SOME MORE NOTES ON DIRECTED UNIVALENCE IN THE BISIMPLICIAL SET MODEL (UNFINISHED)

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1. Directed univalence for discrete types

Recall that bisimplicial sets are indexed by two simplex category variables, the first in the categorical direction and the second in the spacial diretion, By a fibration or weak equivalence in bisimplicial sets, we always refer to the Reedy model structure in the categorical direction with respect to the Kan model structure in the spacial direction.

1.1. **Classifier for covariant fibrations.** Let $\mathcal{U}_{\mathsf{cov}}$ be the universe of covariant fibrations. As a bisimplicial groupoid, it is defined on representable $([m], [n])$ as the groupoid of maps into $\Delta^m \Box \Delta^n$ that are covariant fibrations:

- Reedy fibrations in the spacial direction with respect to the covariant model structure on simplicial sets (that has inner and left outer horns as generating trivial cofibrations) in the categorical direction,
- and Reedy fibrations in the categorical direction with respect to the Kan model structure in the spacial direction,

One then takes a cofibrant replacement as sketched by Shulman or uses the technique of Hofmann-Streicher to represent it by a bisimplicial set.

There is an evident inclusion $\mathcal{U}_{\mathsf{cov}} \to \mathcal{U}$. The universal covariant fibration $\widetilde{\mathcal{U}}_{\mathsf{cov}} \twoheadrightarrow \mathcal{U}_{\mathsf{cov}}$ is defined by pulling back the universal fibration $\widetilde{\mathcal{U}} \to \mathcal{U}$. To note that $\widetilde{\mathcal{U}}_{\mathsf{cov}} \twoheadrightarrow \mathcal{U}_{\mathsf{cov}}$ is actually a covariant fibration, we observe that the class of covariant fibrations is local (in the sense of Cisinski) as the corresponding weak factorization system has generators with representable codomain:

- $i^m \mathbin{\widehat{\Box}} h^n_l$ with $m \geq 0$ and $0 \leq k < m$ and $0 \leq l \leq n$, $n > 0$,
- and $h^m_k \mathbin{\widehat{\Box}} i^n$ with $0 \leq k < m$ and $n \geq 0$,

Note that another profitable set of generators is as follows:

- $i^{m,n} \mathbin{\widehat{\times}} \delta_k$ with $m, n \geq 0$ and $k \in \{0, 1\}$,
- and $i^{m,n} \mathbin{\widehat{\times}} \gamma_0$ with $m, n \geq 0$.

Here, we have $\delta_k = \Delta^0 \Box h^1_k$ the endpoint inclusions for the spacial interval and $\gamma_k = h^1_k \Box \Delta^0$ the endpoint inclusions for the categorical "directed" interval for $k \in \{0, 1\}$. Note that the special interval acts as an interval object for the notion of homotopy. It follows that a fibration p is an inner fibration exactly if the pullback exponential $\widehat{\exp}(\gamma_0, p)$ is a trivial fibration.

In Riehl-Shulman type theory, recall that given $X : \mathcal{U}$ and $C : X \to \mathcal{U}$, we have defined

$$\mathsf{isCov}(C) =_{\mathrm{def}} \prod_{f : \hom_X(x,y)} \prod_{u : C(x)} \mathrm{isContr}\Big(\sum_{v : C(y)} \hom_{C(f)}(u, v) \Big).$$

Lemma 1.1. *Let $Y \twoheadrightarrow X$ be a fibration between fibrant objects, classified by a map $C \colon X \to U$. Then $Y \twoheadrightarrow X$ is a covariant fibration exactly if, in Riehl-Shulman type theory, the global type $\mathsf{isCov}(C)$ is inhabited.*

Proof. A section to $\mathsf{isCov}(C)$ corresponds to a section to the map

$$\sum_{f:\hom_X(x,y)} \sum_{u:C(x)} \mathrm{isContr}\Big(\sum_{v:C(y)} \hom_{C(f)}(u,v)\Big) \to \sum_{f:\hom_X(x,y)} C(x).$$

Recall the equivalence between contractibility of types and trivial fibrations in a type-theoretic model category in the sense of Shulman. So a section to the previous display exists if and only if the projection

$$[f:\hom_X(x,y), u:C(x), v:C(y), m:\hom_{C(f)}(u,v)] \to [f:\hom_X(x,y), u:C(x)]$$

is a trivial fibration (the square brackets abbreviate iterated dependent sums). Recall that Y is the dependent sum of X and C. By the definition of hom-types of type families, the domain is isomorphic to the hom-type of the dependent sum Y. Recall further that hom-types are given by pullback exponentials with the boundary inclusion of the directed interval. Taking into account the dependent sums over endpoints, the previous display is isomorphic to the pullback exponential of $Y \twoheadrightarrow X$ with the left endpoint inclusion $\gamma_0\colon \{(0,0)\} \to \Delta^{1,0}$ of the directed interval. Now recall that the pullback exponential of $Y \twoheadrightarrow X$ with γ_0 is a trivial fibration exactly if $Y \twoheadrightarrow X$ is a covariant fibration. □

We could have used Theorem 8.5 of the Riehl-Shulman paper to shorten the above proof.

Lemma 1.2. *Consider a weak equivalence of fibrations*

$$\begin{tikzcd} Y_0 \arrow[rr, "f", "\sim"'] \arrow[dr, two heads] & & Y_1 \arrow[dl, two heads] \\ & X. & \end{tikzcd}$$

If $Y_0 \twoheadrightarrow X$ is covariant, then so is $Y_1 \twoheadrightarrow X$.

Proof. Suppose we are given a covariant lifting problem against $Y_1 \twoheadrightarrow X$. We use the homotopy inverse to f to transport it to a covariant lifting problem against $Y_0 \to X$, and then use f to transport the diagonal filler there back to a diagonal filler for the original lifting problem against $Y_1 \to X$ that makes the lower right triangle commute strictly and the upper left triangle commute up to homotopy relative to X. Since $Y_0 \twoheadrightarrow X$ is a fibration, we can then correct the diagonal filler to one that solves the lifting problem strictly. □

Note that in case X is fibrant, we can deduce Lemma 1.2 from Lemma 1.1 using (standard) univalence for $\mathcal{U}$ or alternatively, without appealing to univalence, perform internal reasoning analogous to the steps of the proof of Lemma 1.2 (but working with contractibility of solution spaces of lifting problems instead of just ordinary lifts).

Fibrancy of the classifier.

Lemma 1.3. *The canonical map $\mathcal{U}_{\mathsf{cov}} \to \mathcal{U}$ is a fibration.*

Proof. We consider a lifting problem of $i^{m,n} \mathbin{\widehat{\times}} \delta_k$ against $\mathcal{U}_{\mathsf{cov}} \to \mathcal{U}$ with $m, n \geq 0$ and $k \in \{0,1\}$. After unfolding, the problem becomes as follows. Given a pullback square

$$\begin{tikzcd} X' \arrow[rr] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & X \arrow[d, two heads] \\ \Delta^{m,n} +_{\partial\Delta^{n,n}} \partial\Delta^{n,n} \times \Delta^1 \arrow[rr, "i^{m,n} \widehat{\times} \delta_k"'] & & \Delta^{m,n} \times \Delta^{0,1}. \end{tikzcd}$$

where the vertical maps are fibrations, if the left fibration is covariant, then so is the right. We introduce a further pullback along the composite

$$\Delta^{m,n} \times \Delta^{0,1} \longrightarrow \Delta^{m,n} \longrightarrow \Delta^{m,n} +_{\partial\Delta^{n,n}} \partial\Delta^{n,n} \times \Delta^1$$

and obtain the diagram

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads] \\
\Delta^{m,n} \times \Delta^{0,1} \arrow[r] & \Delta^{m,n} +_{\partial\Delta^{n,n}} \partial\Delta^{n,n} \times \Delta^1 \arrow[r, "i^{m,n} \widehat{\times} \delta_k"'] & \Delta^{m,n} \times \Delta^{0,1}.
\end{tikzcd}$$

As a pullback of the middle vertical map, note that $X'' \twoheadrightarrow \Delta^{m,n}$ is a covariant fibration. Now observe that X and X'' are homotopy equivalent over $\Delta^{m,n} \times \Delta^{0,1}$. By Lemma 1.2, it follows that $X \twoheadrightarrow \Delta^{m,n} \times \Delta^{0,1}$ is a covariant fibration. $\square$

Corollary 1.4. *We have that $\mathcal{U}_{\mathsf{cov}}$ is fibrant.*

Covariant directed univalence. Having shown $\mathcal{U}_{\mathsf{cov}}$ fibrant in Corollary 1.4, we can use the internal characterization of the covariant fibration $\widetilde{\mathcal{U}}_{\mathsf{cov}} \to \mathcal{U}_{\mathsf{cov}}$ given by Lemma 1.1. Let us write $i \colon \mathcal{U}_{\mathsf{cov}} \to \mathcal{U}$ for the inclusion constructed earlier, which classifies $\widetilde{\mathcal{U}}_{\mathsf{cov}} \to \mathcal{U}_{\mathsf{cov}}$. We obtain that $\mathsf{isCov}(\mathcal{U}_{\mathsf{cov}}, i)$ is inhabited.

Given discrete types $A, B \colon \mathcal{U}$ let us write

$$\mathsf{CovSpan}(A, B) =_{\mathrm{def}} \sum_{S : A \times B \to \mathcal{U}} \prod_{a : A} \mathrm{isContr}\Big(\sum_{b : B} S(a, b)\Big)$$

for a representation of the type of spans whose left leg is invertible.

Suppose we are given $X : \mathcal{U}$ and $C : X \to \mathcal{U}$ with $h : \mathsf{isCov}(X, C)$. For $x, y : X$, we have a map

$$\hom_X(x, y) \to \mathsf{CovSpan}(C(x), C(y))$$

sending f to the span whose summit is $S(u, v) =_{\mathrm{def}} \hom_{C(f)}(u, v)$ as detailed in the Riehl-Shulman paper.

Note that $\mathsf{CovSpan}(A, B)$ is equivalent to the type $A \to B$. Thus, we also obtain a map

$$\hom_X(x, y) \to (C(x) \to C(y))$$

for $x, y : X$, or equivalently a map

$$X^2 \to \sum_{u : C(X)} \sum_{v : C(Y)} (C(x) \to C(y)).$$

Lemma 1.5 (Covariant directed univalence)**.** *In bisimplicial sets, the map*

$$x, y : \mathcal{U}_{\mathsf{cov}} \vdash \hom_{\mathcal{U}_{\mathsf{cov}}}(x, y) \to (i(x) \to i(y))$$

is an equivalence.

Proof. Let us define

$$\mathsf{CovSpan} =_{\mathrm{def}} \sum_{A, B : \mathcal{U}} \mathsf{isDisc}(A) \times \mathsf{isDisc}(B) \times \mathsf{CovSpan}(A, B).$$

By taking dependent sums over x and y and 2-out-of-3 for equivalences, it will suffices to show that the map

$$\mathcal{U}_{\mathsf{cov}}^2 \to \mathsf{CovSpan}$$

is an equivalence, which will be technically more convent.

Let us define a map in the other direction. Taking an exponential transpose, it will be defined as the classifying map of a covariant fibration $F \twoheadrightarrow \mathbb{2} \times \mathsf{CovSpan}$. We define it by the pushout

$$\begin{tikzcd}
{[(x,y,S):\mathsf{CovSpan}, u:i(x), v:i(y), w:S(u,v)]} \arrow[r, two heads] \arrow[d, tail] & {[(x,y,S):\mathsf{CovSpan}, v:i(y)]} \arrow[d, tail] \arrow[dr, bend left] \\
\mathbb{2} \times [(x,y,S):\mathsf{CovSpan}, u:i(x), v:i(y), w:S(u,v)] \arrow[r] \arrow[drr, bend right, two heads] & F \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted] \\
& & \mathbb{2} \times \mathsf{CovSpan}
\end{tikzcd}$$

where the outer vertical maps select the left endpoint of $\mathbb{2}$.

We claim that the map $F \to \mathbb{2} \times \mathsf{CovSpan}$ is a fibration. Consider a lifting problem

$$\begin{tikzcd}
\Delta^m \square \Lambda^n_l \cup \partial\Delta^m \square \Delta^n \arrow[r] \arrow[d, tail, "i^m \widehat{\square} h^n_l"'] & F \arrow[d] \\
\Delta^m \square \Delta^n \arrow[r, "{\langle f,g \rangle}"] & \mathbb{2} \times \mathsf{CovSpan}
\end{tikzcd} \tag{1.1}$$

with $m \geq 0$ and $0 \leq l \leq n$, $n \geq 1$. We perform a case distinction based on the map $f \colon \Delta^m \square \Delta^n \to \mathbb{2}$, which corresponds to a map $\overline{f} \colon [m] \to [1]$. Write $\overline{f} = !_{[a] \to [0]} \star !_{[b] \to [0]}$ with $a, b \geq -1$.

If $a = -1$, then $\overline{f}$ lifts through the right endpoint $\{1\} \colon [0] \to [1]$ and the lifting problem (1.1) factors via a lifting problem

$$\begin{tikzcd}
\Delta^m \square \Lambda^n_l \cup \partial\Delta^m \square \Delta^n \arrow[r] \arrow[d, tail, "i^m \widehat{\square} h^n_l"'] & {[(x,y,S):\mathsf{CovSpan}, v:i(y)]} \arrow[d] \\
\Delta^m \square \Delta^n \arrow[r] & \mathsf{CovSpan},
\end{tikzcd}$$

which has a solution since the right side is a pullback of the universal fibration $\widetilde{U} \to \mathcal{U}$.

The case $b = -1$ works analogously.

If $a \geq 1$, then the domain $\Delta^m \square \Lambda^n_l \cup \partial\Delta^m \square \Delta^n$ of the left map writes as a union of representables over $\mathbb{2}$ none of which factor via the right endpoint $\gamma^1 \colon 1 \to \mathbb{2}$. The lifting problem (1.1) then factors via a lifting problem

$$\begin{tikzcd}
\Delta^m \square \Lambda^n_l \cup \partial\Delta^m \square \Delta^n \arrow[r] \arrow[d, tail, "i^m \widehat{\square} h^n_l"'] & \mathbb{2} \times [(x,y,S):\mathsf{CovSpan}, u:i(x), v:i(y), w:S(u,v)] \arrow[d, two heads] \\
\Delta^m \square \Delta^n \arrow[r] & \mathbb{2} \times \mathsf{CovSpan},
\end{tikzcd}$$

which has a solution since the right-hand side is given by a type, hence is a fibration.

It remains to consider the case $a = 0$ with $b \geq 0$. In that case, we also construct and solve a lifting problem as in the last paragraph, for which we need a lift

$$\begin{tikzcd}
& \mathbb{2} \times [(x,y,S):\mathsf{CovSpan}, u:i(x), v:i(y), w:S(u,v)] \arrow[d] \\
\Delta^m \square \Lambda^n_l \cup \partial\Delta^m \square \Delta^n \arrow[r] \arrow[ur, "t"] & F.
\end{tikzcd}$$

Recall that the left object is a union of the subobjects $\Delta^{\{0,\ldots,\hat{a},\ldots,m\}} \square \Delta^n$ for $0 \leq a \leq m$ and $\Delta^m \square \Delta^{\{0,\ldots,b,\ldots,n\}}$ for $0 \leq b \leq n$ with $b \neq l$. Since $a = 0$, the only subobject whose map to $\mathbb{2}$

factors via γ^1 is $\Delta^{\{1,\ldots,m\}}\Box\Delta^n$. From the pushout defining F, we thus are able to define t on the remaining subobjects, which we will denote

$$t' \colon D \to 2 \times [(x, y, S) : \mathsf{CovSpan}, u : i(x), v : i(y), w : S(u, v)].$$

Now observe that $D \rightarrowtail \Delta^m\Box\Lambda^n_l \cup \partial\Delta^m\Box\Delta^n$ is a pushout of

$$i^{m-1} \,\widehat{\Box}\, h^n_l \colon \Delta^{m-1}\Box\Lambda^n_l \cup \partial\Delta^{m-1}\Box\Delta^n \to \Delta^{m-1}\Box\Delta^n$$

(here we use $b \geq 0$ to get $m \geq 1$) where the induced map $\Delta^{m-1}\Box\Delta^n \to 2$ factors via γ^1. The extension problem

$$\begin{tikzcd}
D \arrow[r, "t'"] \arrow[d] & 2 \times [(x, y, S) : \mathsf{CovSpan}, u : i(x), v : i(y), w : S(u, v)] \arrow[d] \\
\Delta^m\Box\Lambda^n_l \cup \partial\Delta^m\Box\Delta^n \arrow[r] \arrow[ur, dotted, "t"] & F
\end{tikzcd}$$

is thus solved by a lift in

$$\begin{tikzcd}
\Delta^{m-1}\Box\Lambda^n_l \cup \partial\Delta^{m-1}\Box\Delta^n \arrow[r] \arrow[d, "i^{m-1}\widehat{\Box}h^n_l"] & {[(x, y, S) : \mathsf{CovSpan}, u : i(x), v : i(y), w : S(u, v)]} \arrow[d, two heads] \\
\Delta^{m-1}\Box\Delta^n \arrow[r] \arrow[ur, dotted] & {[(x, y, S) : \mathsf{CovSpan}, v : i(y)]}
\end{tikzcd}$$

which exists because the right map is a context projection, hence also a fibration.

We now claim that the map $F \to 2 \times \mathsf{CovSpan}$ is a covariant fibration. Consider a lifting problem

$$\begin{tikzcd}
\Lambda^n_k\Box\Delta^n \cup \Delta^m\Box\partial\Delta^n \arrow[r] \arrow[d, tail, "h^m_k\widehat{\Box}i^n"'] & F \arrow[d] \\
\Delta^m\Box\Delta^n \arrow[r, "{\langle f,g\rangle}"] & 2 \times \mathsf{CovSpan}
\end{tikzcd} \tag{1.2}$$

with $0 \leq k < n$ and $m \geq 0$. Similar to before, we perform a case distinction based on the map $f \colon \Delta^m\Box\Delta^n \to 2$, which corresponds to a map $\overline{f} \colon [m] \to [1]$ we write $\overline{f} = !_{[a]\to[0]} \star !_{[b]\to[0]}$ with $a, b \geq -1$.

The cases $a = -1$, $b = -1$, and $a \geq 1$ work exactly as before. Once again, we are led to consider the case $a = 0$ with $b \geq 0$. Again, we try to construct a lift t as in

$$\begin{tikzcd}
 & 2 \times [(x, y, S) : \mathsf{CovSpan}, u : i(x), v : i(y), w : S(u, v)] \arrow[d] \\
\Lambda^n_k\Box\Delta^n \cup \Delta^m\Box\partial\Delta^n \arrow[r] \arrow[ur, "t"] & F,
\end{tikzcd}$$

which then enables us to factor the original lifting problem via one against the fibration

$$\begin{tikzcd}
2 \times [(x, y, S) : \mathsf{CovSpan}, u : i(x), v : i(y), w : S(u, v)] \arrow[d, two heads] \\
2 \times \mathsf{CovSpan}.
\end{tikzcd}$$

To construct t, we write the domain of the left map in (1.2) as a union of the subobjects $\Delta^{\{0,\ldots,\hat{a},\ldots,m\}}\Box\Delta^n$ for $0 \leq a \leq m$ with $a \neq k$ and $\Delta^m\Box\Delta^{\{0,\ldots,\hat{b},\ldots,n\}}$ for $0 \leq b \leq n$. As before, from the pushout defining F, we are able to define t on all these subobjects except for $\Delta^{\{1,\ldots,m\}}\Box\Delta^n$ and denote their union D. Note that this critical subobject is only present if $k \geq 0$, i.e. we are filling an inner horn, necessitating $m \geq 2$ Now observe that $D \rightarrowtail \Delta^m \square \Lambda^n_l \cup \partial\Delta^m \square \Delta^n$ is a pushout of

$$h^{m-1}_{k-1} \,\widehat{\square}\, i^{n-1} \colon \Lambda^{m-1}_{k-1} \square \Delta^n \cup \partial\Delta^{m-1} \square \partial\Delta^n \to \Delta^{m-1} \square \Delta^n$$

(note that $m - 1 \geq 1$) where the induced map $\Delta^{m-1} \square \Delta^n \to 2$ factors via γ^1. Thus, as before, we are able to complete the definition of t by using a lifting problem

$$\begin{tikzcd}
\Lambda^{m-1}_{k-1} \square \Delta^n \cup \partial\Delta^{m-1} \square \partial\Delta^n \arrow[r] \arrow[d, "h^{m-1}_{k-1} \widehat{\square} i^{n-1}"] & {[(x, y, S) : \mathsf{CovSpan}, u : i(x), v : i(y), w : S(u, v)]} \arrow[d, two heads] \\
\Delta^{m-1} \square \Delta^n \arrow[r] \arrow[ur, dotted] & {[(x, y, S) : \mathsf{CovSpan}, v : i(y)]}.
\end{tikzcd}$$

This finishes the definition of the inverse map $\mathsf{CovSpan} \to \mathcal{U}^2_{\mathsf{cov}}$.

$\square$

This is a follow-up on some aspects in Emily's notes. We extend the definition of span2arr from a map on points to a map of bisimplicial sets. Questions remain.

2. The map arr2span

Let us work in the bisimplicial model. By interpreting a certain expression in Riehl-Shulman type theory, we have a map

$$\mathrm{arr2span}\colon \mathcal{U}^{\mathbb{2}} \to \sum_{A,B,S:\mathcal{U}} (S \to A \times B).$$

3. The map span2arr

Let us define a map

$$\mathrm{span2arr}\colon \sum_{A,B,S:\mathcal{U}} (S \to A \times B) \to \mathcal{U}^{\mathbb{2}}$$

in the other direction. This is given by the exponential transpose of a map

$$\mathbb{2} \times \sum_{A,B,S:\mathcal{U}} (S \to A \times B) \to \mathcal{U}.$$

This is given by the classifying map for a Reedy fibration

$$Z \twoheadrightarrow \mathbb{2} \times \sum_{A,B,S:\mathcal{U}} (S \to A \times B) \tag{3.1}$$

with small fibers. Abbreviating

$$X =_{\mathrm{def}} [A : \mathcal{U}, B : \mathcal{U}, S : \mathcal{U}, f : S \to A, g : S \to B] =_{\mathrm{def}} \sum_{A,B,S:\mathcal{U}} (S \to A \times B),$$

we construct the pushout

$$\begin{tikzcd}
X.\,\mathrm{El}(S) + X.\,\mathrm{El}(S) \arrow[r, tail] \arrow[d, "X.f+X.g"'] & \mathbb{2} \times X.\,\mathrm{El}(S) \arrow[d] \arrow[ddr, bend left] & \\
X.\,\mathrm{El}(A) + X.\,\mathrm{El}(B) \arrow[r, tail] \arrow[drr, bend right] & Y \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted] & \\
& & \mathbb{2} \times X
\end{tikzcd}$$

and replace the induced map to $\mathbb{2} \times X$ by a Reedy fibration:

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr, tail, "\sim"'] & & \mathbb{2} \times X \\
& Z \arrow[ur, two heads] &
\end{tikzcd} \tag{3.2}$$

where $Y \to Z$ is a Reedy trivial cofibration.

One can observe that $Y \to \mathbb{2} \times X$ has small fibers (relative to the set universe used to construct $\mathcal{U}$). Unfortunately, the bisimplicial set $\mathbb{2} \times X$ is not small, so using the standard construction of the Reedy fibration replacement $Z \to \mathbb{2} \times X$, we do not get small fibers. But this we need to construct the classifying map (3.1).

So we have to use a special construction for this fibration replacement. Maybe we can adapt the construction of fibrant higher inductive types in cubical type theory, where a fiberwise fibrant replacement, which preserves smallness, becomes a fibration? Let us put aside this problem for now.

4. The composition of arr2span followed by span2arr

In the ideal case, we would have that the composite

$$\mathrm{span2arr} \circ \mathrm{arr2span} \colon \mathcal{U}^{\mathbb{2}} \to \mathcal{U}^{\mathbb{2}}$$

is homotopic to the identity. Let us first see how far we get trying to show this. So we aim to construct a homotopy

$$I \times \mathbb{2} \times \mathcal{U}^{\mathbb{2}} \to U$$

where I denotes the representable $\Delta^1 \square \Delta^0$, the interval in the space direction of bisimplicial sets. This will be given by the classifying map of a type

$$H \twoheadrightarrow I \times \mathbb{2} \times \mathcal{U}^{\mathbb{2}}$$

with small fibers. Let us abbreviate

$$[t, F] =_{\mathrm{def}} [t : \mathbb{2}, F : \mathcal{U}^{\mathbb{2}}] =_{\mathrm{def}} \mathbb{2} \times \mathcal{U}^{\mathbb{2}}.$$

On one endpoint of I, H should be given by a certain Reedy fibration replacement $Z' \twoheadrightarrow \mathbb{2} \times [t, F]$ of $Y' \to \mathbb{2} \times [t, F]$ in

$$\begin{tikzcd}
{[t,F].\,\mathrm{El}(\Pi_{\mathbb{2}} F) + [t,F].\Pi_{\mathbb{2}}(\mathrm{El} \circ F)} \arrow[r, tail] \arrow[d, "X.f+X.g"'] & {\mathbb{2} \times [t,F].\Pi_{\mathbb{2}}(\mathrm{El} \circ F)} \arrow[d] \arrow[ddr, bend left] & \\
{[t,F].\,\mathrm{El}(F(0)) + [t,F].\,\mathrm{El}(F(1))} \arrow[r, tail] \arrow[drr, bend right] & Y' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, two heads] & \\
 & & {\mathbb{2} \times [t,F],}
\end{tikzcd}$$

given as a pullback of the chosen Reedy fibration replacement in (3.2). On the other endpoint of I, H should be given by $[t, F].El(F(t)) \twoheadrightarrow [t, F]$, the type classified by $\mathrm{ev} \colon \mathbb{2} \times \mathcal{U}^{\mathbb{2}} \to U$.

Since U is univalent, it suffices to give a homotopy equivalence between the two types

$$\begin{tikzcd}
Z' \arrow[rr, dotted, "\sim"] \arrow[dr, two heads] & & {[t,F].\,\mathrm{El}(F(t))} \arrow[dl, two heads] \\
 & {[t,F].} &
\end{tikzcd}$$

Like Emily detailed, we have a map from Y' to $[t, F].\,\mathrm{El}(F(t))$. We would like to lift this through $Y' \to Z'$, for which we would need that it is Reedy trivial cofibration. But we only know that it is a pullback of the Reedy trivial cofibration $Y \to Z$ along the map $[t, F] \to 2 \times X$, which is not a Reedy fibration.

We can try to define a map from $[t, F].\,\mathrm{El}(F(t))$ to Y' as follows. Suppose we are given $x \in [t, F].\,\mathrm{El}(F(t))_{m,n}$. The first component of x is $t(x) \in \mathbb{2}_{m,n} \simeq \Delta([m], [1])$. If this is $\mathrm{const}(0) \in \Delta([m], [1])$ or $\mathrm{const}(1) \in \Delta([m], [1])$, we map x to an element of $[t, F].\,\mathrm{El}(F(0)) + [t, F].\,\mathrm{El}(F(1))_{m,n}$. Otherwise, we map it to an element f of $[t, F].\Pi_{\mathbb{2}}(\mathrm{El} \circ F)_{m,n}$. How?

References