Why does M form an external lex operation?

Let $i\colon \Delta \to \square$ be a fully faithful extension of Δ with finite products. We have the standard presheaf cwf model of extensional type theory on $\mathcal{P}(\square)$. Inside, we construct the CCHM-style model of homotopy type on $\mathcal{P}(\square)$ with cofibrations given by levelwise decidable inclusions and interval object given by the restriction of i to objects of degree ≤ 1.

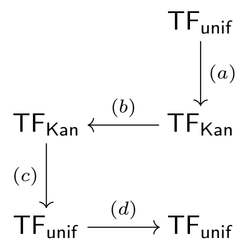
Write $j\colon \Delta_+ \to \Delta$ for the semisimplex category. This is the wide subcategory of Δ with maps restricted to monomorphisms. The composite $ij\colon \Delta_+ \to \square$ gives rise via Kan extension to an adjoint triple of functors $L \dashv U \dashv R$ between $\mathcal{P}(\Delta_+)$ and $\mathcal{P}(\square)$. Note that U and R lift to weak cwf morphisms of presheaf cwf model. We obtain a monad $M = RU$ on $\mathcal{P}(\square)$ that lifts to a pointed weak cwf endomorphism of presheaf cwf model.

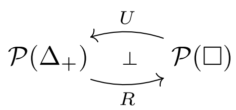
An *external lex operation* is a pointed weak cwf endomorphism. This induces a lex operation in the model in the sense of [CRS]. In fact, most known examples of lex operations seem to come from external lex operations, and those are generally simpler to define.

With this terminology, M is an external lex operation on the presheaf cwf model $\mathcal{P}(\square)$. We wish to lift it to an external lex operation on the inner model of homotopy type theory. For this, we have to explain preservation of fibration structures (compatible with substitution). As usual, it is easier to start with trivial fibrations and then move on to fibrations.

In fact, M can be shown to lift to categories of (trivial) fibrations. For an external lex operation, we only need to know this with morphisms of (trivial) fibrations restricted to pullback squares.

Preservation of trivial fibrations. The following diagram shows what is going on abstractly. The desired preservation is decomposed into steps (a)–(d). Here, vertical arrows live over the identity and horizontal arrows live over the action on arrows of U or R, according to their direction.





Categories of arrows:

- $\mathsf{TF}_{\mathsf{Kan}}$ is trivial Kan fibrations. These are maps with fillers for simplex boundaries.
- $\mathsf{TF}_{\mathsf{unif}}$ is uniform trivial fibrations. These are the contractible types in the model of homotopy type theory in $\mathcal{P}(\square)$.

  They are types $X \in \mathsf{Ty}(\Gamma)$ with an *extension operation*. Internally to the presheaf model, this takes $\phi : \mathsf{Cof}$ and $\gamma : \Gamma$ and extends any ϕ-partial element of $X(\gamma)$ to a total element.

Morphisms:

(a) Every uniform trivial fibration in particular has chosen lifts against simplex boundaries.

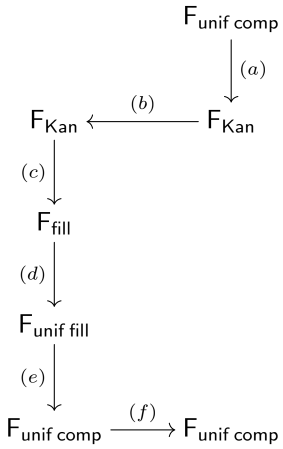
(b) The forgetful functor U creates trivial Kan fibrations. This is because simplex boundaries in $\mathcal{P}(\square)$ are the image under L of simplex boundaries in $\mathcal{P}(\Delta_+)$.

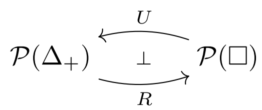
(c) This key step brings in uniformity. Since Δ_+ is a direct category, levelwise decidable inclusions agree with Reedy decidable inclusions. It follows that Reedy split surjections (trivial Kan fibrations) agree with uniform trivial fibrations. Here and below, "agree" means a logical equivalence (maps back and forth) in categories of arrows over $\mathcal{P}(\Delta_+)$.

    Unfolded, this step defines an extension operation by induction over cell dimension.

(d) The forgetful functor U creates levelwise decidable inclusions. Adjoint transposition gives the indicated arrow.

Preservation of fibrations. Now the diagram is a bit more interesting.





We first explain the relevant categories of arrows. We write C for the cylinder functor. In semisimplicial sets, it is $C(X) = \Delta_+[1] \otimes X$. In $\mathcal{P}(\square)$, it is $C(X) = y[1] \times X$.

- $\mathsf{F}_{\mathsf{Kan}}$ is Kan fibrations. These are maps with fillers for horn inclusions.
- $\mathsf{F}_{\mathsf{fill}}$ is prism filling fibrations in semisimplicial sets. These are created from trivial Kan fibrations by pullback monoidal hom with interval endpoints.
- $\mathsf{F}_{\mathsf{unif\ fill}}$ is uniform filling fibrations. These are created from uniform trivial fibrations by pullback monoidal hom with interval endpoints.

  Explicitly, they are types $X \in \mathsf{Ty}(\Gamma)$ with a *filling operation*. This takes a stage A, a levelwise decidable inclusion $\phi \colon S \to \mathrm{y}A$, and a map $\gamma \colon C(\mathrm{y}A) \to \Gamma$. Let $s \colon M \to C(\mathrm{y}A)$ be the subobject given by the union of $C(\phi)$ and a cylinder endpoint. The operation then extends any element of $X[\gamma s]$ to an element of $X[\gamma]$, naturally in A.
- $\mathsf{F}_{\mathsf{unif\ comp}}$ is uniform composition fibrations. These are closely related to uniform filling fibrations. In $\mathcal{P}(\square)$, they agree and form the types in the model of homotopy type theory.

  Explicitly, they are types with a *composition operation*. This works like a filling operation, but only produces an element of $X[\gamma e]$ where $e \colon yA \to CyA$ is the missing endpoint.

Morphisms:

(a) Every horn inclusion is a strong homotopy equivalence. This exhibits it as a retract (with canonical section) of its pushout product with an interval endpoint. This allows us to lift the horn inclusion via a composition with the horn inclusion as chosen cofibration.

(b) The forgetful functor U creates Kan fibrations. This is because horn inclusions in $\mathcal{P}(\square)$ are the image under L of simplex boundaries in $\mathcal{P}(\Delta_+)$.

(c) Every simplex prism filling decomposes into a series of horn fillings.

  This step may equivalently be performed in simplicial sets. That would necessitate splitting step (b) into two parts, stopping in the middle at simplicial sets.

(d) This is again the key step (bringing in uniformity). We derive it from step (c) for trivial fibrations applied to the monoidal pullback hom of the map in question.

(e) Composition is part of filling.

(f) Consider a type $A \in \mathsf{Ty}(\Gamma)$ in semisimplicial sets. Every composition problem for RA in $\mathcal{P}(\square)$ gives rise under transposition to a composition problem for A in semisimplicial sets.

  Note that this argument does not work for filling.