresponds to a function

$$\Sigma_A\colon \mathbf{Ty}(\Gamma.A) \to \mathbf{Ty}(\Gamma)$$

and a natural transformation ϵ as in

$$\begin{tikzcd}
\mathbf{Ty}(\Gamma.A) \arrow[r, dotted, "\Sigma_A"] \arrow[d, "\chi_{\Gamma.A}"'] \arrow[dr, dotted, "\chi_\Gamma\Sigma_A"] & \mathbf{Ty}(\Gamma) \arrow[d, "\chi_\Gamma"] \\
\mathcal{C}/\Gamma.A \arrow[ur, phantom, "\Swarrow\epsilon"] & \mathcal{C}/\Gamma \arrow[l, "(-).A"]
\end{tikzcd}$$

witnessing $\chi_\Gamma\Sigma_A$ as the $\chi_{\Gamma.A}$-relative left adjoint of $(-).A$.

Dually, a choice of dependent products for A and any $B \in \mathbf{Ty}(\Gamma.A)$ corresponds to a function

$$\Pi_A\colon \mathbf{Ty}(\Gamma.A) \to \mathbf{Ty}(\Gamma)$$

and a natural transformation η as in

$$\begin{tikzcd}
\mathbf{Ty}(\Gamma.A) \arrow[r, dotted, "\Pi_A"] \arrow[d, "\chi_{\Gamma.A}"'] \arrow[dr, dotted, "\chi_\Gamma\Pi_A"] & \mathbf{Ty}(\Gamma) \arrow[d, "\chi_\Gamma"] \\
\mathcal{C}/\Gamma.A \arrow[ur, phantom, "\eta\Nearrow"] & \mathcal{C}/\Gamma \arrow[l, "(-).A"]
\end{tikzcd}$$

witnessing $\chi_\Gamma\Pi_A$ as the $\chi_{\Gamma.A}$-relative right adjoint of $(-).A$.

9. Other stuff

9.1. **General tools for discrete fibrations.** where the bottom functor maps a fibration to its total space and the right functor maps an adjunction to the target of its left adjoint.

The category of comprehension categories is defined by the pullback

$$\begin{tikzcd}
\mathbf{CompCat}_{\mathrm{lax}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Endo}_{\mathrm{copt}} \arrow[d] \\
\mathbf{DiscFib} \arrow[r] & \mathbf{Cat}
\end{tikzcd}$$

where the bottom functor maps a fibration to its total space and the right functor maps a copointed endofunctor to the underlying category.

Definition 9.1. The category $\mathbf{A}_{\mathrm{lax}}$ is defined as follows.

- The objects are tuples (P, R, ϵ) with a discrete fibration P having a right adjoint (R, ϵ). Often in our notation, we will leave the counit implicit.
- The morphism structure is created from $\mathbf{Cat}^{\to}$ via the map sending (P, R, ϵ) to P. In particular, a morphism from (P_1, R_1, ϵ_1) to (P_2, R_2, ϵ_2), where we write $P_i \colon \mathcal{E}_i \to \mathcal{B}_i$ for $i \in \{1, 2\}$, is a pair (V, U) of functors making the below square commute:

$$\begin{tikzcd} \mathcal{E}_1 \arrow[r, "V"] \arrow[d, "P_1"] & \mathcal{E}_2 \arrow[d, "P_2"] \\ \mathcal{B}_1 \arrow[r, "U"] & \mathcal{B}_2. \end{tikzcd}$$

We refer to the morphisms of $\mathbf{A}_{\mathrm{lax}}$ as *lax morphisms*.

Consider a lax morphism (V, U) as denoted in Definition 9.1. The natural isomorphism $\mathrm{id} \colon P_2 V \to U P_1$ transposes to a natural transformation $q \colon V R_1 \to R_2 U$. We call (V, U) a *strong morphism* if q is an isomorphism and a *strict morphism* if it is an identity. Observe that strong and strict morphisms are closed under finitary composition.

Definition 9.2. The categories $\mathbf{A}_{\mathrm{pseudo}}$ and $\mathbf{A}_{\mathrm{strict}}$ are the wide subcategories of $\mathbf{A}_{\mathrm{lax}}$ consisting of pseudo and strict morphisms, respectively.

In summary, we have a sequence of wide subcategory inclusions

$$\mathbf{A}_{\mathrm{strict}} \hookrightarrow \mathbf{A}_{\mathrm{pseudo}} \hookrightarrow \mathbf{A}_{\mathrm{lax}}.$$

Observe that this is simply sequence of wide subcategory inclusions

$$\mathbf{Adj}_{\mathrm{strict}} \hookrightarrow \mathbf{Adj}_{\mathrm{pseudo,left\text{-}strict}} \hookrightarrow \mathbf{Adj}_{\mathrm{left\text{-}strict}}$$

restricted to objects whose left adjoint is a discrete fibration.

Definition 9.3. The category $\mathbf{CompCat}_{\mathrm{lax}}$ is defined as follows.

- The objects are tuples $(\mathcal{B}, F, \epsilon)$ with a category $\mathcal{B}$, an endofunctor F on $\mathcal{B}$, and a copointing $\epsilon \colon F \to \mathrm{Id}$ that is cartesian.
- The morphism structure is created from the category of copointed endofunctors and lax morphisms. In particular, a morphism from $(\mathcal{B}_1, F_1, \epsilon_1)$ to $(\mathcal{B}_2, F_2, \epsilon_2)$ is a tuple (U, λ) with a functor $U \colon \mathcal{B}_1 \to \mathcal{B}_2$ and a natural transformation $q \colon U F_1 \to F_2 U$ making the following triangle commute:

$$\begin{tikzcd} U F_1 \arrow[rr, "q"] \arrow[dr, "U\epsilon_1"'] & & F_2 U \arrow[dl, "\epsilon_2 U"] \\ & U. & \end{tikzcd}$$

We refer to the morphisms of $\mathbf{CompCat}_{\mathrm{lax}}$ as *lax morphisms*. Consider a lax morphism (U, q) as denoted in Definition 9.3. We call (U, q) a *strong morphism* if the natural transformation q is cartesian and a *strict morphism* if it is an identity. Observe that strong and strict morphisms are closed under finitary composition.

Definition 9.4. The categories $\mathbf{CompCat}_{\mathrm{pseudo}}$ and $\mathbf{CompCat}_{\mathrm{strict}}$ are the wide subcategories of $\mathbf{CompCat}_{\mathrm{lax}}$ consisting of pseudomorphisms and strict morphisms, respectively.

In summary, we have a sequence of wide subcategory inclusions

$$\mathbf{CompCat}_{\mathrm{strict}} \hookrightarrow \mathbf{CompCat}_{\mathrm{pseudo}} \hookrightarrow \mathbf{CompCat}_{\mathrm{lax}}.$$

Observe that this is simply the sequence of wide subcategory inclusions

$$\mathbf{Endo}_{\mathrm{copt,strict}} \hookrightarrow \mathbf{Endo}_{\mathrm{copt,pseudo}} \hookrightarrow \mathbf{Endo}_{\mathrm{copt}}$$

restricted to objects with cartesian copointings.

We can also describe the categories of Definition 9.4 directly, using the copointed endofunctor $((-)^{\to}, \mathrm{cod}_{\mathsf{cart}})$ where $\mathrm{cod}'_{\mathcal{B}} \colon \mathcal{B}^{\to}_{\mathsf{cart}} \to \mathcal{B}$ on $\mathbf{Cat}$:

- $\mathbf{CompCat}_{\mathrm{strict}}$ is the category of algebras for $((-)^{\to}, \mathrm{cod}_{\mathsf{cart}})$,
- $\mathbf{CompCat}_{\mathrm{pseudo}}$ is the category of algebras and pseudomorphisms for $((-)^{\to}, \mathrm{cod}_{\mathsf{cart}})$ seen as a copointed 2-endofunctor,

Lemma 9.5. *There is an equivalence* $\mathbf{A}_{\mathrm{lax}} \approx \mathbf{CompCat}_{\mathrm{lax}}$. *It restricts to equivalences* $\mathbf{A}_{\mathrm{pseudo}} \simeq \mathbf{CompCat}_{\mathrm{pseudo}}$ *and* $\mathbf{A}_{\mathrm{strict}} \simeq \mathbf{CompCat}_{\mathrm{strict}}$.

Proof. Given $\mathcal{E} \to \mathcal{B}$, we returns the comonad RP. Given a cartesian copointed endofunctor R on $\mathcal{B}$, we let $\mathcal{E}$ be the category of its coalgebras.

In one direction, $\mathcal{E} \to \mathrm{Coalg}(PR)$ needs to be an isomorphism somehow.

In the other direction, we get the identity.

A -> NA

Given $X \in \mathcal{E}$, why does X arise as a □

Lemma 9.6. *Let $\mathcal{B}$ be a category. Then there is an equivalence of categories between:*

(i) discrete fibrations $P \colon \mathcal{E} \to \mathcal{B}$ with a right adjoint (R, ϵ).
(ii) sections to the forgetful functor $\mathrm{cod} \colon \mathcal{B}^{\to}_{\mathsf{cart}} \to \mathcal{B}$.

Proof. We have a functor from (i) to (ii) by sending (P, R, ϵ) to the functor $\epsilon \colon \mathcal{B} \to \mathcal{B}^{\to}_{\mathsf{cart}}$. Its codomain is justified by (3.7). Recall that the codomain of the natural transformation ϵ is the identity, making the functor ϵ a section to cod.

The functor

$$\mathcal{B}^{\to}_{\mathsf{cart},\mathrm{section}} \to \mathcal{B}^{\to}_{\mathsf{cart}}$$

□

Definition 9.7. A *category with families (version A)* $\mathcal{C}$ consists of a category $\mathcal{C}$, a presheaf $\mathbf{Ty}_{\mathcal{C}}$ on $\mathcal{C}$, a presheaf $\mathbf{Tm}_{\mathcal{C}}$ on $\int \mathbf{Ty}_{\mathcal{C}}$, and for each $\Gamma \in \mathcal{C}$ and $A \in \mathbf{Ty}_{\mathcal{C}}(\Gamma)$, a universal element

$$(p_A \colon \Gamma.A \to \Gamma, q_A \in \mathbf{Tm}_{\mathcal{C}}(\mathbf{Ty}_{\mathcal{C}}(p_A)(A)))$$

for the presheaf

$$(\mathcal{C}/\Gamma)^{\mathrm{op}} \xrightarrow{F} (\textstyle\int \mathbf{Ty}_{\mathcal{C}})^{\mathrm{op}} \xrightarrow{\mathbf{Tm}_{\mathcal{C}}} \mathbf{Set}$$

where F is the functor sending $\sigma \colon \Delta \to \Gamma$ to $(\Delta, \mathbf{Ty}_{\mathcal{C}}(\sigma)(A))$.

Definition 9.8. A *category with families (version B)* $\mathcal{C}$ consists of a category $\mathcal{C}$, a presheaf $\mathbf{Ty}_{\mathcal{C}}$ on $\mathcal{C}$, a presheaf $\mathbf{Tm}_{\mathcal{C}}$ on $\int \mathbf{Ty}_{\mathcal{C}}$, and a right adjoint to the Grothendieck construction $\int \mathbf{Tm}_{\mathcal{C}} \to \int \mathbf{Ty}_{\mathcal{C}}$.

Definition 9.9. A *category with families (version B)* $\mathcal{C}$ consists of a category $\mathcal{C}$, a presheaf $\mathbf{Ty}_{\mathcal{C}}$ on $\mathcal{C}$, a presheaf $\mathbf{Tm}_{\mathcal{C}}$ on $\int \mathbf{Ty}_{\mathcal{C}}$, and a section to the functor

$$(\textstyle\int \mathbf{Ty}_{\mathcal{C}})^{\to}_{\mathsf{cart}} \xrightarrow{\mathrm{cod}} (\textstyle\int \mathbf{Ty}_{\mathcal{C}})^{\to}.$$

Definition 9.10. A *category with families (comprehension category version)* (cwf) $\mathcal{C} = (\mathcal{C}, \mathbf{Ty}_{\mathcal{C}}, \chi_{\mathcal{C}})$ consists of a category $\mathcal{C}$, a presheaf $\mathbf{Ty}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, and a cartesian morphism $\chi_{\mathcal{C}} \colon \int \mathbf{Ty}_{\mathcal{C}} \to \mathcal{C}^{\to}$ making the following diagram commute strictly:

$$\begin{tikzcd}
\int \mathbf{Ty}_{\mathcal{C}} \arrow[rr, "\chi_{\mathcal{C}}"] \arrow[dr, "\pi_1"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}_{\mathcal{C}}"] \\
& \mathcal{C}. &
\end{tikzcd}$$

10. Categories of cwf's

Definition 10.1. A *category with families* (cwf) $\mathcal{C} = (\mathcal{C}, \mathbf{Ty}_{\mathcal{C}}, \chi_{\mathcal{C}})$ consists of a category $\mathcal{C}$, a presheaf $\mathbf{Ty}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, and a cartesian morphism $\chi_{\mathcal{C}} \colon \int \mathbf{Ty}_{\mathcal{C}} \to \mathcal{C}^{\to}$ making the following diagram commute strictly:

$$\begin{tikzcd}
\int \mathbf{Ty}_{\mathcal{C}} \arrow[rr, "\chi_{\mathcal{C}}"] \arrow[dr, "\pi_1"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}_{\mathcal{C}}"] \\
& \mathcal{C}. &
\end{tikzcd}$$

Note that $\int \mathbf{Ty}_{\mathcal{C}} \to \mathcal{C}$ is a discrete fibration, hence every morphism in its domain is cartesian. The requirement of $\chi_{\mathcal{C}}$ preserving cartesian arrows can thus be equivalently expressed by giving it signature $\chi_{\mathcal{C}} \colon \int \mathbf{Ty}_{\mathcal{C}} \to \mathcal{C}^{\to}_{\mathsf{cart}}$.

We introduce some shorthand notation. We omit certain subscripts if they are evident from the context. The category $\mathcal{C}$ is also referred to as the *category of contexts and substitutions*. Its objects are usually denoted with uppercase greek letters $\Gamma, \Delta, \Xi, \ldots$. Let $\Gamma \in \mathcal{C}$ be a context and $A \in \mathbf{Ty}(\Gamma)$ be a type over it. We write $p_A \colon \Gamma.A \to \Gamma$ for the *comprehension* or *context extension* of A, its image under χ. Given a substitution $\sigma \colon \Delta \to \Gamma$, we write $A[\sigma] =_{\mathrm{def}} \mathbf{Ty}(\sigma)(A)$ for the *substitution of A by σ*, the image of A under the action of the presheaf $\mathbf{Ty}$ on σ. We also write $p_\sigma \colon p_{A[\sigma]} \to p_A$ for the image of the morphism $(\Delta, A[\sigma]) \xrightarrow{\sigma} (\Gamma, A)$ in $\int \mathbf{Ty}$ under χ:

$$\begin{tikzcd}
\Delta.A[\sigma] \arrow[r, "\sigma.A"] \arrow[d, "p_{A[\sigma]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "p_A"] \\
\Delta \arrow[r, "\sigma"] & \Gamma;
\end{tikzcd}$$

as indicated in the diagram, $\sigma.A =_{\mathrm{def}} \mathrm{dom}_{\mathcal{C}}(p_\sigma)$. When working with objects and morphisms of $\mathcal{C}^{\to}$ or $\mathcal{C}/\Gamma$ (where $\Gamma \in \mathcal{C}$), we may regard them as objects and morphisms of $\mathcal{C}$, leaving the application of the domain functor $\mathrm{dom}_{\mathcal{C}} \colon \mathcal{C}^{\to} \to \mathcal{C}$ or the forgetful functor $\mathcal{C}/\Gamma \to \mathcal{C}$ implicit.

Definition 10.2. A *lax morphism* $F = (F, u_F, v_F)$ of cwf's from $\mathcal{C}$ to $\mathcal{D}$ consists of a functor $F \colon \mathcal{C} \to \mathcal{D}$, a natural transformation $u_F \colon \mathbf{Ty}_{\mathcal{C}} \to \mathbf{Ty}_{\mathcal{D}} F$, and a natural transformation $v_F \colon F^{\to} \chi_{\mathcal{C}} \to \chi_{\mathcal{D}}(\int u_F)$ of functors $\int \mathbf{Ty}_{\mathcal{C}} \to \mathcal{D}^{\to}$ lying strictly over the identity on F (i.e. $\operatorname{cod}_{\mathcal{C}} v_F = \mathrm{id}_{F\pi_1}$):

$$\begin{tikzcd}
\int \mathbf{Ty}_{\mathcal{C}} \arrow[rr, "\chi_{\mathcal{C}}"] \arrow[d, "\int u_F"'] \arrow[ddr] & \arrow[d, phantom, "\Downarrow v_F"] & \mathcal{C}^{\to} \arrow[d, "F^{\to}"] \arrow[ddl] \\
\int \mathbf{Ty}_{\mathcal{D}} \arrow[rr, "\chi_{\mathcal{D}}"] \arrow[dddr] & {} & \mathcal{D}^{\to} \arrow[dddl] \\
& \mathcal{C} \arrow[dd, "F"] & \\
& & \\
& \mathcal{D}. &
\end{tikzcd}$$

This is a *morphism* if u is invertible. It is a *strict morphism* if u is an identity natural transformation.

We introduce some shorthand notation. We omit certain subscripts if they are evident from the context. When applied to a type $A \in \mathbf{Ty}_C(\Gamma)$, we usually write just FA for the image $(u_F)_\Gamma(A)$ of A under the component of the natural transformation u_F at Γ.

Every cwf admits a strict identity morphism. There is an evident notion of composition of lax morphisms; the result is a (strict) morphism if the inputs were. Identity and composition satisfy the expected neutrality and associativity laws. We write $\mathbf{CwF}$ ($\mathbf{CwF}_{\mathrm{lax}}$, $\mathbf{CwF}_{\mathrm{strict}}$) for the category of cwf's with (lax, strict) morphisms.

11. The category of maps of cwf's with right adjoints on types

Definition 11.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a lax morphism of cwf's. A *right adjoint R on types* is a J-relative right adjoint $(R_\Gamma, \epsilon_\Gamma)$ to $F\colon \mathcal{C}/\Gamma \to \mathcal{D}/F\Gamma$ for every $\Gamma \in \mathcal{C}$ where J is the map $\mathbf{Ty}_{\mathcal{D}}(F\Gamma) \to \mathcal{D}/F\Gamma$ given by $\chi_{\mathcal{D}}$:

$$\begin{tikzcd}
& \mathbf{Ty}_{\mathcal{D}}(F\Gamma) \arrow[dl, "J"'] \arrow[dr, dotted, "R_\Gamma"] \arrow[d, phantom, "\overset{\epsilon_\Gamma}{\Rrightarrow}"] & \\
\mathcal{C}/\Gamma \arrow[rr, "F"'] & {} & \mathcal{D}/F\Gamma.
\end{tikzcd}$$

Recall that a relative right adjoint is the same thing as an absolute right Kan lift in the 2-category $\mathbf{Cat}$, i.e. a right Kan extension in $\mathbf{Cat}^{\mathrm{op}}$ (1-cells inverted).

Unfolding the above definition, we obtain the following. For any context $\Gamma \in \mathcal{C}$ and type $B \in \mathbf{Ty}_{\mathcal{D}}(F\Gamma)$, there is a context $R(\Gamma, B)$ over Γ such that maps

$$\begin{tikzcd}
\Delta \arrow[rr, dotted] \arrow[dr, "\sigma"'] & & R(\Gamma, B) \arrow[dl] \\
& \Gamma &
\end{tikzcd}$$

are in bijection with maps

$$\begin{tikzcd}
F\Delta \arrow[rr, dotted] \arrow[dr, "F\sigma"'] & & F\Gamma.B \arrow[dl, "p_B"] \\
& F\Gamma, &
\end{tikzcd}$$

naturally in $\Delta \in \mathcal{C}/\Gamma$.

Consider a morphism $F \to F'$ in $\mathbf{CwF}^{\to}_{\mathrm{lax}}$, i.e. a strictly commuting diagram of lax morphisms

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"'] & \mathcal{D} \arrow[d, "V"] \\
\mathcal{C}' \arrow[r, "F'"] & \mathcal{D}'.
\end{tikzcd} \tag{11.1}$$

Assume that F and F' have right adjoints on types $(R_\Gamma, \epsilon_\Gamma)$ and $(R'_{\Gamma'}, \epsilon'_{\Gamma'})$ with $\Gamma \in \mathcal{C}$ and $\Gamma' \in \mathcal{C}'$, respectively.

Let $\Gamma \in \mathcal{C}$ and consider the commuting diagram, a prism with natural transformations as faces where the front square commutes strictly and the back left square is missing:

$$\begin{tikzcd}
& \mathbf{Ty}_{\mathcal{D}}(F\Gamma) \arrow[dl, "R_\Gamma"'] \arrow[dr, "J"] \arrow[dd, "(u_V)_{F\Gamma}"] \arrow[d, phantom, "\overset{\epsilon_\Gamma}{\Rrightarrow}"] & \\
\mathcal{C}/\Gamma \arrow[rr, "F"] \arrow[dd, "U"'] \arrow[dr, phantom, "d_\Gamma \searrow"] & & \mathcal{D}/F\Gamma \arrow[dd, "V"] \arrow[dl, phantom, "\overset{(v_V)_{F\Gamma}}{\Lleftarrow}"] \\
& \mathbf{Ty}_{\mathcal{D}'}(F'U\Gamma) \arrow[dl, "R'_{U\Gamma}"'] \arrow[dr, "J'"] \arrow[d, phantom, "\overset{\epsilon'_{F\Gamma}}{\Rrightarrow}"] & \\
\mathcal{C}'/U\Gamma \arrow[rr, "F'"'] & & \mathcal{D}'/F'U\Gamma.
\end{tikzcd} \tag{11.2}$$

Recall that absolute Kan lifts are stable under precomposition. By the universal property of the absolute Kan lift $(R'_{F\Gamma}, \epsilon'_{F\Gamma})$ precomposed with $(u_V)_{F\Gamma}$, there is a unique natural transformation in the back left square as indicated making the prism of natural transformations commute.

Definition 11.2. Consider a morphism of $F \to F'$ in $\mathbf{CwF}^{\to}_{\mathrm{lax}}$ as in (11.1) where F and F' have right adjoints on types. We say that $F \to F'$ satisfies the *Beck-Chevalley condition* if the induced natural transformation d_Γ in (11.2) is an isomorphism for all $\Gamma \in \mathcal{C}$.

Proposition 11.3. *Maps that satisify the Beck-Chevalley condition in $\mathbf{CwF}^{\to}_{\mathrm{lax}}$ between objects that have right adjoints on types are closed under finitary composition.*

Proof. Standard. ☐

We write $\mathbf{CwF}^{\to}_{\mathrm{r}}$ ($\mathbf{CwF}^{\to}_{\mathrm{lax,r}}$, $CwF^{\to}_{\mathrm{strict,r}}$) for the category over $\mathbf{CwF}^{\to}$ whose objects are (lax, strict) morphisms of cwf's with right adjoints on types and whose morphisms from F to F' are given by commuting squares $(U, V)\colon F \to F'$ in $\mathbf{CwF}_{\mathrm{lax}}$ that satisfy the Beck-Chevalley condition such that U, V are morphisms of cwf's. If we wish U and V to be lax morphisms or strict morphisms intead, we append a further outermost subscript as in '$(\mathbf{CwF}^{\to}_{\mathrm{r}})_{\mathrm{lax}}$'. We obtain nine different categories in total, all given by Proposition 11.3.

11.1. Global description of right adjoints on types.

Lemma 11.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a lax morphism of cwf's with a right adjoint R on types. Then the relative right adjoints $(R_\Gamma, \epsilon_\Gamma)$ with $\Gamma \in \mathcal{C}$ assemble to a global $(\mathcal{C} \times_{\mathcal{D}} \chi_{\mathcal{D}})$-relative right adjoint (R, ϵ) of $\widehat{\exp}(d_0, F) = \langle \mathrm{cod}, F^{\to} \rangle$ that is a cartesian* *morphism over $\mathcal{C}$:*

$$\begin{tikzcd}
&& \mathcal{C} \times_{\mathcal{D}} \int \mathbf{Ty}_{\mathcal{D}} \arrow[rr] \arrow[d, "\mathcal{C}\times_{\mathcal{D}}\chi_{\mathcal{D}}"] \arrow[ddl] \arrow[dll, dotted, "R"'] \arrow[dl, phantom, "\overset{\epsilon}{\Longrightarrow}"] && \int \mathbf{Ty}_{\mathcal{D}} \arrow[d, "\chi_{\mathcal{D}}"] \arrow[ddl] \\
\mathcal{C}^{\to} \arrow[rr, "\langle \mathrm{cod}, F^{\to}\rangle"'] \arrow[dr] && \mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{\to} \arrow[rr] \arrow[dl] && \mathcal{D}^{\to} \arrow[dl, "\mathrm{cod}_{\mathcal{D}}"] \\
& \mathcal{C} \arrow[rr] && \mathcal{D}.
\end{tikzcd}$$

Proof. This is an instance of Lemma 17.2. ☐

Consider a morphism $F \to F'$ in $\mathbf{CwF}^{\to}_{\mathrm{lax}}$ as in (11.1). Assume that F and F' have right adjoints on types, assembling to global relative right adjoints (R, ϵ) and (R', ϵ') as in Lemma 11.4.

Consider the following prism, where the top and bottom faces are the counits natural transformations of the relative adjoints, the back right square is a pullback of $v_{U_{\mathcal{D}}}$, and the front square commutes strictly:

$$\begin{tikzcd}
&& \mathcal{C} \times_{\mathcal{D}} \int \mathbf{Ty}_{\mathcal{D}} \arrow[dll, "R"'] \arrow[drr] \arrow[dd] \arrow[d, phantom, "\overset{\epsilon}{\Longrightarrow}"] && \\
\mathcal{C}^{\to} \arrow[rrrr] \arrow[dd] \arrow[dr, phantom, "d \Downarrow"] &&&& \mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{\to} \arrow[dd] \arrow[dl, phantom, "\Leftarrow \mathcal{C}\times_{\mathcal{D}} v_{U_{\mathcal{D}}}"] \\
&& \mathcal{C}' \times_{\mathcal{D}'} \int \mathbf{Ty}_{\mathcal{D}'} \arrow[dll, "R'"'] \arrow[drr] \arrow[d, phantom, "\overset{\epsilon'}{\Longrightarrow}"] && \\
(\mathcal{C}')^{\to} \arrow[rrrr] &&&& \mathcal{C}' \times_{\mathcal{D}'} (\mathcal{D}')^{\to}.
\end{tikzcd} \tag{11.3}$$

Recall that absolute Kan lifts are stable under precomposition. By the universal property of the absolute Kan lift (R', ϵ') precomposed with $U_{\mathcal{C}} \times_{U_{\mathcal{D}}} U_{\mathcal{D}}$, there is a unique natural transformation in the back left square as indicated making the prism of natural transformations commute.

Proposition 11.5. *In the context of* (11.3)*, the natural transformation d in* (11.3) *is assembled from the natural transformations d_{Γ} in* (11.2) *for $\Gamma \in \mathcal{C}$.*

Proof. Standard. ☐

Note that Proposition 11.5 implies that $\mathrm{cod}_{B'}\, d = \mathrm{id}$.

Corollary 11.6. *In the context of* (11.3)*, the morphism $F \to F'$ satisfies the Beck-Chevalley condition exactly if the natural transformation d is an isomorphism.*

Given a lax morphism $F \colon \mathcal{C} \to \mathcal{D}$ with a right adjoint on types, there is a way of forming an "intermediate" cwf with contexts coming from $\mathcal{C}$ and types coming from $\mathcal{D}$. This is the context of the next lemma.

Lemma 11.7. *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a lax morphism of cwf's with a right adjoint R on types. Then there is a cwf $\mathcal{E}$ with category of contexts $\mathcal{C}$ and types given by the composite*

$$\mathcal{C}^{\to} \xrightarrow{F} \mathcal{D}^{\to} \longrightarrow \mathbf{Set}.$$

Furthermore, F factors via $\mathcal{E}$ as below:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F", "\mathrm{lax}"'] \arrow[dr, "\mathrm{lax}", "G"'] && \mathcal{D} \\
& \mathcal{E}. \arrow[ur, "\mathrm{lax}", "H"']
\end{tikzcd}$$

Proof. Under the category of elements construction, precomposition of presheaves corresponds to pullback of induced discrete fibrations. So $\int \mathcal{E} \simeq \mathcal{C} \times_{\mathcal{D}} \int \mathbf{Ty}_{\mathcal{D}}$, and under this isomorphism, we may take the functor R of Lemma 11.4 for the comprehension functor $\chi_{\mathcal{E}}$. This completes the definition of the cwf $\mathcal{E}$.

The lax morphism $\mathcal{E} \to \mathcal{D}$ has functor on contexts given by F, natural transformation between types given by the identity on $\mathbf{Ty}_{\mathcal{D}}F$, and natural transformation between comprehensions given by the counit of the relative adjoint R.

The lax morphism $\mathcal{C} \to \mathcal{E}$ has functor on contexts given by $\mathrm{Id}_{\mathcal{C}}$, natural transformations between types given by u_F, and natural transformation between comprehensions given by v_F under the natural isomorphism characterizing R as an absolute right Kan lift. □

Note that even when starting with a morphism of cwf's $\mathcal{C} \to \mathcal{D}$ in Lemma 11.7, the induced lax morphisms $\mathcal{C} \to \mathcal{E}$ and $\mathcal{E} \to \mathcal{D}$ will generally not be morphisms.

12. Glueing over the walking arrow

[Note: old material, not yet converted to shorthand presheaf substitution.]

Let $F\colon (\mathcal{C}, \mathbf{Ty}_{\mathcal{C}}, \mathbf{Tm}_{\mathcal{C}}) \to (\mathcal{D}, \mathbf{Ty}_{\mathcal{D}}, \mathbf{Tm}_{\mathcal{D}})$ be a map of cwf's, required to preserve types and terms strictly. We require that context extension is preserved only up to canonical isomorphism in the following sense. Given $\Gamma_{\mathcal{C}} \in \mathcal{C}$ and $A_{\mathcal{C}} \in \mathbf{Ty}_{\mathcal{C}}(\Gamma_{\mathcal{C}})$, we have a canonical comparison morphism

$$h(\Gamma_{\mathcal{C}}, A_{\mathcal{C}})\colon F(\Gamma_{\mathcal{C}}.A_{\mathcal{C}}) \to F\Gamma_{\mathcal{C}}.FA_{\mathcal{C}}.$$

We require that $h(\Gamma_{\mathcal{C}}, A_{\mathcal{C}})$ is invertible.

We define a new cwf $(\mathcal{E}, \mathbf{Ty}_{\mathcal{E}}, \mathbf{Tm}_{\mathcal{E}})$ as follows. We choose to define comprehension instead of terms. This is an equivalent style of presentation. The terms can be recovered by letting $\mathbf{Tm}_{\mathcal{E}}(\Gamma, A)$ be the set of sections of $p(A)\colon \Gamma.A \to \Gamma$. Functoriality of comprehension makes $\mathbf{Tm}_{\mathcal{E}}$ into a presheaf.

Contexts. We set $\mathcal{E} =_{\mathrm{def}} \mathcal{D} \downarrow F$. Explicitly:

- an object is a tuple $(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha)$ where $\Gamma_{\mathcal{C}} \in \mathcal{C}$, $\Gamma_{\mathcal{D}} \in \mathcal{D}$, and $\alpha\colon \Gamma_{\mathcal{D}} \to F\Gamma_{\mathcal{C}}$,
- a morphism from $(\Delta_{\mathcal{C}}, \Delta_{\mathcal{D}}, \beta)$ to $(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha)$ is a pair $(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}})$ where $\sigma_{\mathcal{C}}\colon \Delta_{\mathcal{C}} \to \Gamma_{\mathcal{C}}$ and $\sigma_{\mathcal{D}}\colon \Delta_{\mathcal{D}} \to \Gamma_{\mathcal{D}}$ such that $F(\sigma_{\mathcal{C}}) \circ \beta = \alpha \circ \sigma_{\mathcal{D}}$,
- identity and composition of morphisms is given componentwise,
- neutrality and associativity laws follow componentwise.

Types. The presheaf $\mathbf{Ty}_{\mathcal{E}}\colon \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ is defined as follows.

- Given $(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha) \in \mathcal{E}$, we let $\mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha)$ be the set of pairs $(A_{\mathcal{C}}, A_{\mathcal{D}})$ where $A_{\mathcal{C}} \in \mathbf{Ty}_{\mathcal{C}}(\Gamma_{\mathcal{C}})$ and $A_{\mathcal{D}} \in \mathbf{Ty}_{\mathcal{D}}(\Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))$.
- Given a morphism
$$(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}})\colon (\Delta_{\mathcal{C}}, \Delta_{\mathcal{D}}, \beta) \to (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha)$$
in $\mathcal{E}$, we define
$$\mathbf{Ty}_{\mathcal{E}}(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}})\colon \mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha) \to \mathbf{Ty}_{\mathcal{E}}(\Delta_{\mathcal{C}}, \Delta_{\mathcal{D}}, \beta)$$
by sending $(A_{\mathcal{C}}, A_{\mathcal{D}})$ to $(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}), \mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))(A_{\mathcal{D}}))$ where
$$\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}})\colon \Delta_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}})(\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}})) \to \Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}})$$
is our notation for comprehension of morphisms in $\mathcal{D}$.
- Coherence with respect to identity and composition of morphisms in $\mathcal{E}$ follow from those of $\mathbf{Ty}_{\mathcal{C}}$ and $\mathbf{Ty}_{\mathcal{D}}$ with respect to morphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively, and the fact that comprehension in $\mathcal{D}$ respects identities and compositions.

Comprehension. Let us define a comprehension functor $\chi_{\mathcal{E}}$ as below:

$$\begin{tikzcd} \int \mathbf{Ty}_{\mathcal{E}} \arrow[rr, dotted, "\chi_{\mathcal{E}}"] \arrow[dr, "\pi_1"'] & & \mathcal{E}^{\to}_{\mathsf{cart}} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{E}. & \end{tikzcd}$$

As usual, we also write $\chi_{\mathcal{E}}(\Gamma, A)$ as $p(A)\colon \Gamma.A \to \Gamma$.

- On objects, we define $\chi_{\mathcal{E}}$ as follows. Suppose we are given $(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha) \in \mathcal{E}$ and $(A_{\mathcal{C}}, A_{\mathcal{D}}) \in \mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha)$. The context extension
$$(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha).(A_{\mathcal{C}}, A_{\mathcal{D}}) \in \mathcal{E}$$
is defined as
$$(\Gamma_{\mathcal{C}}.A_{\mathcal{C}}, \Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}}, \gamma)$$
where
$$\gamma\colon \Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}} \to F(\Gamma_{\mathcal{C}}.A_{\mathcal{C}})$$
is defined as the composition
$$\Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}} \xrightarrow{p(A_{\mathcal{D}})} \Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}) \xrightarrow{\alpha.FA_{\mathcal{C}}} F\Gamma_{\mathcal{C}}.FA_{\mathcal{C}} \xrightarrow{h(\Gamma_{\mathcal{C}}, A_{\mathcal{C}})^{-1}} F(\Gamma_{\mathcal{C}}.A_{\mathcal{C}}).$$
The context projection
$$p(A_{\mathcal{C}}, A_{\mathcal{D}})\colon (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha).(A_{\mathcal{C}}, A_{\mathcal{D}}) \to (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha)$$
is defined as $(p(A_{\mathcal{C}}), p(\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}})) \circ p(A_{\mathcal{D}}))$.
- Let us define $\chi_{\mathcal{E}}$ on morphisms. Suppose we are given a morphism
$$(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}})\colon (\Delta_{\mathcal{C}}, \Delta_{\mathcal{D}}, \beta) \to (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha)$$
in $\mathcal{E}$ and a type $(A_{\mathcal{C}}, A_{\mathcal{D}}) \in \mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha)$. We want to define the morphism of context extensions
$$(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}}).(A_{\mathcal{C}}, A_{\mathcal{D}})\colon (\Delta_{\mathcal{C}}, \Delta_{\mathcal{D}}, \beta).\mathbf{Ty}_{\mathcal{E}}(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}})(A_{\mathcal{C}}, A_{\mathcal{D}}) \to (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha).(A_{\mathcal{C}}, A_{\mathcal{D}}),$$
where the signature unfolds to
$$\begin{aligned} &(\Delta_{\mathcal{C}}.\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}),\\ &\quad \Delta_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\beta)(F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))).\mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))(A_{\mathcal{D}}), \delta) \to\\ &\qquad\qquad (\Gamma_{\mathcal{C}}.A_{\mathcal{C}}, \Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}}, \gamma) \end{aligned}$$
where γ is as before and δ is the composite
$$\begin{tikzcd} \Delta_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))).\mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))(A_{\mathcal{D}})) \arrow[d, "p(\mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))(A_{\mathcal{D}})))"] \\ \Delta_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))) \arrow[d, "\beta.F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))"] \\ F\Delta_{\mathcal{C}}.F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}})) \arrow[d, "h(\Delta_{\mathcal{C}}, \mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))^{-1}"] \\ F(\Delta_{\mathcal{C}}.\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}})). \end{tikzcd}$$
We take
$$(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}}).(A_{\mathcal{C}}, A_{\mathcal{D}}) =_{\mathrm{def}} (\sigma_{\mathcal{C}}.A_{\mathcal{C}}, \sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}}).$$

The second component type-checks as follows:

$$\begin{tikzcd}
\Delta_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\beta)(F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))).\mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))(A_{\mathcal{D}}) \arrow[d, equal] \\
\Delta_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}})(\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}})).\mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))(A_{\mathcal{D}}) \arrow[d, "\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}}"] \\
\Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}}.
\end{tikzcd}$$

To make $(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}}).(A_{\mathcal{C}}, A_{\mathcal{D}})$ into a morphism in $\mathcal{E}$, we need to check that the following diagram commutes:

$$\begin{tikzcd}
\Delta_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))).\mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))(A_{\mathcal{D}})) \arrow[r, "\delta"] \arrow[d, "\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}}"] & F(\Delta_{\mathcal{C}}.\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}})) \arrow[d, "F(\sigma_{\mathcal{C}}.A_{\mathcal{C}})"] \\
\Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}} \arrow[r, "\gamma"] & F(\Gamma_{\mathcal{C}}.A_{\mathcal{C}}).
\end{tikzcd}$$

Since $\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}}$ lives over $\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}})$ with respect to the context projections that form the first factors of γ and δ, respectively, this follows from commutativity of the following diagram:

$$\begin{tikzcd}
\Delta_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))) \arrow[dr, "\beta.F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))"] \arrow[dd, "\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}})"] & & \\
& F\Delta_{\mathcal{C}}.F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}})) \arrow[dr, "h(\Delta_{\mathcal{C}},\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))^{-1}"] \arrow[dd, "F(\sigma_{\mathcal{C}}).FA_{\mathcal{C}}"] & \\
\Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}) \arrow[dr, "\alpha.FA_{\mathcal{C}}"] & & F(\Delta_{\mathcal{C}}.\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}})) \arrow[dd, "F(\sigma_{\mathcal{C}}.A_{\mathcal{C}})"] \\
& F\Gamma_{\mathcal{C}}.FA_{\mathcal{C}} \arrow[dr, "h(\Gamma_{\mathcal{C}},A_{\mathcal{C}})^{-1}"] & \\
& & F(\Gamma_{\mathcal{C}}.A_{\mathcal{C}}).
\end{tikzcd}$$

Here, the left square commutes by functoriality of context extension in $\mathcal{C}$ and commutativity of the right square is equivalent to commutativity of the square

$$\begin{tikzcd}
\Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}) \arrow[r, "h(\Delta_{\mathcal{C}},\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))"] \arrow[d, "F(\sigma_{\mathcal{C}}.A_{\mathcal{C}})"] & F\Delta_{\mathcal{C}}.F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}})) \arrow[d, "F(\sigma_{\mathcal{C}}).FA_{\mathcal{C}}"] \\
F(\Gamma_{\mathcal{C}}.A_{\mathcal{C}}) \arrow[r, "h(\Gamma_{\mathcal{C}},A_{\mathcal{C}})"] & F\Gamma_{\mathcal{C}}.FA_{\mathcal{C}}.
\end{tikzcd}$$

This is a naturality square for the natural transformation relating context extension in $\mathcal{C}$ to context extension in $\mathcal{D}$.

We now need to show that the morphism $(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}}).(A_{\mathcal{C}}, A_{\mathcal{D}})$ lives over $(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}})$ with respect to the context projections, i.e. that the square

$$\begin{tikzcd}
(\Delta_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \beta).\mathbf{Ty}_{\mathcal{E}}(\sigma_{\mathcal{C}}, \sigma_{\mathcal{D}})(A_{\mathcal{C}}, A_{\mathcal{D}}) \arrow[r, "(\sigma_{\mathcal{C}},\sigma_{\mathcal{D}}).(A_{\mathcal{C}},A_{\mathcal{D}})"] \arrow[d, "p(\mathbf{Ty}_{\mathcal{E}}(\sigma_{\mathcal{C}},\sigma_{\mathcal{D}})(A_{\mathcal{C}},A_{\mathcal{D}}))"'] & (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha).(A_{\mathcal{C}}, A_{\mathcal{D}}) \arrow[d, "p(A_{\mathcal{C}},A_{\mathcal{D}})"] \\
(\Delta_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \beta) \arrow[r, "(\sigma_{\mathcal{C}},\sigma_{\mathcal{D}})"] & (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{D}}, \alpha)
\end{tikzcd} \tag{12.1}$$

commutes. By the construction of the category $\mathcal{E}$. This decomposes into commutativity of the squares

$$\begin{tikzcd}
\Delta_{\mathcal{C}}.\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}) \arrow[r, "\sigma_{\mathcal{C}}.A_{\mathcal{C}}"] \arrow[d, "p(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}}))"'] & \Gamma_{\mathcal{C}}.A_{\mathcal{C}} \arrow[d, "p(A_{\mathcal{C}})"] \\
\Delta_{\mathcal{C}} \arrow[r, "\sigma_{\mathcal{C}}"] & \Gamma_{\mathcal{C}}
\end{tikzcd} \tag{12.2}$$

and

$$\begin{tikzcd}
\Delta_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\beta)(F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}})).\mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))(A_{\mathcal{D}}) \arrow[r, "\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}}"] \arrow[d, "p(\mathbf{Ty}_{\mathcal{D}}(\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))(A_{\mathcal{D}}))"'] & \Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}).A_{\mathcal{D}} \arrow[d, "p(A_{\mathcal{D}})"] \\
\Delta_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\beta)(F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}})) \arrow[r, "\sigma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}})"] \arrow[d, "p(\mathbf{Ty}_{\mathcal{D}}(\beta)(F(\mathbf{Ty}_{\mathcal{C}}(\sigma_{\mathcal{C}})(A_{\mathcal{C}})))"'] & \Gamma_{\mathcal{D}}.\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}) \arrow[d, "p(\mathbf{Ty}_{\mathcal{D}}(\alpha)(FA_{\mathcal{C}}))"] \\
\Delta_{\mathcal{D}} \arrow[r, "\sigma_{\mathcal{D}}"] & \Gamma_{\mathcal{D}}.
\end{tikzcd} \tag{12.3}$$

These are instances of induced morphisms between context extensions living over the base context morphisms for $\mathcal{C}$ and $\mathcal{D}$.

Finally, we need to check that the square (12.1) is cartesian, i.e. a pullback. By assumption, the squares (12.2) and (12.3) are cartesian since they are images on morphisms of the comprehension functors in case of $\mathcal{C}$ and pastings thereof in case of $\mathcal{D}$. Since the functor F preserves comprehension up to (canonical) natural isomorphism, the image of (12.2) under F is still a pullback. We now use the fact that limits in the comma category $\mathcal{E} = \mathcal{D} \downarrow F$ are computed componentwise whenever F preserves the limit in the component of $\mathcal{C}$. It follows that the square (12.1) is also a pullback.

- Note that $\chi_{\mathcal{E}}$ preserves identities and compositions because it preserves cartesian arrows.

13. Generalized Glueing

Proposition 13.1 (Generalized glueing)**.** *Consider a morphism F and lax morphism G of cwf's as below:*

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, "F"'] & & \mathcal{B} \arrow[dl, "G", "\text{lax}"'] \\
& \mathcal{C}. &
\end{tikzcd}$$

Assume that G has a right adjoint on types (R, ϵ). Then there is a cwf GenGlue(F, G)*, called the* generalized glueing *of F and G.*

There is a forgetful *strict morphism of cwf's* GenGlue$(F, G) \to \mathcal{A}$ *defined via projection to the first component.*

We obtain the previous glueing construction (oplax limit over the walking arrow) by letting G be an identity.

Proof. We abbreivate $\mathcal{E} =_{\mathrm{def}} \mathrm{GenGlue}(F, G)$.

Contexts. We set $\mathcal{E} =_{\mathrm{def}} G \downarrow F$. Explicitly:

- an object is a tuple $(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha)$ where $\Gamma_{\mathcal{A}} \in \mathcal{A}$, $\Gamma_{\mathcal{B}} \in \mathcal{B}$, and $\alpha \colon G\Gamma_{\mathcal{B}} \to F\Gamma_{\mathcal{A}}$,
- a morphism from $(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}}, \beta)$ to $(\Gamma_{\mathcal{A}}, \Gamma_{B}, \alpha)$ is a pair $(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})$ where $\sigma_{\mathcal{A}} \colon \Delta_{\mathcal{A}} \to \Gamma_{\mathcal{A}}$ and $\sigma_{\mathcal{B}} \colon \Delta_{\mathcal{B}} \to \Gamma_{\mathcal{B}}$ such that $F\sigma_{\mathcal{A}} \circ \beta = \alpha \circ G\sigma_{\mathcal{B}}$:

$$\begin{tikzcd} G\Delta_{\mathcal{B}} \arrow[r, "\beta"] \arrow[d, "G\sigma_{\mathcal{B}}"'] & F\Delta_{\mathcal{A}} \arrow[d, "F\sigma_{\mathcal{A}}"] \\ G\Gamma_{\mathcal{B}} \arrow[r, "\alpha"] & F\Gamma_{\mathcal{A}}, \end{tikzcd} \tag{13.1}$$

- identity and composition of morphisms is given componentwise,
- neutrality and associativity laws follow componentwise.

Types. The presheaf $\mathbf{Ty}_{\mathcal{E}} \colon \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ is defined as follows.

- Given $(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha) \in \mathcal{E}$, we let $\mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha)$ be the set of pairs $(A_{\mathcal{A}}, A_{\mathcal{B}})$ where $A_{\mathcal{A}} \in \mathbf{Ty}_{\mathcal{A}}(\Gamma_{\mathcal{A}})$ and $A_{\mathcal{B}} \in \mathbf{Ty}_{\mathcal{B}}(R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]))$.
- Given a morphism
$$(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}) \colon (\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}}, \beta) \to (\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha)$$
in $\mathcal{E}$, we define
$$\mathbf{Ty}_{\mathcal{E}}(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}) \colon \mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha) \to \mathbf{Ty}_{\mathcal{E}}(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}}, \beta)$$
by sending $(A_{\mathcal{A}}, A_{\mathcal{B}})$ to $(A_{\mathcal{A}}[\sigma_{\mathcal{A}}], A_{\mathcal{B}}[R_{\sigma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha])])$.
- Coherence with respect to identity and composition of morphisms in $\mathcal{E}$ follow from those of $\mathbf{Ty}_{\mathcal{A}}$ and $\mathbf{Ty}_{\mathcal{B}}$ with respect to morphisms in $\mathcal{A}$ and $\mathcal{B}$, respectively, and the fact that $R \colon \mathcal{B} \times_{\mathcal{E}} \int \mathbf{Ty}_{\mathcal{C}} \to \mathcal{B}^{\to}$ is a functor.

Comprehension. Let us define the comprehension functor $\chi_{\mathcal{E}}$ as below:

$$\begin{tikzcd} \int \mathbf{Ty}_{\mathcal{E}} \arrow[rr, dotted, "\chi_{\mathcal{E}}"] \arrow[dr, "\pi_1"'] & & \mathcal{E}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{E}. & \end{tikzcd}$$

- On objects, we define $\chi_{\mathcal{E}}$ as follows. Suppose we are given $(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha) \in \mathcal{E}$ and $(A_{\mathcal{A}}, A_{\mathcal{B}}) \in \mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha)$. The context extension
$$(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha).(A_{\mathcal{A}}, A_{\mathcal{B}}) \in \mathcal{E}$$
is defined as
$$(\Gamma_{\mathcal{A}}.A_{\mathcal{A}}, R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}}, \gamma)$$
where
$$\gamma \colon G(R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}}) \to F(\Gamma_{\mathcal{A}}.A_{\mathcal{A}})$$
is defined as the transport under the natural transformation v_G and natural isomorphism v_F of
$$\gamma' \colon GR_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).GA_{\mathcal{B}} \to F\Gamma_{\mathcal{A}}.FA_{\mathcal{A}}.$$

In turn, this is defined as the composite

$$\begin{tikzcd}
 & GR_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]) \arrow[dr, "\epsilon_{(\Gamma_{\mathcal{B}},(FA_{\mathcal{A}})[\alpha])}"] & & F\Gamma_{\mathcal{A}}.FA_{\mathcal{A}}. \\
GR_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).GA_{\mathcal{B}} \arrow[ur, "p"] & & G\Gamma_{B}.(FA_{\mathcal{A}})[\alpha] \arrow[ur, "\alpha.FA_{\mathcal{A}}"'] &
\end{tikzcd}$$

The context projection

$$p(A_{\mathcal{A}}, A_{\mathcal{B}})\colon (\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha).(A_{\mathcal{A}}, A_{\mathcal{B}}) \to (\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha)$$

is the pair of maps consisting of $p(A_{\mathcal{A}})\colon \Gamma_{\mathcal{A}}.A_{\mathcal{A}} \to \Gamma_{\mathcal{A}}$ and

$$R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}} \xrightarrow{\ p\ } R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]) \longrightarrow \Gamma_{\mathcal{B}}.$$

- Let us define $\chi_{\mathcal{E}}$ on morphisms. Suppose we are given a morphism

$$(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})\colon (\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}}, \beta) \to (\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha)$$

in $\mathcal{E}$ and a type $(A_{\mathcal{A}}, A_{\mathcal{B}}) \in \mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha)$. We want to define the morphism of context extensions

$$(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}).(A_{\mathcal{A}}, A_{\mathcal{B}})\colon (\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}}, \beta).(A_{\mathcal{A}}, A_{\mathcal{B}})[(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})] \to (\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha).(A_{\mathcal{A}}, A_{\mathcal{B}}),$$

where the signature unfolds to

$$\begin{aligned}(\Delta_{\mathcal{A}}.A_{\mathcal{A}}[\sigma_{\mathcal{A}}], R_{\Delta_{\mathcal{B}}}((F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta]).A_{\mathcal{B}}[R_{\sigma_B}((F(A_{\mathcal{A}})[\sigma_{\mathcal{A}}])[\beta])], \delta) \to \\ (\Gamma_{\mathcal{A}}.A_{\mathcal{A}}, R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}}, \gamma)\end{aligned}$$

where γ is as before and δ is the transport under the natural transformation v_G and natural isomorphism v_F of γ', given by the composite

$$\begin{tikzcd}
 & GR_{\Delta_{\mathcal{B}}}((F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta]) \arrow[dr, "\epsilon_{(\Gamma_{\mathcal{B}},(F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta])}"] & & F\Delta_{\mathcal{A}}.F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]). \\
GR_{\Delta_{\mathcal{B}}}((F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta]).GA_{\mathcal{B}}[R_{\sigma_B}((F(A_{\mathcal{A}})[\sigma_{\mathcal{A}}])[\beta])] \arrow[ur, "p"] & & G\Delta_{B}.(F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta] \arrow[ur, "\beta.F(A_{\mathcal{A}}[\alpha])"'] &
\end{tikzcd}$$

We take

$$(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}).(A_{\mathcal{A}}, A_{\mathcal{B}}) =_{\text{def}} (\sigma_{\mathcal{A}}.A_{\mathcal{A}}, R_{\sigma_B}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}}).$$

The second component uses

$$R_{\sigma_B}((FA_{\mathcal{A}})[\alpha])\colon R_{\Delta_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha][G\sigma_{\mathcal{B}}]) \to R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha])$$

and type-checks because of the following chain of equations:

$$\begin{aligned}(F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta] &= (FA_{\mathcal{A}})[F\sigma_{\mathcal{A}}][\beta] \\ &= (FA_{\mathcal{A}})[F\sigma_{\mathcal{A}} \circ \beta] \\ &= (FA_{\mathcal{A}})[\alpha \circ G\sigma_{\mathcal{B}}] \\ &= (FA_{\mathcal{A}})[\alpha][G\sigma_{\mathcal{B}}].\end{aligned}$$

To make $(\sigma_\mathcal{A}, \sigma_\mathcal{B}).(A_\mathcal{A}, A_\mathcal{B})$ into a morphism in $\mathcal{E}$, we need to check that the following diagram commutes:

$$\begin{tikzcd}
G(R_{\Delta_\mathcal{B}}((F(A_\mathcal{A}[\sigma_\mathcal{A}]))[\beta]).A_\mathcal{B}[R_{\sigma_B}((F(A_\mathcal{A}[\sigma_\mathcal{A}]))[\beta])]) \arrow[dr, "\delta"] \arrow[dd, "G(R_{\sigma_B}((FA_\mathcal{A})[\alpha]).A_\mathcal{B})"] & \\
& F(\Delta_\mathcal{A}.A_\mathcal{A}[\sigma_\mathcal{A}]) \arrow[dd, "F(\sigma_\mathcal{A}.A_\mathcal{A})"] \\
G(R_{\Gamma_\mathcal{B}}((FA_\mathcal{A})[\alpha]).A_\mathcal{B}) \arrow[dr, "\gamma"] & \\
& F(\Gamma_\mathcal{A}.A_\mathcal{A}).
\end{tikzcd}$$

Using naturality squares of the natural transformation v_G and natural isomorphism v_F, this is equivalent to commutativity of the following diagram:

$$\begin{tikzcd}
GR_{\Delta_\mathcal{B}}((F(A_\mathcal{A}[\sigma_\mathcal{A}]))[\beta]).G(A_\mathcal{B}[R_{\sigma_B}((F(A_\mathcal{A}[\sigma_\mathcal{A}]))[\beta])]) \arrow[dr, "\delta'"] \arrow[dd, "GR_{\sigma_B}((FA_\mathcal{A})[\alpha]).GA_\mathcal{B}"] & \\
& F\Delta_\mathcal{A}.F(A_\mathcal{A}[\sigma_\mathcal{A}]) \arrow[dd, "F\sigma_\mathcal{A}.FA_\mathcal{A}"] \\
GR_{\Gamma_\mathcal{B}}((FA_\mathcal{A})[\alpha]).GA_\mathcal{B} \arrow[dr, "\gamma'"] & \\
& F\Gamma_\mathcal{A}.FA_\mathcal{A}.
\end{tikzcd}$$

Since $GR_{\sigma_B}((FA_\mathcal{A})[\alpha]).GA_\mathcal{B}$ lives over $GR_{\sigma_B}((FA_\mathcal{A})[\alpha])$ with respect to the context projections that form the first factors of γ' and δ', respectively, this follows from commutativity of the following diagram:

$$\begin{tikzcd}
GR_{\Delta_\mathcal{B}}((F(A_\mathcal{A}[\sigma_\mathcal{A}]))[\beta]) \arrow[r, "\epsilon(\Gamma_\mathcal{B},(F(A_\mathcal{A}[\sigma_\mathcal{A}]))[\beta])"] \arrow[d, "GR_{\sigma_B}((FA_\mathcal{A})[\alpha])"] & G\Delta_B.(F(A_\mathcal{A}[\sigma_\mathcal{A}]))[\beta] \arrow[r, "\beta.F(A_\mathcal{A}[\alpha])"] \arrow[d, "G\sigma_B.(FA_\mathcal{A})[\alpha]"] & F\Delta_\mathcal{A}.F(A_\mathcal{A}[\sigma_\mathcal{A}]) \arrow[d, "F\sigma_\mathcal{A}.FA_\mathcal{A}"] \\
GR_{\Gamma_\mathcal{B}}((FA_\mathcal{A})[\alpha]) \arrow[r, "\epsilon(\Gamma_\mathcal{B},(FA_\mathcal{A})[\alpha])"] & G\Gamma_B.(FA_\mathcal{A})[\alpha] \arrow[r, "\alpha.FA_\mathcal{A}"] & F\Gamma_\mathcal{A}.FA_\mathcal{A}.
\end{tikzcd}$$

Here, the left square is a naturality square for the counit ϵ of R and the right square comes substituting the type $FA_\mathcal{A}$ along the morphisms of the commuting square 16.1.

We now need to show that the morphism $(\sigma_\mathcal{A}, \sigma_\mathcal{B}).(A_\mathcal{A}, A_\mathcal{B})$ lives over $(\sigma_\mathcal{A}, \sigma_\mathcal{B})$ with respect to the context projections, i.e. that the square

$$\begin{tikzcd}
(\Delta_\mathcal{A}, \Gamma_\mathcal{B}, \beta).(A_\mathcal{A}, A_\mathcal{B})[(\sigma_\mathcal{A}, \sigma_\mathcal{B})] \arrow[r, "(\sigma_\mathcal{A},\sigma_\mathcal{B}).(A_\mathcal{A},A_\mathcal{B})"] \arrow[d, "p((A_\mathcal{A},A_\mathcal{B})[(\sigma_\mathcal{A},\sigma_\mathcal{B})])"'] & (\Gamma_\mathcal{A}, \Gamma_\mathcal{B}, \alpha).(A_\mathcal{A}, A_\mathcal{B}) \arrow[d, "p(A_\mathcal{A},A_\mathcal{B})"] \\
(\Delta_\mathcal{A}, \Gamma_\mathcal{B}, \beta) \arrow[r, "(\sigma_\mathcal{A},\sigma_\mathcal{B})"] & (\Gamma_\mathcal{A}, \Gamma_\mathcal{B}, \alpha)
\end{tikzcd} \tag{13.2}$$

commutes. By the construction of the category $\mathcal{E}$, this decomposes into commutativity of the square

$$\begin{tikzcd}
\Delta_{\mathcal{A}}.\mathbf{Ty}_{\mathcal{A}}(\sigma_{\mathcal{A}})(A_{\mathcal{A}}) \arrow[r, "\sigma_{\mathcal{A}}.A_{\mathcal{A}}"] \arrow[d, "p"'] & \Gamma_{\mathcal{A}}.A_{\mathcal{A}} \arrow[d, "p"] \\
\Delta_{\mathcal{A}} \arrow[r, "\sigma_{\mathcal{A}}"] & \Gamma_{\mathcal{A}},
\end{tikzcd} \tag{13.3}$$

which is a morphism in the image of $\chi_{\mathcal{A}}$, and the square

$$\begin{tikzcd}
R_{\Delta_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha][G\sigma_{\mathcal{B}}]).A_{\mathcal{B}}[R_{\Delta_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha][G\sigma_{\mathcal{B}}])] \arrow[r, "R_{\sigma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}}"] \arrow[d, "p"] & R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}} \arrow[d, "p"] \\
R_{\Delta_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha][G\sigma_{\mathcal{B}}]) \arrow[r, "R_{\sigma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha])"] \arrow[d] & R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]) \arrow[d] \\
\Delta_{\mathcal{B}} \arrow[r, "\sigma_{\mathcal{B}}"] & \Gamma_{\mathcal{B}},
\end{tikzcd} \tag{13.4}$$

which is a pasting of a morphism in the image of $\chi_{\mathcal{B}}$ and a morphism in the image of R.

Finally, we need to check that the square (16.2) is cartesian, i.e. a pullback. By assumption, the squares (16.3) and (16.4) are cartesian since they are (pastings of) images of cartesian arrows under cartesian morphisms into the arrow categories of $\mathcal{A}$ and $\mathcal{B}$, respectively. Since the functor F preserves comprehension up to the natural isomorphism v_F, the image of (16.3) under F is still a pullback. We now use the fact that limits in the comma category $\mathcal{E} = G \downarrow F$ are computed componentwise whenever F preserves the limit in the second component. It follows that the square (16.2) is also a pullback.

- Note that $\chi_{\mathcal{E}}$ preserves identities and compositions because it preserves cartesian arrows.

We obtain the strict morphism of cwf's $\mathcal{E} \to \mathcal{A}$ by projecting contexts and types to their first component. □

We will now show that the generalized glueing construction is suitably functorial.

Proposition 13.2 (Generalized glueing: action on morphisms)**.** *Consider a diagram of morphisms of cwf's as below:*

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, "F"] \arrow[dd, "U_{\mathcal{A}}"'] & & \mathcal{B} \arrow[dl, "G", "\text{lax}"'] \arrow[dd, "U_{\mathcal{B}}"] \\
& \mathcal{C} \arrow[dd, "U_{\mathcal{C}}"] & \\
\mathcal{A}' \arrow[dr, "F'"'] & & \mathcal{B}' \arrow[dl, "G'", "\text{lax}"'] \\
& \mathcal{C}'. &
\end{tikzcd}$$

Assume that G and G' have right adjoints on types (R, ϵ) and (R', ϵ'), respectively, and that $(U_{\mathcal{B}}, U_{\mathcal{C}})\colon G \to G'$ satisfies the Beck-Chevalley condition. Then there is an induced morphism cwf's

$$\mathrm{GenGlue}(U, V, W)\colon \mathrm{GenGlue}(F, G) \to \mathrm{GenGlue}(F', G')$$

commuting with the forgetful strict morphisms as follows:

$$\begin{tikzcd} \mathrm{GenGlue}(F, G) \arrow[r, "\text{strict}"] \arrow[d] & \mathcal{A}' \arrow[d] \\ \mathrm{GenGlue}(F', G') \arrow[r, "\text{strict}"] & \mathcal{A}. \end{tikzcd}$$

Proof. We abbreviate

$$\begin{aligned} \mathcal{E} &=_{\mathrm{def}} \mathrm{GenGlue}(F, G), \\ \mathcal{E}' &=_{\mathrm{def}} \mathrm{GenGlue}(F, G), \\ T &=_{\mathrm{def}} \mathrm{GenGlue}(U, V, W). \end{aligned}$$

As in (11.2) and (11.3), we write d for the natural isomorphism of the Beck-Chevalley condition.

Contexts. Recall that the categories of contexts of source and target of T are given by $\mathcal{E} = G \downarrow F$ and $\mathcal{E}' = G' \downarrow F'$. The functor T between the categories of contexts is induced by functoriality of the comma category construction.

Explicitly:

- An object

$$(\Gamma_{\mathcal{A}}, \Gamma_B, \alpha) \in \mathcal{E}$$

  with $\alpha\colon G\Gamma_{\mathcal{B}} \to F\Gamma_{\mathcal{A}}$ is sent to the object

$$(U_{\mathcal{A}}\Gamma_{\mathcal{A}}, U_{\mathcal{B}}\Gamma_{\mathcal{B}}, U_{\mathcal{C}}\alpha) \in \mathcal{E}'$$

  with $U_{\mathcal{C}}\alpha\colon G'U_{\mathcal{B}}\Gamma_{\mathcal{B}} \to F'U_{\mathcal{A}}\Gamma_{\mathcal{A}}$.
- A morphism

$$(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})\colon (\Delta_{\mathcal{A}}, \Delta_B, \beta) \to (\Gamma_{\mathcal{A}}, \Gamma_B, \alpha)$$

  of $\mathcal{E}$ is sent to the morphism

$$(U_{\mathcal{A}}\sigma_{\mathcal{A}}, U_{\mathcal{B}}\sigma_{\mathcal{B}})\colon (U_{\mathcal{A}}\Delta_{\mathcal{A}}, U_{\mathcal{B}}\Delta_B, U_{\mathcal{C}}\beta) \to (U_{\mathcal{A}}\Gamma_{\mathcal{A}}, U_{\mathcal{B}}\Gamma_B, U_{\mathcal{C}}\alpha),$$

- preservation of identities and compositions follows from the that of $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$.

Types. We need to define a natural transformation $u_T\colon \mathbf{Ty}_{\mathcal{E}} \to \mathbf{Ty}_{\mathcal{E}'}T$.

- Given $(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha) \in \mathcal{E}$, let us define the component

$$(u_T)_{(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha)}\colon \mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha) \to \mathbf{Ty}_{\mathcal{E}'}(U_{\mathcal{A}}\Gamma_{\mathcal{A}}, U_{\mathcal{B}}\Gamma_{\mathcal{B}}, U_{\mathcal{C}}\alpha).$$

  This will be given as a dependent pairing of functions. On the first component, the function is

$$\mathbf{Ty}_{\mathcal{A}}(\Gamma_{\mathcal{A}}) \xrightarrow{\quad U_{\mathcal{A}} \quad} \mathbf{Ty}_{\mathcal{A}'}(U_{\mathcal{A}}\Gamma_{\mathcal{A}}).$$

Given $A \in \mathbf{Ty}_{\mathcal{A}}(\Gamma_{\mathcal{A}})$, the function on the second component is the composite

$$\begin{tikzcd}
\mathbf{Ty}_{\mathcal{B}}(R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha])) \arrow[dr, "U_{\mathcal{B}}"] & & & \\
& \mathbf{Ty}_{\mathcal{B}'}(U_{\mathcal{B}}R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha])) \arrow[dr, "(-)[d^{-1}_{(\Gamma_{\mathcal{B}},(FA_{\mathcal{A}})[\alpha])}]"] & & \\
& & \mathbf{Ty}_{\mathcal{B}'}(R'_{U_{\mathcal{B}}\Gamma_{\mathcal{B}}}U_{\mathcal{B}}((FA_{\mathcal{A}})[\alpha])) \arrow[dr, equals] & \\
& & & \mathbf{Ty}_{\mathcal{B}'}(R'_{U_{\mathcal{B}}\Gamma_{\mathcal{B}}}((F'U_{\mathcal{A}}A_{\mathcal{A}})[U_{\mathcal{C}}\alpha])).
\end{tikzcd}$$

- Given a morphism

$$(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}) \colon (\Delta_{\mathcal{A}}, \Delta_B, \beta) \to (\Gamma_{\mathcal{A}}, \Gamma_B, \alpha)$$

of $\mathcal{E}$, let us verify that u_T is natural at (σ_A, σ_B), i.e. that the square

$$\begin{tikzcd}
\mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha) \arrow[r, "(u_T)_{(\Gamma_{\mathcal{A}},\Gamma_{\mathcal{B}},\alpha)}"] \arrow[d, "(-)[(\sigma_A,\sigma_B)]"'] & \mathbf{Ty}_{\mathcal{E}'}(U_{\mathcal{A}}\Gamma_{\mathcal{A}}, U_{\mathcal{B}}\Gamma_{\mathcal{B}}, U_{\mathcal{C}}\alpha) \arrow[d, "(-)[(U_{\mathcal{A}}\sigma_A, U_{\mathcal{B}}\sigma_B)]"] \\
\mathbf{Ty}_{\mathcal{E}}(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}}, \beta) \arrow[r, "(u_T)_{(\Delta_{\mathcal{A}},\Delta_{\mathcal{B}},\beta)}"] & \mathbf{Ty}_{\mathcal{E}'}(U_{\mathcal{A}}\Delta_{\mathcal{A}}, U_{\mathcal{B}}\Delta_{\mathcal{B}}, U_{\mathcal{C}}\beta)
\end{tikzcd}$$

commutes. On the first component, this is commutativity of the square

$$\begin{tikzcd}
\mathbf{Ty}_{\mathcal{A}}(\Gamma_{\mathcal{A}}) \arrow[r, "U_{\mathcal{A}}"] \arrow[d, "(-)[\sigma_A]"'] & \mathbf{Ty}_{\mathcal{A}'}(U_{\mathcal{A}}\Gamma_{\mathcal{A}}) \arrow[d, "(-)[U_{\mathcal{A}}\sigma_A]"] \\
\mathbf{Ty}_{\mathcal{A}}(\Delta_{\mathcal{A}}) \arrow[r, "U_{\mathcal{A}}"] & \mathbf{Ty}_{\mathcal{A}'}(U_{\mathcal{A}}\Delta_{\mathcal{A}}),
\end{tikzcd}$$

i.e. naturality of $u_{U_{\mathcal{A}}}$ at $\sigma_{\mathcal{A}}$. On the second component, given $A_{\mathcal{A}} \in \mathbf{Ty}_{\mathcal{A}}(\Gamma_{\mathcal{A}})$, this is commutativity of the outer rectangle in the commuting diagram

$$\begin{tikzcd}
\mathbf{Ty}_{\mathcal{B}}(R_{\Delta_{\mathcal{B}}}((F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta])) \arrow[dr, "U_{\mathcal{B}}"] \arrow[dd, "(-)[R_{\sigma_B}((FA_{\mathcal{A}})[\alpha])]"'] & & & \\
& \mathbf{Ty}_{\mathcal{B}'}(U_{\mathcal{B}}R_{\Delta_{\mathcal{B}}}((F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta])) \arrow[dd, "(-)[U_{\mathcal{B}}R_{\sigma_B}((FA_{\mathcal{A}})[\alpha])]"'] \arrow[dr, "(-)[d^{-1}_{(\Delta_{\mathcal{B}},(F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta])}]"] & & \\
\mathbf{Ty}_{\mathcal{B}}(R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha])) \arrow[dr, "U_{\mathcal{B}}"'] & & \mathbf{Ty}_{\mathcal{B}'}(R'_{U_{\mathcal{B}}\Delta_{\mathcal{B}}}U_{\mathcal{B}}((F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta])) \arrow[dd, "(-)[R'_{U_{\mathcal{B}}\sigma_{\mathcal{B}}}U_{\mathcal{B}}((FA_{\mathcal{A}})[\alpha])]"'] \arrow[dr, equals] & \\
& \mathbf{Ty}_{\mathcal{B}'}(U_{\mathcal{B}}R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha])) \arrow[dr, "(-)[d^{-1}_{(\Gamma_{\mathcal{B}},(FA_{\mathcal{A}})[\alpha])}]"'] & & \mathbf{Ty}_{\mathcal{B}'}(R'_{U_{\mathcal{B}}\Delta_{\mathcal{B}}}((F'U_{\mathcal{A}}(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[U_{\mathcal{C}}\beta])) \arrow[dd, "(-)[R'_{U_{\mathcal{B}}\sigma_{\mathcal{B}}}((F'U_{\mathcal{A}}A_{\mathcal{A}})[U_{\mathcal{C}}\alpha])]"'] \\
& & \mathbf{Ty}_{\mathcal{B}'}(R'_{U_{\mathcal{B}}\Gamma_{\mathcal{B}}}U_{\mathcal{B}}((FA_{\mathcal{A}})[\alpha])) \arrow[dr, equals] & \\
& & & \mathbf{Ty}_{\mathcal{B}'}(R'_{U_{\mathcal{B}}\Gamma_{\mathcal{B}}}((F'U_{\mathcal{A}}A_{\mathcal{A}})[U_{\mathcal{C}}\alpha])).
\end{tikzcd}$$

The left square is the naturality square for $u_{U_{\mathcal{B}}}$ at the morphism $R_{\sigma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha])$. The middle square is the naturality square for d^{-1} at the morphism $(\sigma_{\mathcal{B}}, (FA_{\mathcal{A}})[\alpha])$ of $\mathcal{B} \times_{\mathcal{C}} \int \mathbf{Ty}_{\mathcal{C}}$. The right square commutes by rewriting using strict equalities of functors.

Comprehension. We need to define a natural transformation

$$\begin{tikzcd}
\int \mathbf{Ty}_{\mathcal{E}} \arrow[r, "\chi_{\mathcal{E}}"] \arrow[d, "\int u_T"'] \arrow[dr, phantom, "\swarrow v_T"] & \mathcal{E}^{\to} \arrow[d, "T^{\to}"] \\
\int \mathbf{Ty}_{\mathcal{E}'} \arrow[r, "\chi_{\mathcal{E}'}"] & (\mathcal{E}')^{\to}
\end{tikzcd}$$

such that $\mathrm{cod}_{\mathcal{E}'} v_t = \mathrm{id}$.

- Given $(\Gamma_A, \Gamma_B, \alpha) \in \mathcal{E}$ and $(A_{\mathcal{A}}, A_{\mathcal{B}}) \in \mathbf{Ty}_{\mathcal{E}}(\Gamma_A, \Gamma_B, \alpha)$, let us define the component
$$(v_T)_{(A_{\mathcal{A}}, A_{\mathcal{B}})} \colon T^{\to}\chi_{\mathcal{E}}(A_{\mathcal{A}}, A_{\mathcal{B}}) \to \chi_{\mathcal{E}'} \int u_T(A_{\mathcal{A}}, A_{\mathcal{B}})$$
of v_T at $((\Gamma_A, \Gamma_B, \alpha), (A_{\mathcal{A}}, A_{\mathcal{B}}))$.

  On the component of $\mathcal{A}'$, we need to find a morphism of arrows
$$\begin{tikzcd}
U_{\mathcal{A}}(\Gamma_{\mathcal{A}}.A_{\mathcal{A}}) \arrow[r, dotted] \arrow[d, "U_{\mathcal{A}} p"] & U_{\mathcal{A}}\Gamma_{\mathcal{A}}.U_{\mathcal{A}}A_{\mathcal{A}} \arrow[d, "p"] \\
U_{\mathcal{A}}\Gamma_{\mathcal{A}} \arrow[r, dotted] & U_{\mathcal{A}}\Gamma_{\mathcal{A}}.
\end{tikzcd}$$
We simply choose the component of $v_{U_{\mathcal{A}}}$ at $(\Gamma_{\mathcal{A}}, A_{\mathcal{A}})$. This has an identity at the bottom.

  On the component of $\mathcal{B}'$, we need to find dotted morphisms making the following outer square commute:
$$\begin{tikzcd}
U_{\mathcal{B}}(R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}}) \arrow[r, dotted] \arrow[d, "U_{\mathcal{A}} p"] & R'_{U_{\mathcal{B}}\Gamma_{\mathcal{B}}}((F(U_{\mathcal{A}}A_{\mathcal{A}}))[U_{\mathcal{C}}\alpha]).(U_{\mathcal{B}}A_{\mathcal{B}})[d^{-1}_{\Gamma_{\mathcal{B}},(FA_{\mathcal{A}})[\alpha]}] \arrow[d, "p"] \\
U_{\mathcal{B}}R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]) \arrow[r, dashed, "d_{(\Gamma_{\mathcal{B}},(FA_{\mathcal{A}})[\alpha])}"] \arrow[d] & R'_{U_{\mathcal{B}}\Gamma_{\mathcal{B}}}((FU_{\mathcal{A}}A_{\mathcal{A}})[U_{\mathcal{C}}\alpha]) \arrow[d] \\
U_{\mathcal{B}}\Gamma_{\mathcal{B}} \arrow[r, dotted] & U_{\mathcal{B}}\Gamma_{\mathcal{B}}.
\end{tikzcd}$$
We decompose the square into two smaller squares as indicated. The lower square is the component of d at $(\Gamma_{\mathcal{B}}, (FA_{\mathcal{A}})[\alpha])$. This has an identity at the bottom. The upper square decomposes as follows:
$$\begin{tikzcd}
U_{\mathcal{B}}(R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}}) \arrow[r] \arrow[d, "U_{\mathcal{A}} p"] & U_{\mathcal{B}}R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).U_{\mathcal{B}}A_{\mathcal{B}} \arrow[r] \arrow[d, "p"] & R'_{U_{\mathcal{B}}\Gamma_{\mathcal{B}}}((F(U_{\mathcal{A}}A_{\mathcal{A}}))[U_{\mathcal{C}}\alpha]).(U_{\mathcal{B}}A_{\mathcal{B}})[d^{-1}_{\Gamma_{\mathcal{B}},(FA_{\mathcal{A}})[\alpha]}] \arrow[d, "p"] \\
U_{\mathcal{B}}R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]) \arrow[r, equal] & U_{\mathcal{B}}R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]) \arrow[r, "d_{(\Gamma_{\mathcal{B}},(FA_{\mathcal{A}})[\alpha])}"] & R'_{U_{\mathcal{B}}\Gamma_{\mathcal{B}}}((FU_{\mathcal{A}}A_{\mathcal{A}})[U_{\mathcal{C}}\alpha]).
\end{tikzcd}$$
The left square is the component of $v_{U_{\mathcal{B}}}$ at $(R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]), A_{\mathcal{B}})$. The right square is the action of $\chi_{\mathcal{B}'}$ on the inverse of $(d^{-1}_{(\Gamma_{\mathcal{B}},(FA_{\mathcal{A}})[\alpha])}, U_{\mathcal{B}}A_{\mathcal{B}})$.

  We now need to show that the morphisms in $(\mathcal{A}')^{\to}$ and $(\mathcal{B}')^{\to}$ are compatible with the morphism in $(\mathcal{C}')^{\to}$ of the objects $T^{\to}\chi_{\mathcal{E}}(A_{\mathcal{A}}, A_{\mathcal{B}})$ and $\chi_{\mathcal{E}'} \int u_T(A_{\mathcal{A}}, A_{\mathcal{B}})$. Because these share the same codomain and the bottom morphisms of the squares in $\mathcal{A}'$ and $\mathcal{B}'$ are identities, it only remains to show compatibility on domains. [To be finished.]
- [Naturality of comprehension needs to be checked.]

□

We define

$$\mathrm{GenGlueInput} =_{\mathrm{def}} \mathbf{CwF}^{\to} \times_{\mathbf{CwF}} \mathbf{CwF}^{\to}_{\mathrm{lax,r}}$$

the category of input data for the generalized glueing construction (the pullback is with respect to

Theorem 13.3. *Generalized glueing forms a functor*

$$\mathrm{GenGlue}\colon \mathrm{GenGlueInput} \to \mathbf{CwF}$$

Proof. The action on objects and morphisms is defined in Proposition 13.1 and Proposition 13.2, respectively. It remains to verify that identities and compositions are preserved. [To be checked.] □

14. Thoughts on organizing generalized glueing better

Let us work with discrete fibrations instead of presheaves of types. Likely we do not even need discreteness. Then we can formulate the construction at the level of split comprehension categories.

Let

$$\begin{tikzcd} \mathbf{Ty}_{\mathcal{A}} \arrow[rr, "\chi_{\mathcal{A}}"] \arrow[dr] & & \mathcal{A}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{A} & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Ty}_{\mathcal{B}} \arrow[rr, "\chi_{\mathcal{B}}"] \arrow[dr] & & \mathcal{B}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{B} & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbf{Ty}_{\mathcal{C}} \arrow[rr, "\chi_{\mathcal{C}}"] \arrow[dr] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

be split comprehension categories. Let $F\colon \mathcal{A} \to \mathcal{C}$ be a morphism of split comprehension categories and $G\colon \mathcal{B} \to \mathcal{C}$ a lax morphism of split comprehension categories with a right adjoint on types. We will try to define the glued split comprehension category $\mathcal{E}$ in an as categorical way as possible.

The category of contexts of $\mathcal{E}$ is given by

$$G \downarrow F \simeq (\mathcal{B} \times \mathcal{A}) \times_{\mathcal{C}^2} \mathcal{C}^{\to}.$$

We have a functor $H\colon \mathcal{E} \times_{\mathcal{A}} \mathbf{Ty}_{\mathcal{A}} \to \mathcal{B}$ defined as the following composite:

$$\mathcal{E} \times_{\mathcal{A}} \mathbf{Ty}_{\mathcal{A}} \longrightarrow (\mathcal{B} \times \mathbf{Ty}_{\mathcal{C}}) \times_{\mathcal{C}^2} \mathcal{C}^{\to} \longrightarrow \mathcal{B} \times_{\mathcal{C}} \mathbf{Ty}_{\mathcal{C}} \longrightarrow \mathcal{B}^{\to} \longrightarrow \mathcal{B}.$$

We let $\mathbf{Ty}_{\mathcal{E}} \to \mathcal{E}$ be the pullback of $\mathbf{Ty}_{\mathcal{B}} \to \mathcal{B}$ along H followed by postcomposition with the pullback of $\mathbf{Ty}_{\mathcal{A}} \to \mathcal{A}$ along $\mathcal{E} \to \mathcal{A}$. [In a sense, the functor $\mathcal{E} \to \mathcal{A}$ is on the same footing as the functor H.]

Now we need to define $\mathbf{Ty}_{\mathcal{E}} \to \mathcal{E}^{\to}$ over $\mathcal{E}$.

Maps $\mathcal{D} \to \mathcal{E}$ are in correspondence with maps $U\colon \mathcal{D} \to \mathcal{A}$ and $V\colon \mathcal{D} \to \mathcal{B}$ with a natural transformation $GV \to FU$. An analogous correspondence holds for maps $\mathcal{D} \to \mathcal{E}^{\to}$.

We have a map

$$\mathbf{Ty}_{E} \longrightarrow \mathcal{E} \times_{\mathcal{A}} \mathbf{Ty}_{A} \longrightarrow \mathbf{Ty}_{\mathcal{A}} \longrightarrow \mathcal{A}^{\to}.$$

What about $\mathbf{Ty}_{\mathcal{E}} \to \mathcal{B}^{\to}$?

14.1. **A different approach.** Given

$$\begin{tikzcd} \mathcal{A} \arrow[dr, "F"'] & & \mathcal{B} \arrow[dl, "G", "\text{lax}"'] \\ & \mathcal{C}. & \end{tikzcd}$$

We view it as

$$\begin{tikzcd} \mathcal{A} \arrow[dr, "F"'] & & G \downarrow \mathcal{C} \arrow[dl] \\ & \mathcal{C}. & \end{tikzcd}$$

We show that $G \downarrow \mathcal{C}$ is a cwf. And then show that $G \downarrow \mathcal{C} \to \mathcal{C}$ is a special kind of map of cwf that can be pulled back (a discrete "opfibration").

A type in $G \downarrow \mathcal{C}$ over $(Y, f \colon FY \to X)$ consists of a type A in $\mathcal{C}$ over X and a type B in $\mathcal{B}$ over $R(A[f])$.

Definition 14.1. A lax morphism $F \colon \mathcal{C} \to \mathcal{D}$ of cwf's is called a *(discrete) (op)fibration* if the morphism (u_F, F) from $\int \mathbf{Ty}_\mathcal{C} \to \mathcal{C}$ to $\int \mathbf{Ty}_\mathcal{D} \to \mathcal{D}$ in $\mathbf{Cat}^\to$ is a Reedy (discrete) (op)fibration, viewing the walking arrow as an inverse category.

Concretely, this means that $\mathcal{C} \to \mathcal{D}$ and $\int \mathbf{Ty}_\mathcal{C} \to \mathcal{C} \times_\mathcal{D} \int \mathbf{Ty}_\mathcal{D}$ are a (discrete) (op)fibrations.

Remark 14.2. In $\mathbf{CwF}$, pullbacks along discrete opfibrations exist and are computed componentwise in $\mathbf{Cat}$. Discrete opfibrations are stable under pullback.

15. Pullbacks of cwf's

Lemma 15.1. *The forgetful functor* $\mathbf{Endo}_{\text{copt,pseudo}} \to \mathbf{Func}_{\text{pseudo}}$ *creates pullbacks in some sense.*

$$\epsilon \colon F^\mathcal{D} \to \mathrm{Id}_\mathcal{D}$$

The category of discrete fibrations has all limits. The category $\mathbf{CwF}_{\text{strict}}$ has all limits.

Lemma 15.2. *The category* $\mathbf{CwF}_{\text{pseudo}}$ *has pullbacks along strict isofibrations. Furthermore, the pullback of a strict isofibration can be chosen as a strict isofibration.*

Like for Lemma 18.6, isofibrations are invariant under isomorphisms of maps in $\mathbf{Func}_{\text{pseudo}}$, but being strict is not, explaining the last point of the statement.

Proof. We work with the presentation of $\mathbf{CwF}_{\text{pseudo}}$ given by Definition 5.5.

Consider a solid cospan in $\mathbf{CwF}_{\text{pseudo}}$ with P a strict isofibration as follows:

$$\begin{tikzcd} \mathcal{D} \arrow[r, dotted, "G"] \arrow[d, dotted, "Q"', "\text{isofib}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "P"', "\text{isofib}"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{A}. \end{tikzcd}$$

We have to construct a pullback as indicated such that Q a strict isofibration.

We work with the presentation of $\mathbf{CwF}_{\text{pseudo}}$ given by Definition 5.5. The contexts and types of the cwf $\mathcal{D}$ are constructed from those of $\mathcal{A}, \mathcal{B}, \mathcal{D}$ by taking the pullback in $\mathbf{DiscFib}$ (so that the pullback is preserved under the forgetful map $\mathbf{CwF}_{\text{pseudo}} \to \mathbf{DiscFib}$).

Next, we construct the cartesian copointed endofunctor on $(N^\mathcal{D}, \epsilon^\mathcal{D})$ on $\mathbf{Ty}^\mathcal{D}$.

$(N^{\mathcal{B}}, \epsilon^{\mathcal{B}})$
$(N^{\mathcal{A}}, \epsilon^{\mathcal{A}})$
$(N^{\mathcal{C}}, \epsilon^{\mathcal{C}})$

$$\begin{tikzcd}
\mathcal{D} \arrow[rr] \arrow[dr, dotted, "N^{\mathcal{D}}"] \arrow[dd] & & \mathcal{B} \arrow[dr, "N^{\mathcal{B}}"] \arrow[dd] & \\
& \mathcal{D} \arrow[rr] \arrow[dd] \arrow[r, phantom, "\beta \Nearrow"] & & \mathcal{B} \arrow[dd, two heads, "\text{isofib}"] \\
\mathcal{C} \arrow[rr] \arrow[dr, "N^{\mathcal{C}}"'] & & \mathcal{A} \arrow[dr, "N^{\mathcal{A}}"] & \\
& \mathcal{C} \arrow[rr] \arrow[r, phantom, "\alpha \Nearrow"] & & \mathcal{A}.
\end{tikzcd}$$

□

Definition 15.3. A morphism $P \colon \mathcal{E} \to \mathcal{B}$ in **CwF** is an *isofibration* if it is strict and the underlying functor P is a split isofibration of categories.

Proposition 15.4. *The category* **CwF** *has pullbacks along isofibrations. Isofibrations are stable under pullback.*

Proof. We are given a cospan

$$\begin{tikzcd}
\mathcal{D} \arrow[r, dotted, "\overline{F}"] \arrow[d, dotted, "\overline{P}"', "\text{isofib}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "P"', "\text{isofib}"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{A}.
\end{tikzcd}$$

in the category **CwF** as indicated by the solid arrows, and we wish to construct its pullback as indicated by the whole diagram.

To define the category of context $\mathcal{D}$, we take the pullback as above on underlying categories:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, dotted, "\overline{F}"] \arrow[d, dotted, two heads, "\overline{P}"', "\text{isofib}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, two heads, "P"', "\text{isofib}"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{A}.
\end{tikzcd} \tag{15.1}$$

Importantly, since $P \colon \mathcal{B} \to \mathcal{A}$ is an isofibration in **Cat**, so is $\overline{P} \colon \mathcal{D} \to \mathcal{C}$ and the pullback (15.1) is also a pullback in **Cat** as a weak 2-category (a homotopy pullback).

To define types $\mathbf{Ty}_{\mathcal{D}}$, we take the following pullback of presheaves over $\mathcal{D}^{\mathrm{op}}$:

$$\begin{tikzcd}
\mathbf{Ty}_{\mathcal{D}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Ty}_{\mathcal{B}}\overline{F} \arrow[d] \\
\mathbf{Ty}_{\mathcal{C}}\overline{P} \arrow[r] & \mathbf{Ty}_{\mathcal{A}}F\overline{P}.
\end{tikzcd}$$

Note that this makes

$$\begin{tikzcd}
\int \mathbf{Ty}_{\mathcal{D}} \arrow[r] \arrow[d, two heads, "\text{isofib}"] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathbf{Ty}_{\mathcal{B}} \arrow[d, two heads, "\text{isofib}"] \\
\int \mathbf{Ty}_{\mathcal{C}} \arrow[r] & \int \mathbf{Ty}_{\mathcal{A}}
\end{tikzcd} \tag{15.2}$$

a pullback of categories with isofibrations as indicated. Furthermore, there is an induced morphism in **Cat** from the square (15.2) to the square (15.1) which extends the given morphism between cospans.

It remains to define comprehension

$$\chi_{\mathcal{D}}\colon \int \mathbf{Ty}_{\mathcal{D}} \to \mathcal{D}^{\to}.$$

$$\begin{tikzcd}
\int \mathbf{Ty}_{\mathcal{D}} \arrow[rr] \arrow[dr, dotted] \arrow[dd] & & \int \mathbf{Ty}_{\mathcal{B}} \arrow[dr] \arrow[dd] & \\
& \mathcal{D}^{\to} \arrow[rr] \arrow[dd] & & \mathcal{B}^{\to} \arrow[dd, two heads, "\text{isofib}"] \\
\int \mathbf{Ty}_{\mathcal{C}} \arrow[rr] \arrow[dr] & & \int \mathbf{Ty}_{\mathcal{A}} \arrow[dr] & \\
& \mathcal{C}^{\to} \arrow[rr] & & \mathcal{A}^{\to}.
\end{tikzcd}$$

Remark 15.5 (Excursion on pullbacks of isofibrations)**.** Let $P\colon \mathcal{E} \to \mathcal{B}$ be a split isofibration and $F\colon \mathcal{B}' \to \mathcal{B}$ a functor:

$$\begin{tikzcd}
& \mathcal{E} \arrow[d, two heads, "P"', "\text{isofib}"] \\
\mathcal{B}' \arrow[r, "F"] & \mathcal{B}.
\end{tikzcd} \tag{15.3}$$

The category of *semi-strict cones* over the cospan (F, P) is defined as follows:

- The objects are tuples $(\mathcal{A}, G, Q, \theta)$ with a category $\mathcal{A}$, functors $Q\colon \mathcal{B}' \to \mathcal{E}$, $G\colon \mathcal{A} \to \mathcal{E}$, and a natural isomorphism $\theta\colon FQ \to PG$:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "G"] \arrow[d, two heads, "Q"'] \arrow[dr, phantom, "\swArrow\theta"] & \mathcal{E} \arrow[d, two heads, "P"', "\text{isofib}"] \\
\mathcal{B}' \arrow[r, "F"'] & \mathcal{B}.
\end{tikzcd}$$

- Morphisms from $(\mathcal{A}_1, G_1, Q_1, \theta_1)$ to $(\mathcal{A}_2, G_2, Q_2, \theta_2)$ are pairs (H, α) with a functor $H\colon \mathcal{A}_1 \to \mathcal{A}_2$ such that $Q_1 = HQ_2$ and a natural isomorphism $\alpha\colon G_1 \to G_2H$ such that $\theta_1 = \theta_2 H \circ P\alpha$:

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[drr, bend left, "G_1"] \arrow[dr, "H"] \arrow[ddr, bend right, "Q_1"'] \arrow[dr, phantom, shift right=3, "\swArrow\theta_1\ \nearrow\mathrm{id}"] & & \\
& \mathcal{A}_2 \arrow[r, "G_2"] \arrow[d, "Q_2"'] \arrow[u, phantom, shift left=4, "\Downarrow\alpha"] \arrow[dr, phantom, "\swArrow\theta_2"] & \mathcal{E} \arrow[d, two heads, "P"', "\text{isofib}"] \\
& \mathcal{B}' \arrow[r, "F"'] & \mathcal{B};
\end{tikzcd}$$

  and furthermore, for $X \in \mathcal{A}_1$, we have that $\alpha_X\colon G_1X \to G_2HX$ is the chosen lift through P of the isomorphism $(\theta_2 HX)^{-1} \circ \theta_1 X\colon PG_1X \to PG_2HX$ when given the lift G_1X of the domain. Note that this is both a constraint on G_2H and α, and that α is completely determined by this constraint.
- The identity on $(\mathcal{A}, G, Q, \theta)$ is given by $(\mathrm{Id}_{\mathcal{A}}, \mathrm{id})$. The composition

$$(\mathcal{A}_1, G_1, Q_1, \theta_1) \xrightarrow{(H_1, \alpha_1)} (\mathcal{A}_2, G_2, Q_2, \theta_2) \xrightarrow{(H_2, \alpha_2)} (\mathcal{A}_3, G_3, Q_3, \theta_3)$$

is given by $(H_2 H_1, \alpha_2 H_1 \circ \alpha_1)$. In both cases, we use that P is split.
- Neutrality and associativity laws are easily verified.

We take the ordinary pullback of the cospan (15.3) in **Cat**:

$$\begin{tikzcd}
\mathcal{E}' \arrow[r, "\overline{F}"] \arrow[d, two heads, "\overline{P}"', "\text{isofib}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, two heads, "P"', "\text{isofib}"] \\
\mathcal{B}' \arrow[r, "F"] & \mathcal{B}.
\end{tikzcd}$$

Note that $\overline{P}$ will again be a split isofibration with cleavage transferred from P (and $\overline{F}$ preserving the cleavage). Thus, we can regard $(\mathcal{E}', \overline{F}, \overline{P}, \mathrm{id})$ as a semi-strict cone over (F, P). We claim that it is terminal.

Consider an arbitrary object $(\mathcal{A}, G, Q, \theta)$. Our aim is to construct a unique morphism

$$(\mathcal{A}, G, Q, \theta) \xrightarrow{(H,\alpha)} (\mathcal{E}', \overline{F}, \overline{P}, \mathrm{id})$$

as indicated below:

$$\begin{tikzcd}
\mathcal{A} \arrow[drr, bend left, "G"] \arrow[dr, dotted, "H"] \arrow[ddr, bend right, "Q"', "\text{isofib}"] \arrow[dr, phantom, "\Downarrow\theta"'] & & \\
& \mathcal{E}' \arrow[r, "\overline{F}"] \arrow[d, "\overline{P}"', "\text{isofib}"] \arrow[ur, phantom, "\Downarrow\alpha"] \arrow[dl, phantom, "\Uparrow\mathrm{id}"] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & \mathcal{E} \arrow[d, two heads, "P"', "\text{isofib}"] \\
& \mathcal{B}' \arrow[r, "F"'] & \mathcal{B}.
\end{tikzcd}$$

In detail, we consider the set of pairs (H, α) with a functor $H \colon A \to \mathcal{E}'$ such that $\overline{P}H = Q$ and a natural isomorphism $\alpha \colon G \to \overline{F}H$ such that $P\alpha = \theta$ and furthermore, for $X \in \mathcal{A}$, we have that $\alpha_X \colon GX \to \overline{F}HX$ is the chosen lift through P of $\theta X \colon PGX \to P\overline{F}HX$ when given the lift GX of the domain.

Writing $H = \langle Q', G' \rangle$ with $Q' \colon \mathcal{A} \to \mathcal{B}'$ and $G' \colon \mathcal{A} \to \mathcal{E}$ such that $FQ' = PG'$, this is in bijection with the set of pairs (G', α) with a functor $G' \colon \mathcal{A} \to \mathcal{E}$ such that $PG' = P\overline{F}H$ and a natural isomorphism $\alpha \colon G \to G'$ such that $P\alpha = \theta$ and, furthermore, for $X \in \mathcal{A}$, we have that $\alpha_X \colon GX \to G'X$ is the chosen lift through P of $\theta X \colon PGX \to PG'X$ when given the lift GX of the domain.

The latter condition determines $(G'X, \alpha X)$ uniquely for every $X \in \mathcal{X}$, hence determines (G', α) uniquely. To see that the set is inhabited, define $(G'X, \alpha X)$ for every $X \in \mathcal{X}$ according to the chosen lift through P of $\theta X \colon PGX \to PG'X$ when given the lift GX of the domain. The action on morphisms of G' is transported from that of G using the family of isomorphisms α, making α natural. Preservation of identities and compositions for G' follows from that for G.

Then given a category $\mathcal{A}$, functors $G' \colon \mathcal{A} \to \mathcal{B}'$, $H \colon \mathcal{A} \to \mathcal{E}$ and a natural isomorphism $\alpha \colon FG \to HP$, there is a unique functor $H' \colon \mathcal{A} \to \mathcal{E}'$ and a natural isomorphism $\alpha' \colon \overline{F}H' \to H$ such that $G' = \overline{P}H'$ and $\alpha = P\alpha'$.

$\int \mathbf{Ty}_{\mathcal{D}} \to \mathcal{D}$ a pullback of that same data for $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in $\mathbf{Cat}^{\to}$, and that this is also a weak pullback. $\square$

To define types

$$\mathbf{Ty}_{\mathcal{D}} \colon \mathcal{D}^{\to} \to \mathbf{Set},$$

we take the pullback of presheaves as follows:

$$\begin{tikzcd} \mathbf{Ty}_{\mathcal{D}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Ty}_{\mathcal{B}}\overline{F} \arrow[d] \\ \mathbf{Ty}_{\mathcal{C}}\overline{P} \arrow[r] & \mathbf{Ty}_{\mathcal{A}}F\overline{P}. \end{tikzcd}$$

Note that this makes $\int \mathbf{Ty}_{\mathcal{D}} \to \mathcal{D}$ a pullback of that same data for $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in $\mathbf{Cat}^{\to}$, and that this is also a weak pullback. Now we need to define comprehension

$$\chi_{\mathcal{D}} \colon \int \mathbf{Ty}_{\mathcal{D}} \to \mathcal{D}^{\to}.$$

For this, we use the weak pullback

$$\begin{tikzcd} \mathcal{D}^{\to} \arrow[r, "\overline{F}^{\to}"] \arrow[d, "\overline{P}^{\to}"', "\text{strict}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B}^{\to} \arrow[d, "P^{\to}"', "\text{strict}"] \\ \mathcal{C}^{\to} \arrow[r, "F^{\to}"] & \mathcal{A}^{\to}. \end{tikzcd}$$

The comprehension functors of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ assemble to a morphism in the (weak) 2-category of spans from the cospan of categories of types to the cospan in the above diagram. By 2-functoriality of weak pullbacks, we thus obtain a functor $\chi_{\mathcal{D}}$ as required.

Explicitly, given an object (Θ, Γ, Δ) where $\Gamma = F\Theta$ and $\Gamma = P\Delta$ of $\mathcal{D}$ and an element (C, A, B) of $\mathbf{Ty}_{\mathcal{D}}(\Theta, \Gamma, \Delta)$ where $A = FC$ and $A = PB$, we have

$$\chi_{\mathcal{D}}((\Theta, \Gamma, \Delta), (C, A, B)) = (\chi_{\mathcal{C}}(\Theta, C), \chi_{\mathcal{A}}(\Gamma, A), \chi_{\mathcal{B}}(\Delta, B)).$$

$$\begin{tikzcd} \mathbf{Ty}_{\mathcal{D}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Ty}_{\mathcal{B}} \arrow[d] \\ \mathbf{Ty}_{\mathcal{C}} \arrow[r] & \mathbf{Ty}_{\mathcal{A}} \end{tikzcd}$$

16. Abstracted glueing

Proposition 16.1 (Abstracted glueing). *Let $G \colon \mathcal{B} \to \mathcal{C}$ be a lax morphism of cwf's that has a right adjoint on types (R, ϵ). Then there is a cwf* $\mathrm{AGlue}(G)$ *and a discrete opfibration* $\mathrm{AGlue}(G) \to \mathcal{C}$.

Proof. We abbreviate $\mathcal{E} =_{\text{def}} \mathrm{AGlue}(G)$.

Contexts. We set $\mathcal{E} =_{\text{def}} G \downarrow \mathcal{C}$. Explicitly:

- an object is a tuple $(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha)$ where $\Gamma_{\mathcal{C}} \in \mathcal{C}$, $\Gamma_{\mathcal{B}} \in \mathcal{B}$, and $\alpha \colon G\Gamma_{\mathcal{B}} \to \Gamma_{\mathcal{C}}$,
- a morphism from $(\Delta_{\mathcal{C}}, \Delta_{\mathcal{B}}, \beta)$ to $(\Gamma_{\mathcal{C}}, \Gamma_{B}, \alpha)$ is a pair $(\sigma_{\mathcal{C}}, \sigma_{\mathcal{B}})$ where $\sigma_{\mathcal{C}} \colon \Delta_{\mathcal{C}} \to \Gamma_{\mathcal{C}}$ and $\sigma_{\mathcal{B}} \colon \Delta_{\mathcal{B}} \to \Gamma_{\mathcal{B}}$ such that $\sigma_{\mathcal{C}} \circ \beta = \alpha \circ G\sigma_{\mathcal{B}}$:

$$\begin{tikzcd} G\Delta_{\mathcal{B}} \arrow[r, "\beta"] \arrow[d, "G\sigma_{\mathcal{B}}"'] & \Delta_{\mathcal{C}} \arrow[d, "F\sigma_{\mathcal{A}}"] \\ G\Gamma_{\mathcal{B}} \arrow[r, "\alpha"] & \Gamma_{\mathcal{C}}, \end{tikzcd} \tag{16.1}$$

- identity and composition of morphisms is given componentwise,
- neutrality and associativity laws follow componentwise.

Types. The presheaf $\mathbf{Ty}_{\mathcal{E}} \colon \mathcal{E}^{\text{op}} \to \mathbf{Set}$ is defined as follows.

- Given $(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha) \in \mathcal{E}$, we let $\mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha)$ be the set of pairs $(A_{\mathcal{C}}, A_{\mathcal{B}})$ where $A_{\mathcal{C}} \in \mathbf{Ty}_{\mathcal{C}}(\Gamma_{\mathcal{C}})$ and $A_{\mathcal{B}} \in \mathbf{Ty}_{\mathcal{B}}(R_{\Gamma_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha]))$.

- Given a morphism
$$(\sigma_{\mathcal{C}}, \sigma_{\mathcal{B}})\colon (\Delta_{\mathcal{C}}, \Delta_{\mathcal{B}}, \beta) \to (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha)$$
in $\mathcal{E}$, we define
$$\mathbf{Ty}_{\mathcal{E}}(\sigma_{\mathcal{C}}, \sigma_{\mathcal{B}})\colon \mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha) \to \mathbf{Ty}_{\mathcal{E}}(\Delta_{\mathcal{C}}, \Delta_{\mathcal{B}}, \beta)$$
by sending $(A_{\mathcal{C}}, A_{\mathcal{B}})$ to $(A_{\mathcal{C}}[\sigma_{\mathcal{C}}], A_{\mathcal{B}}[R_{\sigma_B}(A_{\mathcal{C}}[\alpha])])$.
- Coherence with respect to identity and composition of morphisms in $\mathcal{E}$ follow from those of $\mathbf{Ty}_{\mathcal{C}}$ and $\mathbf{Ty}_{\mathcal{B}}$ with respect to morphisms in $\mathcal{C}$ and $\mathcal{B}$, respectively, and the fact that $R\colon \mathcal{B} \times_{\mathcal{E}} \int \mathbf{Ty}_{\mathcal{C}} \to \mathcal{B}^{\to}$ is a functor.

Comprehension. Let us define the comprehension functor $\chi_{\mathcal{E}}$ as below:

$$\begin{tikzcd}
\int \mathbf{Ty}_{\mathcal{E}} \arrow[rr, dotted, "\chi_{\mathcal{E}}"] \arrow[dr, "\pi_1"'] & & \mathcal{E}^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{E}. &
\end{tikzcd}$$

- On objects, we define $\chi_{\mathcal{E}}$ as follows. Suppose we are given $(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha) \in \mathcal{E}$ and $(A_{\mathcal{C}}, A_{\mathcal{B}}) \in \mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha)$. The context extension
$$(\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha).(A_{\mathcal{A}}, A_{\mathcal{B}}) \in \mathcal{E}$$
is defined as
$$(\Gamma_{\mathcal{C}}.A_{\mathcal{C}}, R_{\Gamma_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha]).A_{\mathcal{B}}, \gamma)$$
where
$$\gamma\colon G(R_{\Gamma_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha]).A_{\mathcal{B}}) \to \Gamma_{\mathcal{C}}.A_{\mathcal{C}}$$
is defined as the composite
$$\begin{tikzcd}
& GR_{\Gamma_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha]) \arrow[dr, "\epsilon_{(\Gamma_{\mathcal{B}}, A_{\mathcal{A}}[\alpha])}"] & & \\
GR_{\Gamma_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha]).GA_{\mathcal{B}} \arrow[ur, "p"] & & G\Gamma_{B}.A_{\mathcal{C}}[\alpha] \arrow[r, "\alpha.A_{\mathcal{C}}"] & \Gamma_{\mathcal{C}}.A_{\mathcal{C}}.
\end{tikzcd}$$
The context projection
$$p(A_{\mathcal{C}}, A_{\mathcal{B}})\colon (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha).(A_{\mathcal{C}}, A_{\mathcal{B}}) \to (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha)$$
is the pair of maps consisting of $p(A_{\mathcal{C}})\colon \Gamma_{\mathcal{C}}.A_{\mathcal{C}} \to \Gamma_{\mathcal{C}}$ and
$$R_{\Gamma_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha]).A_{\mathcal{B}} \xrightarrow{p} R_{\Gamma_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha]) \longrightarrow \Gamma_{\mathcal{B}}.$$
- Let us define $\chi_{\mathcal{E}}$ on morphisms. Suppose we are given a morphism
$$(\sigma_{\mathcal{C}}, \sigma_{\mathcal{B}})\colon (\Delta_{\mathcal{C}}, \Delta_{\mathcal{B}}, \beta) \to (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha)$$
in $\mathcal{E}$ and a type $(A_{\mathcal{C}}, A_{\mathcal{B}}) \in \mathbf{Ty}_{\mathcal{E}}(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha)$. We want to define the morphism of context extensions
$$(\sigma_{\mathcal{C}}, \sigma_{\mathcal{B}}).(A_{\mathcal{C}}, A_{\mathcal{B}})\colon (\Delta_{\mathcal{C}}, \Delta_{\mathcal{B}}, \beta).(A_{\mathcal{C}}, A_{\mathcal{B}})[(\sigma_{\mathcal{C}}, \sigma_{\mathcal{B}})] \to (\Gamma_{\mathcal{C}}, \Gamma_{\mathcal{B}}, \alpha).(A_{\mathcal{C}}, A_{\mathcal{B}}),$$
where the signature unfolds to
$$(\Delta_{\mathcal{C}}.A_{\mathcal{C}}[\sigma_{\mathcal{C}}], R_{\Delta_{\mathcal{B}}}(A_{\mathcal{C}}[\sigma_{\mathcal{C}} \circ \beta]).A_{\mathcal{B}}[R_{\sigma_B}(A_{\mathcal{C}}[\sigma_{\mathcal{C}} \circ \beta])], \delta) \to$$
$$(\Gamma_{\mathcal{C}}.A_{\mathcal{C}}, R_{\Gamma_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha]).A_{\mathcal{B}}, \gamma)$$

where γ is as before and δ is the composite

$$\begin{tikzcd}
 & GR_{\Delta_{\mathcal{B}}}(A_{\mathcal{C}}[\sigma_{\mathcal{C}} \circ \beta]) \arrow[dr, "\epsilon(\Gamma_{\mathcal{B}}, A_{\mathcal{C}}[\sigma_{\mathcal{C}} \circ \beta])"] & & \Delta_{\mathcal{C}}.A_{\mathcal{C}}[\sigma_{\mathcal{C}}]. \\
GR_{\Delta_{\mathcal{B}}}(A_{\mathcal{C}}[\sigma_{\mathcal{A}} \circ \beta]).GA_{\mathcal{B}}[R_{\sigma_B}(A_{\mathcal{C}}[\sigma_{\mathcal{A}} \circ \beta])] \arrow[ur, "p"] & & G\Delta_B.A_{\mathcal{C}}[\sigma_{\mathcal{C}} \circ \beta] \arrow[ur, "\beta.A_{\mathcal{C}}[\alpha]"'] &
\end{tikzcd}$$

We take

$$(\sigma_{\mathcal{C}}, \sigma_{\mathcal{B}}).(A_{\mathcal{C}}, A_{\mathcal{B}}) =_{\mathrm{def}} (\sigma_{\mathcal{C}}.A_{\mathcal{A}}, R_{\sigma_B}(A_{\mathcal{C}}[\alpha]).A_{\mathcal{B}}).$$

The second component uses

$$R_{\sigma_B}(A_{\mathcal{C}}[\alpha])\colon R_{\Delta_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha \circ G\sigma_{\mathcal{B}}]) \to R_{\Gamma_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha]).$$

To make $(\sigma_{\mathcal{C}}, \sigma_{\mathcal{B}}).(A_{\mathcal{C}}, A_{\mathcal{B}})$ into a morphism in $\mathcal{E}$, we need to check that the following diagram commutes:

$$\begin{tikzcd}
G(R_{\Delta_{\mathcal{B}}}(A_{\mathcal{C}}[\sigma_{\mathcal{C}} \circ \beta]).A_{\mathcal{B}}[R_{\sigma_B}(A_{\mathcal{C}}[\sigma_{\mathcal{C}} \circ \beta])]) \arrow[d, "G(R_{\sigma_B}(A_{\mathcal{C}}[\alpha]).A_{\mathcal{B}})"] \arrow[r, "\delta"] & \Delta_{\mathcal{C}}.A_{\mathcal{C}}[\sigma_{\mathcal{C}}] \arrow[d, "\sigma_{\mathcal{C}}.A_{\mathcal{C}}"] \\
G(R_{\Gamma_{\mathcal{B}}}(A_{\mathcal{C}}[\alpha]).A_{\mathcal{B}}) \arrow[r, "\gamma"] & \Gamma_{\mathcal{C}}.A_{\mathcal{C}}.
\end{tikzcd}$$

Since $GR_{\sigma_B}(A_{\mathcal{C}}[\alpha]).GA_{\mathcal{B}}$ lives over $GR_{\sigma_B}(A_{\mathcal{C}}[\alpha])$ with respect to the context projections that form the first factors of γ' and δ', respectively, this follows from commutativity of the following diagram:

$$\begin{tikzcd}
GR_{\Delta_{\mathcal{B}}}((F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta]) \arrow[r, "\epsilon(\Gamma_{\mathcal{B}}, (F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta])"] \arrow[d, "GR_{\sigma_B}((FA_{\mathcal{A}})[\alpha])"] & G\Delta_B.(F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]))[\beta] \arrow[r, "\beta.F(A_{\mathcal{A}}[\alpha])"] \arrow[d, "G\sigma_B.(FA_{\mathcal{A}})[\alpha]"] & F\Delta_{\mathcal{A}}.F(A_{\mathcal{A}}[\sigma_{\mathcal{A}}]) \arrow[d, "F\sigma_{\mathcal{A}}.FA_{\mathcal{A}}"] \\
GR_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]) \arrow[r, "\epsilon(\Gamma_{\mathcal{B}}, (FA_{\mathcal{A}})[\alpha])"] & G\Gamma_B.(FA_{\mathcal{A}})[\alpha] \arrow[r, "\alpha.FA_{\mathcal{A}}"] & F\Gamma_{\mathcal{A}}.FA_{\mathcal{A}}.
\end{tikzcd}$$

Here, the left square is a naturality square for the counit ϵ of R and the right square comes substituting the type $FA_{\mathcal{A}}$ along the morphisms of the commuting square 16.1.

We now need to show that the morphism $(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}).(A_{\mathcal{A}}, A_{\mathcal{B}})$ lives over $(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})$ with respect to the context projections, i.e. that the square

$$\begin{tikzcd}
(\Delta_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \beta).(A_{\mathcal{A}}, A_{\mathcal{B}})[(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})] \arrow[r, "(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}).(A_{\mathcal{A}}, A_{\mathcal{B}})"] \arrow[d, "p((A_{\mathcal{A}}, A_{\mathcal{B}})[(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})])"'] & (\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha).(A_{\mathcal{A}}, A_{\mathcal{B}}) \arrow[d, "p(A_{\mathcal{A}}, A_{\mathcal{B}})"] \\
(\Delta_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \beta) \arrow[r, "(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})"] & (\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}, \alpha)
\end{tikzcd} \tag{16.2}$$

commutes. By the construction of the category $\mathcal{E}$, this decomposes into commutativity of the square

$$\begin{tikzcd}
\Delta_{\mathcal{A}}.\mathbf{Ty}_{\mathcal{A}}(\sigma_{\mathcal{A}})(A_{\mathcal{A}}) \arrow[r, "\sigma_{\mathcal{A}}.A_{\mathcal{A}}"] \arrow[d, "p"'] & \Gamma_{\mathcal{A}}.A_{\mathcal{A}} \arrow[d, "p"] \\
\Delta_{\mathcal{A}} \arrow[r, "\sigma_{\mathcal{A}}"] & \Gamma_{\mathcal{A}},
\end{tikzcd} \tag{16.3}$$

which is a morphism in the image of $\chi_{\mathcal{A}}$, and the square

$$\begin{tikzcd}
R_{\Delta_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha][G\sigma_{\mathcal{B}}]).A_{\mathcal{B}}[R_{\Delta_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha][G\sigma_{\mathcal{B}}])] \arrow[r, "R_{\sigma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}}"] \arrow[d, "p"] & R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]).A_{\mathcal{B}} \arrow[d, "p"] \\
R_{\Delta_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha][G\sigma_{\mathcal{B}}]) \arrow[r, "R_{\sigma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha])"] \arrow[d] & R_{\Gamma_{\mathcal{B}}}((FA_{\mathcal{A}})[\alpha]) \arrow[d] \\
\Delta_{\mathcal{B}} \arrow[r, "\sigma_{\mathcal{B}}"] & \Gamma_{\mathcal{B}},
\end{tikzcd} \tag{16.4}$$

which is a pasting of a morphism in the image of $\chi_{\mathcal{B}}$ and a morphism in the image of R.

Finally, we need to check that the square (16.2) is cartesian, i.e. a pullback. By assumption, the squares (16.3) and (16.4) are cartesian since they are (pastings of) images of cartesian arrows under cartesian morphisms into the arrow categories of $\mathcal{A}$ and $\mathcal{B}$, respectively. Since the functor F preserves comprehension up to the natural isomorphism v_F, the image of (16.3) under F is still a pullback. We now use the fact that limits in the comma category $\mathcal{E} = G \downarrow F$ are computed componentwise whenever F preserves the limit in the second component. It follows that the square (16.2) is also a pullback.

- Note that $\chi_{\mathcal{E}}$ preserves identities and compositions because it preserves cartesian arrows.

We obtain the strict morphism of cwf's $\mathcal{E} \to \mathcal{A}$ by projecting contexts and types to their first component. □

17. Appendix: Fibered right adjoints

Lemma 17.1. *Let L be an opcartesian morphism over $\mathcal{B}$ and $\mathcal{C} \to \mathcal{B}$ an opfibration in the strictly commuting diagram below:*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "L"] \arrow[dr, two heads, "\text{opfib}"'] & & \mathcal{D} \arrow[dl, two heads] \\
& \mathcal{B}. &
\end{tikzcd}$$

Let $L_A \colon \mathcal{C}_A \to \mathcal{D}_A$ have a right adjoint (R_A, ϵ_A) for every $A \in \mathcal{B}$. Then these local right adjoints assemble to a global right adjoint (R, ϵ) of L that is a cartesian morphism over $\mathcal{B}$.

Proof. Standard.

Consider the inclusion

$$L_A \downarrow_{\mathcal{D}_A} Y \to L \downarrow_{\mathcal{D}} Y \tag{17.1}$$

for $A \in \mathcal{B}$ and $Y \in \mathcal{D}_A$. The given right adjoints (R_A, ϵ_A) to L_A for $A \in \mathcal{B}$ consist of a terminal object in the domain of (17.1) for every $Y \in \mathcal{D}_A$. The desired right adjoint (R, ϵ) to L consists of a terminal object in the codomain of (17.1) for every

$A \in \mathcal{B}$ and $Y \in \mathcal{D}_A$. It thus suffices to show that the inclusion (17.1) preserves terminal objects. But it is a right adjoint since L preserves opcartesian arrows.

By construction of RY via the initial object of $L_A \downarrow_{\mathcal{D}_A} Y$, we see that the action of objects of R commutes strictly with those of the functors to $\mathcal{B}$. The unit and counit maps are those of $L_A \dashv R_A$ and thus vertical (map to identities in $\mathcal{B}$). By naturality of counits, then R is a functor over $\mathcal{B}$ since L is.

Let us verify that R is a cartesian morphism. Let $Y \to Z$ be a cartesian arrow in $\mathcal{D}$ over $B \to C$ in $\mathcal{B}$. To show that $RY \to RZ$ is cartesian, consider $A \to B$ in $\mathcal{B}$ and a lift $X \to RZ$ of $A \to C$ to $\mathcal{C}$. Our goal is to show that there is a unique lift $X \to RY$ of $A \to B$ making the evident triangle in $\mathcal{C}$ commute, as seen in the left diagram below:

$$\begin{tikzcd}
X \arrow[dr, dotted] \arrow[drr, bend left] & & & \mathcal{C} \arrow[dd] \\
& RY \arrow[r] & RZ & \\
A \arrow[dr] \arrow[drr, bend left] & & & \mathcal{B}, \\
& B \arrow[r] & C &
\end{tikzcd}
\qquad
\begin{tikzcd}
LX \arrow[dr, dotted] \arrow[drr, bend left] & & & \mathcal{D} \arrow[dd] \\
& Y \arrow[r, "\mathsf{cart}"'] & Z & \\
A \arrow[dr] \arrow[drr, bend left] & & & \mathcal{B}. \\
& B \arrow[r] & C &
\end{tikzcd}$$

Using the adjunction $L \dashv R$, such lifts correspond bijectively to lifts as in the right diagram, which has a unique solution since $Y \to Z$ was assumed cartesian. $\square$

Lemma 17.1 has a straightforward generalization to relative right adjoints.

Lemma 17.2. *Let L be an opcartesian morphism and J a cartesian morphism over $\mathcal{B}$ and $\mathcal{C} \to \mathcal{B}$ an opfibration in the strictly commuting diagram below:*

$$\begin{tikzcd}
& & \mathcal{D}' \arrow[d, "J"] \arrow[dll, dotted, "R"'] \arrow[ddl] \\
\mathcal{C} \arrow[rr, "L"] \arrow[dr, "\text{opfib}"'] & & \mathcal{D} \arrow[dl] \\
& \mathcal{B}. &
\end{tikzcd}$$

Let $L_A \colon \mathcal{C}_A \to \mathcal{D}_A$ have a J_A-relative right adjoint (R_A, ϵ_A) for every $A \in \mathcal{B}$. Then these local relative right adjoints assemble to a global J-relative right adjoint (R, ϵ) of L that is a cartesian morphism over $\mathcal{B}$.

Proof. Consider the inclusion

$$L_A \downarrow_{\mathcal{D}_A} J_A Y \to L \downarrow_{\mathcal{D}} JY \tag{17.2}$$

for $A \in \mathcal{B}$ and $Y \in \mathcal{D}'_A$. The given right J_A-relative adjoints (R_A, ϵ_A) to L_A for $A \in \mathcal{B}$ consist of a terminal object in the domain of (17.2) for every $Y \in \mathcal{D}'_A$. The desired J-relative right adjoint (R, ϵ) to L consists of a terminal object in the codomain of (17.2) for every $A \in \mathcal{B}$ and $Y \in \mathcal{D}'_A$. It thus suffices to show that the inclusion (17.2) preserves terminal objects. But it is a right adjoint since L preserves opcartesian arrows.

By construction of RY via the initial objects of $L_A \downarrow_{\mathcal{D}_A} J_A Y$, we see that the action of objects of R commutes strictly with those of the functors to $\mathcal{B}$. The counit maps are those of $L_A \dashv R_A$ and thus vertical (map to identities in $\mathcal{B}$). By naturality of counits, then R is a functor over $\mathcal{B}$ since L is.

Let us verify that R is a cartesian morphism. Let $Y \to Z$ be a cartesian arrow in $\mathcal{D}'$ over $B \to C$ in $\mathcal{B}$. To show that $RY \to RZ$ is cartesian, consider $A \to B$ in $\mathcal{B}$ and a lift $X \to RZ$ of $A \to C$ to $\mathcal{C}$. Our goal is to show that there is a unique lift $X \to RY$ of $A \to B$ making the evident triangle in $\mathcal{C}$ commute, as seen in the left diagram below:

$$
\begin{tikzcd}
X \arrow[dr, dotted] \arrow[drr, bend left] & & & \\
& RY \arrow[r] & RZ & \mathcal{C} \arrow[dd] \\
A \arrow[dr] \arrow[drr, bend left] & & & \\
& B \arrow[r] & C & \mathcal{B},
\end{tikzcd}
\qquad
\begin{tikzcd}
LX \arrow[dr, dotted] \arrow[drr, bend left] & & & \\
& JY \arrow[r, "\mathsf{cart}"'] & JZ & \mathcal{D} \arrow[dd] \\
A \arrow[dr] \arrow[drr, bend left] & & & \\
& B \arrow[r] & C & \mathcal{B}.
\end{tikzcd}
$$

Using the adjunction $L \dashv R$, such lifts correspond bijectively to lifts as in the right diagram, which has a unique solution as $JY \to JZ$ is cartesian since $Y \to Z$ was assumed cartesian and J is a cartesian morphism. □

18. Appendix: Isofibrations

Isofibrations are functors which admit lifts of isomorphisms given a lift of one of their endpoints.

Definition 18.1. A functor $P\colon \mathcal{E} \to \mathcal{B}$ is an *isofibration* if, given an object $X \in \mathcal{E}$ and an isomorphism $f\colon PX \to B$ in $\mathcal{B}$, there is an isomorphism $u\colon X \to Y$ in $\mathcal{E}$ such that $Pu = f$ (and hence $PY = B$).

If there is always a unique such isomorphism u, we call P a *discrete isofibration.*

We may succinctly express that P is an isofibration by saying that it has the right lifting property against the inclusion $\{\bullet\} \hookrightarrow \{\bullet \simeq \bullet\}$. For P a discrete isofibration, the right lifting property is replaced by right orthogonality.

A *cleavage* for an isofibration P is a choice of lifts of isomorphisms as in Definition 18.1. We then speak of a *cloven* isofibration. For us, all isofibrations will be cloven by default, and we will not mentioned this anymore.

Lemma 18.2. *Given composable functors P and Q, if Q is a discrete isofibration, then the following are equivalent:*

(i) P is a (discrete) isofibration,
(ii) QP is a (discrete) isofibration.

Proof. This is an instance of an abstract fact about (weak) factorization systems. □

Corollary 18.3. *Let $(U, V)\colon P \to Q$ be a morphism between discrete fibrations. If V is an isofibration, then so is U.*

Proof. Observe that discrete fibrations are discrete isofibrations and use Lemma 18.2. □

Isofibrations are stable under pullback in **Cat** since they are defined by a right lifting property. Recall that isofibrations are the fibrations of the canonical model structure on **Cat**, in which every object is fibrant. Pullbacks along isofibrations are thus also homotopy pullbacks, 2-pullbacks in the 2-category **Cat**. This 1-categorical and 2-categorical behaviour gives rise to mixed properties as illustrated by the following statement.

Lemma 18.4. *Consider a pullback of an isofibration as follows:*

$$\begin{tikzcd}
\mathcal{E}' \arrow[r, "F'"] \arrow[d, two heads, "P'"', "\text{isofib}"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E} \arrow[d, two heads, "P"', "\text{isofib}"] \\
\mathcal{B}' \arrow[r, "F"] & \mathcal{B}.
\end{tikzcd}$$

Then for any 2-cone

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "G"] \arrow[d, "Q"'] \arrow[dr, phantom, "\overset{\simeq}{\swarrow}\theta"] & \mathcal{E} \arrow[d, two heads, "P"', "\text{isofib}"] \\
\mathcal{B}' \arrow[r, "F"'] & \mathcal{B},
\end{tikzcd}$$

there is a functor $H\colon \mathcal{A} \to \mathcal{E}'$ such that $Q = P'H$ with a natural isomorphism $\alpha\colon G \to F'H$ such that $P\alpha = \theta$:

$$\begin{tikzcd}
\mathcal{A} \arrow[drr, bend left, "G"] \arrow[dr, dotted, "H"] \arrow[ddr, bend right, "Q"'] & & \\
& \mathcal{E}' \arrow[r, "F'"] \arrow[d, two heads, "P'"', "\text{isofib}"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E} \arrow[d, two heads, "P"', "\text{isofib}"] \\
& \mathcal{B}' \arrow[r, "F"'] & \mathcal{B}.
\end{tikzcd} \tag{18.1}$$

Proof. For each object $X \in \mathcal{A}$, we have an isomorphism $\theta_X : PGX \to FQX$ in $\mathcal{B}$. Since P is an isofibration, this lifts to an isomorphism we call $\alpha_X : GX \to G'X$. This defines the action on objects of a functor $G' : \mathcal{A} \to \mathcal{E}$ and of a natural transformation $\alpha\colon G \to G'$. The action on morphisms of G' is transported from that of G using the family of isomorphisms α, making α natural. Preservation of identities and compositions by G' follows from that by G.

Since $PG' = FQ$, we can now define $H = \langle Q, G' \rangle$. This makes the left triangle in (18.1) commute and makes α have the correct signature in the top triange (18.1) by construction. We have $P\alpha = \theta$ by construction of α. $\square$

Corollary 18.5. *Consider a diagram in* **Cat**

$$\begin{tikzcd}
& & \mathcal{B}' \arrow[dr] \arrow[dd] & \\
& & & \mathcal{B} \arrow[dd, two heads, "P"', "\text{isofib}"] \\
\mathcal{C}' \arrow[rr] \arrow[dr] & & \mathcal{A}' \arrow[dr] & \\
& \mathcal{C} \arrow[rr] \arrow[ur, phantom, "\overset{\alpha}{\nearrow}\simeq"] & & \mathcal{A}.
\end{tikzcd} \tag{18.2}$$

where the right square commutes and the bottom squares commutes up to a natural isomorphism α as indicated. Then there is an induced functor between the pullbacks *as in*

$$
\begin{tikzcd}
\mathcal{D}' \arrow[rr] \arrow[dd, "Q'"'] \arrow[dr, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathcal{B}' \arrow[dr] \arrow[dd] \arrow[dl, phantom, "\overset{\beta}{\Nearrow}\simeq"] & \\
& \mathcal{D} \arrow[rr, crossing over] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathcal{B} \arrow[dd, two heads, "P"', "\text{isofib}"] \\
\mathcal{C}' \arrow[rr] \arrow[dr] & & \mathcal{A}' \arrow[dr] \arrow[dl, phantom, "\overset{\alpha}{\Nearrow}\simeq"] & \\
& \mathcal{C} \arrow[rr] & & \mathcal{A}.
\end{tikzcd}
\tag{18.3}
$$

where the left square commutes and the top square commutes up to a natural isomorphism β as indicated such that $P\beta = \alpha Q'$.

Proof. Apply Lemma 18.4 to the 2-cone given by the object $\mathcal{D}'$ over the cospan $\mathcal{C} \to \mathcal{A} \leftarrow \mathcal{B}$. □

Call a map $(U, V, VF \xrightarrow{\alpha} GU)\colon F \to G$ in $\mathbf{Func}_{\text{pseudo}}$ *good* if V is an isofibration.

Lemma 18.6. *The category $\mathbf{Func}_{\text{pseudo}}$ has pullbacks along good strict maps. Furthermore, the pullback of a good strict map can be chosen as a good strict map.*

Being good is invariant under isomorphisms of maps in $\mathbf{Func}_{\text{pseudo}}$. Being strict is not, however, explaining the last point of the statement.

Proof. Consider a span in $\mathbf{Func}_{\text{pseudo}}$ with one map being good and strict as in (18.2). We take pullbacks in $\mathbf{Cat}$ separately in domain and codomain parts and construct a commuting square

$$
\begin{tikzcd}
(\mathcal{D}', \mathcal{D}, F^{\mathcal{D}}) \arrow[r, "{(G', G, \beta)}"] \arrow[d, "{(Q', Q)}"', "\text{good,strict}"] & (\mathcal{B}', \mathcal{B}, F^{\mathcal{B}}) \arrow[d, "{(P', P)}"', "\text{good,strict}"] \\
(\mathcal{C}', \mathcal{C}, F^{\mathcal{C}}) \arrow[r, "{(F', F, \alpha)}"'] & (\mathcal{A}', \mathcal{A}, F^{\mathcal{A}})
\end{tikzcd}
\tag{18.4}
$$

as in (18.3) using Corollary 18.5. Note that omit writing the identity natural isomorphism for strict morphisms.

Let us verify that the square (18.4) is a pullback in $\mathbf{Func}_{\text{pseudo}}$. For this, consider a cone as given by the outer square in the diagram

$$
\begin{tikzcd}
(\mathcal{E}', \mathcal{E}, F^{\mathcal{E}}) \arrow[dr, dotted, "{(H', H, \eta)}"] \arrow[drr, bend left, "{(T', T, \delta)}"] \arrow[ddr, bend right, "{(R', R, \gamma)}"'] & & \\
& (\mathcal{D}', \mathcal{D}, F^{\mathcal{D}}) \arrow[r, "{(G', G, \beta)}"] \arrow[d, "{(Q', Q)}"', "\text{good,strict}"] & (\mathcal{B}', \mathcal{B}, F^{\mathcal{B}}) \arrow[d, "{(P', P)}"', "\text{good,strict}"] \\
& (\mathcal{C}', \mathcal{C}, F^{\mathcal{C}}) \arrow[r, "{(F', F, \alpha)}"'] & (\mathcal{A}', \mathcal{A}, F^{\mathcal{A}}).
\end{tikzcd}
\tag{18.5}
$$

We have to show that there is a unique dotted map (H', H, η) as indicated. Note that H' and H are uniquely determined by the universal property of the pullbacks in $\mathbf{Cat}$ shown in the front and back squares of the cube (18.3). Thus, it remains to show that there is a unique natural isomorphism $\eta\colon HF^{\mathcal{E}} \to F^{\mathcal{D}}H'$ such that $\beta H' \circ G\eta = \delta$ and $Q\eta = \gamma$.

Since P is an isofibration, the pullback

$$\begin{tikzcd} \mathcal{D} \arrow[r, "G"] \arrow[d, "Q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{B} \arrow[d, two heads, "P"', "\text{isofib}"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{A} \end{tikzcd} \tag{18.6}$$

is also a 2-pullback in the 2-category **Cat**. Let us look at the cone

$$\begin{tikzcd} \mathcal{E}' \arrow[drr, bend left, "TF^{\mathcal{E}'}"] \arrow[ddr, bend right, "RF^{\mathcal{E}'}"'] & & \\ & & \mathcal{B} \arrow[d, two heads, "P"', "\text{isofib}"] \\ & \mathcal{C} \arrow[r, "F"'] & \mathcal{A}. \end{tikzcd}$$

By the universal property of 2-pullbacks, the category of morphisms of 2-cones from this cone to the one of (18.6) (with the identity natural isomorphism) is contractible (equivalent to the terminal category). We have two such morphisms, one given by

$$\begin{tikzcd} \mathcal{E}' \arrow[drr, bend left, "TF^{\mathcal{E}'}"] \arrow[ddr, bend right, "RF^{\mathcal{E}'}"'] \arrow[dr, "HF^{\mathcal{E}'}"] & & \\ & \mathcal{D} \arrow[r, "G"] \arrow[d, "Q"'] & \mathcal{B} \arrow[d, two heads, "P"', "\text{isofib}"] \\ & \mathcal{C} \arrow[r, "F"'] & \mathcal{A} \end{tikzcd}$$

with identity natural isomorphisms everywhere, and the other given by

$$\begin{tikzcd} \mathcal{E}' \arrow[drr, bend left, "TF^{\mathcal{E}'}"] \arrow[ddr, bend right, "RF^{\mathcal{E}'}"'] \arrow[dr, "F^{\mathcal{D}}H'"] & \arrow[d, phantom, "\Downarrow\phi"] & \\ \arrow[r, phantom, "\psi\Rightarrow"] & \mathcal{D} \arrow[r, "G"] \arrow[d, "Q"'] & \mathcal{B} \arrow[d, two heads, "P"', "\text{isofib}"] \\ & \mathcal{C} \arrow[r, "F"'] & \mathcal{A} \end{tikzcd}$$

where ϕ is the composite

$$TF^{\mathcal{E}'} \xrightarrow{\delta} F^{\mathcal{B}}T' =\!=\!= F^{\mathcal{B}}G'H' \xrightarrow{\beta^{-1}H'} GF^{\mathcal{D}}H'$$

and ψ is the composite

$$RF^{\mathcal{E}'} \xrightarrow{\gamma} F^{\mathcal{C}}R' =\!=\!= F^{\mathcal{C}}Q'H' =\!=\!= QF^{\mathcal{D}}H'.$$

A 2-cell between these morphisms is given by a morphism $\eta\colon HF^{\mathcal{E}} \to F^{\mathcal{D}}H'$ such that $G\eta = \phi$ and $Q\eta = \psi$, and it is invertible precisely if η is iso. By what we have said about contractibility of the category of morphisms of 2-cones to the 2-pullback (18.6), there is a unique such 2-cell and it is invertible. This proves our claim. □