KOCK'S FAT Δ IS A DIRECT REPLACEMENT OF Δ

1. Ambifibrations

Let $\mathcal{B}$ be a category with an orthogonal factorization system $(\mathcal{L}, \mathcal{R})$. We consider a functor $P \colon \mathcal{E} \to \mathcal{B}$ and its restrictions

$$\begin{array}{ccc} \mathcal{E}_{\mathcal{L}} & \longrightarrow & \mathcal{E} \\ {\scriptstyle P_{\mathcal{L}}}\downarrow & \lrcorner & \downarrow{\scriptstyle P} \\ \mathcal{L} & \longrightarrow & \mathcal{B}, \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{E}_{\mathcal{R}} & \longrightarrow & \mathcal{E} \\ {\scriptstyle P_{\mathcal{R}}}\downarrow & \lrcorner & \downarrow{\scriptstyle P} \\ \mathcal{R} & \longrightarrow & \mathcal{B}. \end{array}$$

Let $\mathcal{W} \subseteq \mathcal{E}$ be the subcategory mapped to isomorphisms via P.

We call P an *ambifibration* if $P_{\mathcal{L}}$ a Grothendieck opfibration and $P_{\mathcal{R}}$ a Grothendieck fibration. The orthogonal factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{B}$ then has lifts $(\mathcal{E}_{\mathcal{L}}, \mathcal{E}_{\mathcal{R}}^{\mathsf{cart}})$ and $(\mathcal{E}_{\mathcal{L}}^{\mathsf{cocart}}, \mathcal{E}_{\mathcal{R}})$ to $\mathcal{E}$. Every map in $\mathcal{E}$ factors uniquely and functorially as a map in $\mathcal{E}_{\mathcal{L}}^{\mathsf{cocart}}$ followed by a map in $\mathcal{W}$ followed by a map in $\mathcal{E}_{\mathcal{R}}^{\mathsf{cart}}$.

Proposition 1.1. *Assume that $P_{\mathcal{L}}$ be a Grothendieck opfibration, and that P and*

$$\mathcal{E}_{\mathcal{R}}^{[1]} \xrightarrow{\widehat{\exp}(d_1, P_{\mathcal{R}})} \mathcal{E} \times_{\mathcal{B}} \mathcal{R}^{\to} \tag{1.1}$$

have weakly contractible fibers. Then the homotopical functor $(\mathcal{E}, \mathcal{W}) \to (\mathcal{B}, \mathsf{Iso})$ is a Dwyer-Kan equivalence.

A word on terminology. By fibers of a functor, we mean essential fibers (also called homotopy fibers) rather than strict fibers. For the current proposition and its proof, this will make no difference as $P_{\mathcal{L}}$ is an isofibration. It also follows that the below (strict) pullbacks of categories are also bicategorical pullbacks.

Weak equivalences in categories are created from weak equivalences in simplicial sets by the nerve. A category is weakly contractible if the canonical map to the terminal category is a weak equivalence. A homotopical functor (a map between relative categories) is a Dwyer-Kan equivalence if it induces an equivalence of $(\infty, 1)$-categories upon simplicial localization.

Proof of Proposition 1.1. By [Hin16, 1.3.6 Key Lemma], it suffices to show that $P^{[n]} \colon \mathcal{E}^{[n]} \to \mathcal{B}^{[n]}$ has weakly contractible fibers for any $n \geq 0$. We proceed by induction. The case $n = 0$ is given by assumption.

Assume now that $P^{[n]}$ has weakly contractible fibers and consider a chain $C \colon [n+1] \to \mathcal{B}$. Let $C' =_{\mathsf{def}} C \circ d_{n+1} \colon [n] \to \mathcal{B}$ denote its first n maps. To show that $(P^{[n+1]})^{-1}(C)$ is weakly contractible, it will suffice to verify that the projection

$$(P^{[n+1]})^{-1}(C) \longrightarrow (P^{[n]})^{-1}(C') \tag{1.2}$$

is a weak equivalence.

Let $f \colon U \to V$ denote the last map of C. Note that (1.2) is a pullback of

$$(P^{[1]})^{-1}(f) \longrightarrow P^{-1}(U), \tag{1.3}$$

and thus a discrete Grothendieck fibration (this is true for any functor P). Applying Quillen's Theorem A, it then suffices to show that the fiber of (1.3) over any $X \in P^{-1}(U)$ is weakly contractible.

Let

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & V \\ & {\scriptstyle l}\searrow \quad \nearrow{\scriptstyle r} & \\ & M & \end{array}$$

be the unique $(\mathcal{L}, \mathcal{R})$-factorization of f. Since $P_{\mathcal{L}}$ is a Grothendieck opfibration, there is a cocartesian lift $l' \colon X \to N$ of l. Using the universal property of cocartesian arrows, the fiber of (1.3) over X is equivalent to the fiber of

$$(P_{\mathcal{R}}^{[1]})^{-1}(r) \longrightarrow P^{-1}(M)$$

over N. But this is a pullback of (1.1) and thus has weakly contractible fibers by assumption. □

2. The fat Δ

Joachim Kock defines a category $\mathbf{\Delta}$ (fat simplex category) as follows. Objects are epimorphisms in Δ. A map (u, i) from $f \colon [a] \twoheadrightarrow [m]$ to $g \colon [b] \twoheadrightarrow [n]$ is given by a map $u \colon [m] \to [n]$ and a monomorphism $i \colon [a] \rightarrowtail [b]$ in Δ making the following square commute:

$$\begin{array}{ccc} [a] & \overset{i}{\rightarrowtail} & [b] \\ \downarrow & & \downarrow \\ [m] & \underset{u}{\longrightarrow} & [n]. \end{array}$$

Composition is defined in the evident way.

The canonical inclusion of the semisimplex into the simplex category factors via $\mathbf{\Delta}$:

$$\begin{array}{ccc} \Delta^{+} & \longrightarrow & \mathbf{\Delta} \\ & \searrow & \downarrow {\scriptstyle P} \\ & & \Delta. \end{array}$$

Here, the top functor is the "inclusions" $\Delta^{+} \to \mathbf{\Delta}$ sending $[m]$ to $\mathrm{id} \colon [m] \to [m]$. The right functor is the "projection" $P \colon \mathbf{\Delta} \to \Delta$ sending $[a] \twoheadrightarrow [m]$ to $[m]$.

Recall that Δ is a Reedy category with Reedy factorization $(\mathcal{L}, \mathcal{R})$ given by epimorphisms and monomorphisms. Note that P has cocartesian lifts of epimorphisms $\mathcal{L}$ and cartesian lifts of monomorphisms $\mathcal{R}$. Thus, the projection $\mathbf{\Delta} \to \Delta$ is an ambifibration.

3. The directed homotopy type of the fat Δ

We will consider $\mathbf{\Delta}$ as a homotopical category. The weak equivalences $\mathcal{W} \subseteq \mathbf{\Delta}$ are created from the identities in Δ via P.

Proposition 3.1. *The projection $P \colon (\mathbf{\Delta}, \mathcal{W}) \to (\Delta, \mathrm{Iso})$ is a Dwyer-Kan equivalence of homotopical categories.*

Proof. Since P is an ambifibration, we may apply Proposition 1.1. For this, we need to verify that all categories (3.1) and (3.2) defined below are weakly contractible.

Given $[n] \in \Delta$, note that

$$P^{-1}([n]) = (\Delta_{+})^{n+1} \tag{3.1}$$

Similarly, given a monomorphism $r\colon [m] \to [n]$ in Δ and a lift $u \in P^{-1}([m])$, i.e. an epimorphism $s\colon [a] \to [m]$, the fiber of $(P^{[1]})^{-1}(r) \to P^{-1}([m])$ over u is isomorphic to

$$\prod_{i\in[n]} r^{-1}(i) \downarrow_{\Delta_{+,\mathrm{aug}}} \Delta_+ = \prod_{i\in[n]} \begin{cases} r^{-1}(i)/\Delta_+ & \text{if } i \in \mathrm{im}(r) \\ \Delta_+ & \text{else.} \end{cases} \tag{3.2}$$

Coslice categories are weakly contractible as they have an initial object. Finite products of weakly contractible categories are again weakly contractible. To finish the proof, it thus suffices to observe that Δ_+ is weakly contractible by Proposition 3.2. □

Proposition 3.2. Δ_+ *is weakly contractible.*

The below proof was given by Charles Rezk in a Math Overflow post; a comment indicated it is (separately?) due to Maltsiniotis in "La Théorie de l'homotopie de Grothendieck".

Proof. The identity functor on Δ_+ and the constant functor returning $[0]$ are related by a cospan of natural transformations using the join inclusions:

$$\mathrm{Id} \longrightarrow \mathrm{Id} \star \mathrm{const}([0]) \longleftarrow \mathrm{const}([0]).$$

□

Old and redundant proof of Proposition 3.2. We will show that $\Delta_+^{\leq n}$ is weakly contractible for even n. Since $\{0\} \to N\Delta_+$ arises as a sequential colimit of trivial cofibrations

$$N\Delta_+^{\leq 0} \overset{\sim}{\rightarrowtail} N\Delta_+^{\leq 2} \overset{\sim}{\rightarrowtail} \cdots,$$

then Δ_+ will be weakly contractible as well.

We will show that the inclusion $\Delta_{+,\mathrm{aug}}^{\leq n} \to \Delta_+^{\leq n}$ is a weak equivalence for n even. For $n \geq 0$, the domain is weakly contractible since it has an initial object. By 2-out-of-3, the codomain will then be weakly contractible as well.

For the latest claim, we proceed by induction. The base case is $n = -2$ where domain and codomain are both empty. For the induction step, by stability of trivial cofibrations under pushout and 2-out-of-3, it will suffice to show that the Leibniz application of $N(-)^{\leq n} \to N(-)^{\leq n+2}$ to $\Delta_+ \to \Delta_{+,\mathrm{aug}}$, i.e. the inclusion

$$N\Delta_+^{\leq n+2} \cup N\Delta_{+,\mathrm{aug}}^{\leq n} \rightarrowtail N\Delta_{+,\mathrm{aug}}^{\leq n+2} \tag{3.3}$$

is a trivial cofibration.

Consider the functor

$$[1]^{n+3} \xrightarrow{\simeq} (\Delta_{+,\mathrm{aug}}^{\leq 0})^{n+3} \longrightarrow \Delta_{+,\mathrm{aug}}^{\leq n+2} \tag{3.4}$$

where the second map is given by $(n+3)$-fold join. Consider further the simplicial subset

$$U \xrightarrow{d_0 \widehat{\times} \cdots \widehat{\times} d_1} N[1]^{n+3} \tag{3.5}$$

of the cube with all faces removed that contain the initial vertex. The inclusion (3.3) is the quotient of (3.5) under the nerve of the functor (3.4):

$$\begin{array}{ccc} U & \rightarrowtail & N[1]^{n+3} \\ {\scriptstyle \mathrm{epi}}\downarrow & & \downarrow{\scriptstyle \mathrm{epi}} \\ N\Delta_+^{\leq n+2} \cup N\Delta_{+,\mathrm{aug}}^{\leq n} & \rightarrowtail & N\Delta_{+,\mathrm{aug}}^{\leq n+2}. \end{array}$$

The above commuting square is the image of a similar one in cubical sets (without symmetries or connections) under realization of cubes as simplices. Since cubical realization is a Quillen equivalence, let us switch to the cubical picture.

Note that the missing faces of the cube $[1]^{n+3}$ all get identified under the above quotient. So the bottom map misses precisely two elements: an $(n+2)$-dimensional cube a and a $[n+3]$-dimensional cube b. Note that $n+3$ of the faces of b are given by a. Since $n+3$ is odd, these form a trivial cofibration.[1] □

References

[Hin16] Vladimir Hinich. Dwyer-kan localization revisited. *Homology, Homotopy and Applications*, 18(1):27–48, 2016.

[1]This needs to be justified. The argument should be as for Dunce's hat and be purely synthetical. A similar statement for horn inclusions with an odd number of missing faces, all identified, is written up. More flexible shapes would make the write-up nicer.