GENERALIZED PUSHFORWARD

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Here is an approach for chasing pushforwards in diagrams that gives a cleaner handle on distributivity. I came up with this while writing up some categorical resoning in the homotopy canonicity project with Chris Kapulkin.

Normally, pushforwards are defined as follows.

Definition 0.1 (Pushforward)**.** Consider a category $\mathcal{C}$ with a map $p\colon Y \to X$ that admits base change. Consider a map $q\colon Z \to Y$. Introduce the category $\mathsf{Sp}(p, q)$ whose objects are extensions of this data to diagrams

$$\begin{tikzcd} Z \arrow[dr, "q"'] & T \arrow[l, "v"'] \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "f"] \\ & Y \arrow[r, "p"] & X. \end{tikzcd}$$

We write such an object as a tuple (S, T, f, f', u, v). A *pushforward* along p of q is a terminal object in this category.

Pushfoward of a composite has an explanation in terms of pushforward of the factors. This is known as distributivity of dependent products over dependent sums or the type-theoretic axiom of choice. It requires a base change along an evaluation. To give a more compositional explanation of this phenomenon, we introduce the following generalization.

Definition 0.2 (Generalized pushforward)**.** Consider a category $\mathcal{C}$ with a diagram

$$\begin{tikzcd} A' \arrow[d, "a"] & & \\ A & Y \arrow[l, "v"'] \arrow[r, "p"] & X \end{tikzcd}$$

where p admits base change. Introduce the category $\mathsf{Sp}'(v, p, a)$ whose objects are extensions of this data to a diagram

$$\begin{tikzcd} A' \arrow[d, "a"] & T \arrow[l, "v'"] \arrow[r, "p'"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "f"] \\ A & Y \arrow[l, "v"'] \arrow[r, "p"] & X. \end{tikzcd}$$

We write such an object as a tuple (S, T, f, f', u, v'). A *generalized pushforward* of a along (v, p) is a terminal object in this category.

Remark 0.3. Consider the slice over an object X. The exponential over X of $q\colon Z \to X$ with $p\colon Y \to X$ is the generalized pushforward for

$$\begin{tikzcd} Z \arrow[d, "q"] & & \\ X & Y \arrow[l, "p"'] \arrow[r, "p"] & X. \end{tikzcd}$$

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Remark 0.4. Note that Definition 0.1 is the special case of Definition 0.2 where v is the identity on Y. Conversely, given a base change

$$\begin{tikzcd}
A' & Z \arrow[l] \arrow[d, "b"] \arrow[dl, phantom, "\lrcorner", very near start] \\
A' \arrow[from=1-1, "a"'] & Y, \arrow[l, "v"']
\end{tikzcd}$$

generalized pushfoward of a along (v, p) reduces to pushforward of b along p.

Lemma 0.5. . *Consider a functor $q\colon \mathcal{E} \to \mathcal{C}$. Consider a triangle of maps*

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Z \\
& Y \arrow[ur] &
\end{tikzcd}$$

in $\mathcal{E}$ such that the map $Y \to Z$ is cartesian. Then the map $X \to Y$ is cartesian exactly if the map $X \to Z$ is cartesian.

Proof. Standard. ☐

Definition 0.6. Consider an isofibration $q\colon \mathcal{E} \to \mathcal{C}$. A morphism $h\colon X \to Y$ is *locally cartesian* with respect to q if it is terminal in its fiber over $\mathcal{E}^{[1]} \times_{\mathcal{E}} \mathcal{C}$.

Note that a cartesian morphism is locally cartesian. The reverse does not hold in general.

Lemma 0.7. . *Consider a cocartesian fibration $\mathcal{E} \to \mathcal{C}$. Then a morphism is cartesian exactly if it is locally cartesian.*

Proof. Consider a triangle

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Z \\
& Y \arrow[ur] &
\end{tikzcd}$$

in $\mathcal{C}$. Consider lifts $X' \to Z'$ and $Y' \to Z'$ where the latter is locally cartesian. We have to show that there is a unique lift $X' \to Y'$ of $X \to Y$ making the triangle

$$\begin{tikzcd}
X' \arrow[rr] \arrow[dr, dotted] & & Z' \\
& Y' \arrow[ur] &
\end{tikzcd}$$

commute. Take a cocartesian lift $X' \to Y''$ of the map $X \to Y$. This induces a unique map $Y'' \to Z'$ over $Y \to Z$ such that the triangle

$$\begin{tikzcd}
X' \arrow[rr] \arrow[dr] & & Z' \\
& Y'' \arrow[ur] &
\end{tikzcd}$$

commutes. The desired lift corresponds to a lift $Y'' \to Y'$ of the identity on Y making the triangle

$$\begin{tikzcd}
X'' \arrow[rr] \arrow[dr] & & Z' \\
& Y' \arrow[ur] &
\end{tikzcd}$$

commute. This exists uniquely since $Y' \to Z'$ is locally cartesian. ☐

Lemma 0.8. *Consider a category $\mathcal{C}$. Introduce the category $\mathcal{E}$ whose objects are spans*

$$\begin{tikzcd} A & Y \arrow[l, "v"'] \arrow[r, "p"] & X \end{tikzcd}$$

where p admits base change and morphisms are morphisms of spans

$$\begin{tikzcd} A' \arrow[d] & Y' \arrow[l, "v'"'] \arrow[r, "p'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d] \\ A & Y \arrow[l, "v"] \arrow[r, "p"] & X \end{tikzcd}$$

where the right square is a pullback. We have the projection $q\colon \mathcal{E} \to \mathcal{C}$ to the left object. Given a morphism in $\mathcal{E}$, the following are equivalent:

(1) the morphism is cartesian with respect to q.
(2) the morphism is locally cartesian with respect to q,
(3) the morphism is a generalized pushforward,

Proof. Conditions (2) and (3) unfold to the same thing. The equivalence between conditions (1) and (2) is given by Lemma 0.7. For this, we note that q is a cocartesian fibration. □

Generalized pushforward enjoys a pasting property, similar to pullback pasting. This is in fact the main motivation for the generalization.

Corollary 0.9. *Consider a category C with diagrams*

$$\begin{tikzcd} A'' \arrow[d, "a'"] & Y'' \arrow[l, "v''"'] \arrow[r, "p''"] \arrow[d, "y"'] & S'' \arrow[d, "f"] \\ A' \arrow[d, "a"] & Y' \arrow[l, "v'"'] \arrow[r, "p'"] \arrow[d, "y"'] & S \arrow[d, "f"] \\ A & Y \arrow[l, "v"'] \arrow[r, "p"] & X \end{tikzcd}$$

where p admits base change. Assume that the bottom squares form a generalized pushforward. Then the top squares form a generalized pushforward exactly if the vertically composite squares form a generalized pushforward.

Proof. We re-express the claim using the equivalence between conditions (1) and (3) of Lemma 0.8. The claim becomes Lemma 0.5. □

Remark 0.10. Consider the special case of Corollary 0.9 where v is an identity. We tranlate to ordinary pushforwards using Remark 0.4:

- The bottom generalized pushforward is an ordinary pushfoward.
- The top generalized pushforward reduces to pushfoward along p' of the base change of a' along v'.

We thus obtain the standard characterization of pushforward of composites (distributivity of dependent products over dependent sums).

Remark 0.11. It is possible to generalize this notion from the codomain fibration to other fibrations. Then the pasting property (Corollary 0.9) becomes a statement about a tower of fibrations.