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High-Energy Asymptotics of Erdős–Rényi Random Graphs

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supervised by Prof. Dr. Peter Müller

Declaration of Authorship

In good faith, I hereby declare that this work represents the results of my own investigations, except as correspondingly acknowledged, and has not been submitted for a degree, either in part or as whole, for assessment at any university.

Christian Sattler
Munich, 16th of April 2010

„Nothing to see here.”

Abstract

In this thesis, we consider Erdős–Rényi random graphs with N vertices and edge probability $\frac{p}{N}$ for fixed $p > 0$. We examine the limiting spectral properties of the Laplacian $\Delta^{(N)}$ as $N \to \infty$ at the upper asymptotical end. Specifically, we show that the integrated density of states $\sigma(E)$ of $\Delta^{(N)}$ has, *surprisingly*, logarithmically the same asymptotical behavior for $E \to \infty$ as the Poissonian vertex degree limiting distribution, i. e. that $-\ln(1-\sigma(E))$ behaves as $E \cdot \ln(E)$ for $E \to \infty$.

Contents

1 Introduction

The topic of this thesis is rooted within spectral graph theory, a rising field of research for the past few decades [7]. Generally, one considers certain linear operators defined on Hilbert spaces associated to finite or infinite graphs, hoping that crucial information about the graph is encoded in the spectral properties of the chosen operators. Here, the operator we are interested in is the (combinatorial) Laplacian, a natural discretization of the well-known continuous version. The graphs we will look at have a random component in that we consider ensembles of finite graphs as given by the Erdős–Rényi random graph model. In this model, for each $N \geq 2$ a random subgraph of the complete graph on N vertices is chosen with each edge having independent probability $c(N)$ for a chosen N-dependent function c. One then takes the asymptotical limit in the size N of the graph to get deterministic results. A detailed description of the model and problem follows below.

The Erdős–Rényi random graph model has been proposed as a model for certain kinds of random networks, for example the chaining of large polymer molecules [5]. Many other networks, though, exhibit a scale-free behaviour, incompatible with the (scaled) Poissonian vertex degree distribution of the Erdős–Rényi model. Mathematically, the topic at hand is concretely related to two different well-established fields:

First, there is the broad field of study of spectral properties of large random matrices. Starting with the observation [29] of Wigner that, as the size of the matrix grows, the spectral density of Wigner matrices, consisting of independent and identically distributed entries up to the symmetric or Hermitian constraint and certain normalization conditions, converges under very general assumptions to the density of a universal distribution, now famous as the Wigner semicircle law because of the shape of the graph of its density, much work has gone into finding spectral statistics which exhibit the same kind of universality behavior and broadening the class of matrices for which this universality holds.

For the Gaussian Unitary Ensemble (GUE), a special class of Hermitian matrices where the entries are required to be independent and identically Gaussian distributed, Dyson has established [8] an explicit form for the joint distribution function of the eigenvalues, enabling him to calculate the local eigenvalue correlation functions (informally, the distribution of the gaps of nearby eigenvalues) in the limit. This local eigenvalue statistics, know as the Dyson sine-kernel, having been readily generalized to the Gaussian Orthogonal Ensemble (GOE) and Gaussian Symplectic Ensemble (GSE), is also expected to be universal for a much broader class of matrix ensembles, though there is a certain lack of tools compensating for the loss of invariance of distribution under conjugation with orthogonal, unitary or symplectic matrices, which one has for the corresponding Gaussian ensembles.

Much recent work, culminating in a collaborative effort [10] of two different groups, has gone into developing crucial techniques and establishing the sine-kernel and related local eigenvalue correlation results for the Wigner matrix ensemble, with, aside from the usual variance normalization condition, the only assumpton that the distributions of the entries have subexponential decay. Even more recent work has dealt with the broader class of generalized Wigner matrices, where the individual entries, though still required to be independent, do not have do be identically distributed. Using a strong local form of the semicircle law, the previously mentioned results for Wigner matrices could be shown to also hold for generalized Wigner matrix ensembles [11, 12]. See also [9] for an up-to-date survey of the current state of affairs in this area.

One can try to fit the adjacency matrix of a random graph as in the Erdős–Rényi model into the general framework of random matrices, noting that the matrix entries have independent Bernoulli distribution and at least fulfill the basic symmetry constraint. Assuming the inverse of the edge probability $c(N)$ to scale slower than the size N of the graph, this makes the adjancency matrix non-sparse, and one can indeed apply standard methods for random matrices and establish the Wigner semicircle law for a correspondingly scaled version of the adjancency matrix (see e.g. [16]). For $c(N)N$ of constant order, as is the case for the model we are considering, the adjacency matrix is too sparse for the machinery to work and one ends up with only a weak existence property for the spectral distribution, the density of which having as support the full real line (see [18, 20]). For the Laplacian, things are even worse since there is a large non-independent contribution to the matrix in the form of its diagonal elements, which fulfill the side condition of being the negated sum of all the other elements in the same row. Not surprisingly though, in the case of constant edge probability, this distortion can still be managed somewhat, and the centered and normalized Laplacian can be shown to follow the Wigner semicircle law convolved with a Gaussian normal distribution component (see [4]). In our setting of inversely proportional edge probability and graph size, however, having to deal with the Laplacian side condition to the matrix as well as sparsity makes application of standard methods from the field of random matrices seem inaccessible.

Another nearby field, more directly related to mathematical physics, is the study of operators on graphs arising in physical models aspiring to model certain properties (e.g. conductance, heat dissipation) of randomly disordered systems (e.g. crystals formed from two different materials, solid matter with impurities). The random component may present itself in many forms, as an additional random diagonal component to the graph operator, for example as a random potential to the Laplacian in the parabolic Anderson model on $\mathbb{Z}^d$ or percolated subgraphs thereof [2], which leads to the generally theory of random Schrödinger operators [6, 22], or as a random percolation process, usually performed on a lattice such as $\mathbb{Z}^d$, where either bonds or sites (i.e. vertices or edges) are randomly selected such as to form a random subgraph of the original lattice. Regarding the latter, examining spectral properties of the Laplacian for bond-percolation on regular infinite graphs [28, 21] bears probably the most resemblence to our setting.

Having found certain spectral properties of the operator under consideration (e.g. support of the spectrum, existence/local properties/asymptotics of the integrated density of state, localization of eigenvectors) to show a deterministic behavior, oblivious to the random process underlying the situation, one can derive global information about the system in question, often with direct physical ramifications. [1]

Note that the primary difference between the Erdős–Rényi random graph model and most of these physical models lies not in the fact that in the former we consider at no point infinite graphs, but an infinite ensemble of graphs of increasing size. The most differentiating feature is the complete lack of geometric structure in the Erdős–Rényi case. This leads to a rather different approach, consisting more of combinatorial reasoning than of underlying geometrical intuition and methods from stochastic ergodicity. Incorporating aspects of the Erdős–Rényi model into a bond-percolation setting, for example with the probability of an additional edge between two vertices depending algebraically on their distance, would invariably require a change of methods.

The Erdős–Rényi random graph model was introduced and first analyzed in terms of some elemental asymptotic combinatorial properties by Erdős and Rényi in their seminal work [13]. In fact, they considered two different kinds of models: In the first, one considers a graph on N labeled vertices uniformly chosen from all graphs with $m(N)$ edges. In the second, the edges themselves are independently chosen with probability $c(N)$. One is then interested in properties of the scaling limit $N \to \infty$ of such a random graph ensemble, where $m(N)$ or $c(N)$, respectively, follow a given asymptotics with respect to N. It turns out that both models generally behave the same way if one sets $c(N) = \binom{N}{2}^{-1} m(N)$. For an overview of general properties of Erdős–Rényi

[1]With the underlying random process being in fact *responsible* for the deterministic spectral properties of the system, one could even go so far as to argue that random microscopic disorder is a *natural necessity* for macroscopic stability of most systems.

random, see also the more recent accounts [3, 17].

For reasons of technical simplicity, we choose the second model to work with, though we are confident this does not affect our results. Also, we restrict ourselves to the case where $c(N)$ scales inversely with N, i.e. $c(N) = \frac{p}{N}$ for some p. In the asymptotic limit, this is equivalent to requiring the average vertex degree to converge to p. The most well-known property of this model, already established in [13], is the emergence of a giant cluster for $p \geq 1$, giving rise to fundamentally different behavior patterns for the subcritical regime $p \in]0,1[$, the critical point $p = 1$ and the supercritical regime $p \in]1,\infty[$.

We are interested in asymptotic spectral properties of the (combinatorial) Laplacian. In [19], the authors established that under the above circumstances, the integrated density of states of the Laplacian exhibits a Lifshitz tail [23, 24, 25] with Lifshitz exponent $\frac{1}{2}$ at the lower asymptotic end for the subcritical regime $p < 1$. This thesis was originally planned to correspondingly examine the upper asymptotic end of the integrated density of states in the subcritical regime, but is was quickly noted that the methods employed could be simplified, making them oblivious to the choice of p. In fact, it is to be expected that the result, logarithmically identical asymptotic behavior of the integrated density of states of the Laplacian and the vertex degree limiting distribution, which in this case is Poissonian and has logarithmic asymptotics $-E \cdot \ln(E)$, holds for a much more general notion of random graph model, the charactertic feature of which would most likely be the lack of any substantial concept of geometric locality.

1.1 Acknowledgement

I would like to thank my advisor Prof. Dr. Peter Müller for his continued extraordinary patience and permament readiness to supportingly discuss all kinds of matters, whether related to this thesis or not.

2 Terminology and definitions

2.1 Graphs and other basic notions

A *graph* $G = (V, E)$ consists of a set V of vertices and a set E of edges between them. Formally, an *edge* is an unordered pair of distinct vertices, i.e. a two-element subset of V. [2] We denote an edge between vertices $u, v \in V$ by $[u, v]$. As said earlier, $[u, v] = [v, u]$ denote the same edge and there is no edge from a vertex to itself. Hence, when being given an edge $[u, v]$, the assumption $u \neq v$ is always implicit.

Vertices $u, v \in V$ are called *adjacent* if there is an edge $[u, v] \in E$ between them. A vertex $u \in V$ is *incident* to an edge $e \in E$ if there is $v \in V$ such that $e = [u, v]$. Similarly, two edges $e, f \in E$ are called *incident* if there are $u, v, w \in V$ such that $e = [u, v]$ and $f = [v, w]$. The set $N_G(v)$ of *neighbors* of a vertex v consists of all vertices w such that v is adjacent to w. The *degree* $\deg_G(v)$ of a vertex $v \in V$ denotes the number of edges the vertex is incident to, i.e. the cardinality of $N_G(v)$. Finally, the equivalence classes of V under the transitive-reflexive closure of the adjacency relation are called the *clusters* of G.

An example is the complete graph on an arbitrary vertex set V, which has as edges all two-element subsets of V, i.e. an edge $[u, v] \in E$ for all distinct $u, v \in V$. All distinct vertices $u, v \in V$ are adjacent, the set of neighbors of $v \in V$ is $V \setminus \{v\}$, and there is only one cluster, V.

When dealing with an expression $f(u, v)$ depending on an edge $[u, v]$, e.g. in the context of a sum $\sum_{[u,v] \in E} f(u, v)$ or a set $\{f(u, v) \mid [u, v] \in E\}$, in order for the whole expression to be well-defined, the expression $f(u, v)$ will always have to be symmetric with respect to u and v, i.e. invariant under transposition of u and v, since we identified $[u, v]$ and $[v, u]$. By the same reason, it should also be clear that there appear no distinct terms $f(u, v)$ and $f(v, u)$, e.g. in the previously mentioned sum or set.

A *graph morphism* from a graph $G_1 = (V_1, E_1)$ to a graph $G_2 = (V_2, E_2)$ is a mapping $f : V_1 \to V_2$ which preserves the edge relation in that $[f(u), f(v)] \in E_2$ for any edge $[u, v] \in E_1$. The homomorphism is injective/surjective if the mapping on the vertex sets is injective/surjective, respectively. A *subgraph* is a morphism where the vertex mapping is the identity restricted to some subset of the original vertex set. The subgraph *induced* by a subset U of the vertex set V is the subgraph with vertex set U and maximal edge set F, i.e. $F = \{[u, v] \in E \mid u, v \in U\}$.

[2] In this thesis, we will only consider *simple* graphs, i.e. undirected graphs without loops or multiple edges between the same vertices.

To a graph G with a countable set V of vertices, we associate the Hilbert space

$$l^2(V) := \left\{ \phi : V \to \mathbb{C} \;\middle|\; \sum_{v \in V} |\phi(v)|^2 < \infty \right\}$$

of square-summable functions from V to $\mathbb{C}$. The canonical inner product $\langle\ .\ ,\ .\ \rangle$ is given by

$$\langle \psi, \phi \rangle := \sum_{v \in V} \overline{\psi(v)} \phi(v)$$

for $\phi, \psi \in l^2(V)$.

In what follows, we will only consider graphs G with a finite set V of vertices. The condition about square-summability in the definition of $l^2(V)$ will then vanish, and $l^2(V)$ becomes an ordinary finite-dimensional vector space with an inner product. Note that we have a canonical basis $\{e_v \mid v \in V\}$ of $l^2(V)$, where

$$e_v(u) := \delta_{uv} := \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{else} \end{cases}$$

for $u, v \in V$.

There are a number of linear operators on $l^2(V)$ associated to a graph $G = (V, E)$ which are fundamental to spectral graph theory. With respect to the canonical basis $\{e_v \mid v \in V\}$ of $l^2(V)$, we can think of them as matrices with rows and columns indexed by vertices of V.

First, the *adjacency matrix* A_G is given by

$$(A_G)_{uv} = \begin{cases} 1 & \text{if } [u, v] \in E, \\ 0 & \text{else} \end{cases}$$

for vertices $u, v \in V$. Alternatively, we may characterize A_G via

$$\langle \psi, A_G \phi \rangle = \sum_{\substack{u, v \in V, \\ [u,v] \in E}} \overline{\psi(u)} \phi(v)$$

for $\phi, \psi \in l^2(X)$. Clearly, since adjacency is a symmetric relation, the adjacency operator is self-adjoint.

Next, the *degree matrix* D_G is simply the diagonal matrix with diagonal entries $(D_G)_{vv} = \deg_G(v)$ for $v \in V$. Again, alternatively we may write

$$\langle \psi, D_G \phi \rangle = \sum_{u \in V} \deg_G(u) \cdot \overline{\psi(u)} \phi(u) = \sum_{\substack{u, v \in V, \\ [u,v] \in E}} \overline{\psi(u)} \phi(u)$$

for $\phi, \psi \in l^2(V)$. Obviously, this is also a self-adjoint operator.

Finally, the *Laplacian* Δ_G is defined as $\Delta_G := D_G - A_G$. As the difference of self-adjoint operators, Δ_G itself is self-adjoint. By the previous characterizations of A_G and D_G, we may view Δ_G as a matrix with entries

$$(\Delta_G)_{uv} := \begin{cases} \deg_G(v) & \text{if } u = v, \\ -1 & \text{if } [u,v] \in E, \\ 0 & \text{else} \end{cases}$$

for $u, v \in V$. Alternatively, we have

$$\begin{aligned}
\langle \psi, \Delta_G \phi \rangle &= \langle \psi, (D_G - A_G)\phi \rangle \\
&= \langle \psi, D_G \phi \rangle - \langle \psi, A_G \phi \rangle \\
&= \sum_{\substack{u,v \in V, \\ [u,v] \in E}} \overline{\psi(u)} \phi(u) - \sum_{\substack{u,v \in V, \\ [u,v] \in E}} \overline{\psi(u)} \phi(v) \\
&= \sum_{\substack{u,v \in V, \\ [u,v] \in E}} \overline{\psi(u)} \cdot (\phi(u) - \phi(v)) \\
&= \sum_{[u,v] \in E} \overline{\psi(u)} \cdot (\phi(u) - \phi(v)) + \overline{\psi(v)} \cdot (\phi(v) - \phi(u)) \\
&= \sum_{[u,v] \in E} \overline{(\psi(u) - \phi(v))} \cdot (\phi(u) - \phi(v))
\end{aligned}$$

for $\phi, \psi \in l^2(V)$.

By this last characterization, we see that

$$\begin{aligned}
\langle \phi, \Delta_G \phi \rangle &= \sum_{[u,v] \in E} \overline{(\phi(u) - \phi(v))} \cdot (\phi(u) - \phi(v)) \\
&= \sum_{[u,v] \in E} |\phi(u) - \phi(v)|^2 \geq 0
\end{aligned} \tag{2.1}$$

for $\phi \in l^2(V)$. Hence, Δ_G is non-negative [3] definite.

Let $V_1, \ldots, V_k \subseteq V$ be the clusters of G and let $G_1, \ldots, G_k$ be the subgraphs of G induced by $V_1, \ldots, V_k$, respectively. Since there are no edges from a vertex in one cluster to a vertex in another cluster, the degree of a vertex $v \in V_i$, with $1 \leq i \leq k$, is the same in G_i and G, i.e. $\deg_{G_i}(v) = \deg_G(v)$. Letting $\pi_i : l^2(V) \to l^2(V_i)$,

$$\phi \mapsto \phi|_{V_i},$$

denote the canonical projection, where $\phi|_{V_i}$ denotes the restriction of the function ϕ to the domain V_i, and $\iota_i : l^2(V_i) \to l^2(V)$ such that

$$\iota_i(\phi)(v) := \begin{cases} \phi(v) & \text{for } v \in V_i, \\ 0 & \text{else} \end{cases}$$

[3] positive

for all $v \in V$ denote the canonical embedding for $1 \leq i \leq k$, [4] we hence see that $D_{G_i} = \pi_i \circ D_G \circ \iota_i$ for $1 \leq i \leq k$ as well as

$$D_G = \sum_{i=1}^{k} \iota_i \circ D_{G_i} \circ \pi_i.$$

Similarly, $A_{G_i} = \pi_i \circ A_G \circ \iota_i$ for $1 \leq i \leq k$ as well as

$$A_G = \sum_{i=1}^{k} \iota_i \circ A_{G_i} \circ \pi_i,$$

and, by linearity, $\Delta_{G_i} = \pi_i \circ \Delta_G \circ \iota_i$ for $1 \leq i \leq k$ as well as

$$\Delta_G = \sum_{i=1}^{k} \iota_i \circ \Delta_{G_i} \circ \pi_i.$$

Phrased in terms of the usual matrix formalism, this just means that the adjacency/degree/Laplacian matrix of a graph can be viewed as a block diagonal matrix with each block matrix being the adjacency/degree/Laplacian matrix, respectively, of the subgraph induced by the corresponding cluster.

We are interested in the eigenvalues of Δ_G for particular graphs G. By our previous calculation (2.1), we see that eigenvectors with corresponding eigenvalue zero, i.e. vectors $\phi \in l^2(V)$ such that equality holds in the inequality (2.1), must map adjacent vertices to the same value and are hence given by vectors $\phi \in l^2(V)$ which map vertices of the same cluster to the same value. The multiplicity of the eigenvalue zero, equal to the dimension of the corresponding eigenspace, is hence given by the number of clusters of G.

Let T be a self-adjoint operator on a finite Hilbert space H of dimension $n \in \mathbb{N} := \{1, 2, \ldots\}$ and let $\lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of T, counted with multiplicity. We define the *eigenvalue counting function* $\gamma_T : \mathbb{R} \to \{0, 1, \ldots, n\}$ of T as

$$\begin{aligned} \gamma_T(E) &:= |\{i \in \{1, \ldots, n\} \mid \lambda_i \leq E\}| \\ &= \begin{cases} \max_{\substack{i \in \{1,\ldots,n\}, \\ \lambda_i \leq E}} (i) & \text{if } E \geq \lambda_1, \\ 0 & \text{else,} \end{cases} \\ &= \begin{cases} \min_{\substack{i \in \{1,\ldots,n\}, \\ \lambda_i > E}} (i-1) & \text{if } E < \lambda_n, \\ n & \text{else} \end{cases} \end{aligned}$$

[4] Note that $\pi_i \circ \iota_i = \mathrm{id}_{l^2(V_i)}$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k} \iota_i \circ \pi_i = \mathrm{id}_{l^2(V)}$, i.e. we have a canonical decomposition $l^2(V) = \bigoplus_{i=1}^{k} l^2(V_i)$.

for $E \in \mathbb{R}$. Note, since Δ_G is non-negative definite, that $\gamma_{\Delta_G}(E) = 0$ for $E < 0$.

We will also need the *Rayleigh quotient* $R_T(\phi)$ of a non-zero vector $\phi \in H$ with respect to T, which is defined as

$$R_T(\phi) := \frac{\langle \phi, T\phi \rangle}{\langle \phi, \phi \rangle} = \frac{\langle \phi, T\phi \rangle}{|\phi|^2},$$

where $\langle \,.\, , \,.\, \rangle$ denotes the inner product of the Hilbert space H. Some basic facts concerning relations between Rayleigh quotients and eigenvalues of an operator are provided in the appendix.

Writing $U \lhd H$ will denote U as a subspace of H, with $\dim(U)$ denoting the dimension of U.

Finally, we will make use of some basic stochastic notions and standard notation regarding measure spaces, probability measures, random variables, (mathematical) expectation and distributions. In particular, $B_{n,p} : \mathbb{Z} \to [0, 1]$ will denote the *binomial distribution* with parameters $n \in \mathbb{N}_0 := \{0, 1, \dots\}$ and $p \in [0, 1]$, i.e.

$$B_{n,p}(k) := \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

for $k \geq 0$ and $B_{n,p}(k) := 0$ for $k < 0$. Note that $\binom{n}{k} = 0$, and hence $B_{n,p}(k) = 0$, for $k > n$. Further, $\pi_\lambda : \mathbb{Z} \to [0, 1]$ will denote the *Poisson distribution* with real parameter $\lambda \geq 0$, i.e.

$$\pi_{\lambda(k)} := \frac{\lambda^k}{k!} \cdot e^{-\lambda}$$

for $k \geq 0$ and $\pi_\lambda(k) := 0$ for $k < 0$. Note that $\pi_0(0) = \frac{0^0}{0!} \cdot e^{-0} = 0^0 = 1$. Some facts relating the binomial distribution to the Poisson distribution are proved in the appendix.

2.2 Erdős–Rényi random graph model

Fix $p \in]0, \infty]$ for the remainder of this thesis, aside from the appendix. We are interested in *Erdős–Rényi random graphs* on $N \in \mathbb{N}$ vertices, $N > p$, with edge probability $\frac{p}{N}$, random subgraphs of the complete graph $K^{(N)}$ on N vertices $1, \dots, N$ where each edge is chosen with equal probability $\frac{p}{N}$. Formally, we construct $\binom{N}{2}$ copies $\Omega_{i,j}$, for $1 \leq i < j \leq N$, of the discrete probability space with two elements 1 and 0 of weights $\frac{p}{N}$ and $1 - \frac{p}{N}$, respectively, each copy representing one edge $[i, j]$ of $K^{(N)}$. Their measure space product is canonically isomorphic to a discrete measure space $\Omega^{(N)}$ on the set of subgraphs of $K^{(N)}$, with the events $\{1\}$ and $\{0\}$ of each of the copies $\Omega_{i,j}$, for $1 \leq i < j \leq N$, corresponding to the set of subgraphs of $K^{(N)}$ with edge $[i,j]$ present or absent, respectively.

Let $\mathbb{P}^{(N)}$ denote the measure of this probability space $\Omega^{(N)}$. Since each original copy has Bernoulli distribution with parameter $\frac{p}{N}$, the measure $\mathbb{P}^{(N)}$ on the product has corresponding binomial distribution with parameters $\binom{N}{2} = \frac{N(N-1)}{2}$ and $\frac{p}{N}$, assigning each subgraph $\mathcal{G}^{(N)}$ with edge set $\mathcal{E}^{(N)}$ weight

$$\mathbb{P}^{(N)}\left(\left\{\mathcal{G}^{(N)}\right\}\right) = \left(\frac{p}{N}\right)^{\left|\mathcal{E}^{(N)}\right|}\left(1 - \frac{p}{N}\right)^{\frac{N(N-1)}{2} - \left|\mathcal{E}^{(N)}\right|}.$$

In what follows, $\mathcal{G}^{(N)}$ will always denote the identical random variable on the probability space $\Omega^{(N)}$ and $\mathcal{E}^{(N)}$ will denote its edge set. If not indicated otherwise, graph theoretical notions as introduced in the previous sections, such as degree or neighbor sets, will always relate to $\mathcal{G}^{(N)}$ when written without graph subscript argument.

For each edge $[i,j]$, where $1 \leq i,j \leq N$ and $i \neq j$, we introduce a random variable

$$g_{i,j} := \begin{cases} 1 & [i,j] \in \mathcal{E}^{(N)}, \\ 0 & \text{else}, \end{cases}$$

valued 1 or 0 according to whether this edge is present or absent, respectively, in $\mathcal{G}^{(N)}$. For consisteny, we also set $g_{i,i} := 0$ for $1 \leq i \leq N$. Note that we follow the usual abuse in notation when talking about random variables in that we treat random variables as if they were normal variables, that is, values instead of functions from a measure space, and silently omit the source measure space argument. By construction of $\Omega^{(N)}$, the $g_{i,j}$, with $1 \leq i < j \leq N$, are independent and Bernoulli distributed under $\mathbb{P}^{(N)}$ with parameter $\frac{p}{N}$.

The permutations π of the set $\{1,\ldots,N\}$ are in bijective correspondence with graph automorphisms on $K^{(N)}$, sending vertex i to $\pi(i)$, for $1 \leq i \leq N$, and edge $[i,j]$ to $[\pi(i),\pi(j)]$, for $1 \leq i < j \leq N$. Consequently, each such automorphism induces a permutation of the subgraphs of $K^{(N)}$. Since the (discrete) density of $\mathbb{P}^{(N)}$ only depends on the number of edges of its subgraph argument, which is invariant under each such automorphism, we hence have a measure space automorphism j_π on $\Omega^{(N)}$ induced by π. This means that for any event A of $\Omega^{(N)}$, we have

$$\mathbb{P}^{(N)}(A) = \left(\mathbb{P}^{(N)} \circ j_\pi\right)(A) = \mathbb{P}^{(N)}\left(j_\pi(A)\right).$$

In particular, for any formula $\phi\left[X_{1,2}, X_{1,3}, \ldots, X_{N-1,N}\right]$ parameterized by $\{0,1\}$-valued variables $X_{i,j}$, where $1 \leq i < j \leq N$, we have

$$\begin{aligned}&\mathbb{P}^{(N)}\left(\left\{\omega \mid \phi\left[g_{1,2}(\omega), g_{1,3}(\omega), \ldots, g_{N-1,N}(\omega)\right]\right\}\right)\\ &= \mathbb{P}^{(N)}\left(\left\{\omega \mid \phi\left[g_{\pi(1),\pi(2)}(\omega), g_{\pi(1),\pi(3)}(\omega), \ldots, g_{\pi(N-1),\pi(N)}(\omega)\right]\right\}\right),\end{aligned}$$

or in the usual lazy notation,

$$\begin{aligned}&\mathbb{P}^{(N)}\left(\phi\left[g_{1,2}, g_{1,3}, \ldots, g_{N-1,N}\right]\right)\\ =\ &\mathbb{P}^{(N)}\left(\phi\left[g_{\pi(1),\pi(2)}, g_{\pi(1),\pi(3)}, \ldots, g_{\pi(N-1),\pi(N)}\right]\right).\end{aligned}$$

This basic fact of symmetry will be readily used in the course of this thesis, sometimes without further mention.

For $n \in \mathbb{N}$, we will identify $l^2(\{1,\ldots,n\})$ with $\mathbb{C}^n$. Instead of writing $\phi \in l^2(\{1,\ldots,n\})$ and $\phi(i)$ for $i \in \{1,\ldots,n\}$, we will henceforth write $\phi \in \mathbb{C}^n$ and ϕ_i for the component of ϕ corresponding to i. In particular, when working with a graph with vertex set $\{1,\ldots,N\}$ like $\mathcal{G}^{(N)}$, the usual operators on $l^2(\{1,\ldots,N\})$ associated to the graph will be viewed as operators on $\mathbb{C}^N$ or, more specifically, elements of $\mathbb{C}^{N\times N}$, the (vector) space of matrices with N rows and N columns.

Since $\mathcal{G}^{(N)}$ itself is a random variable, the adjacency matrix $A^{(N)}$, degree matrix $D^{(N)}$ and Laplacian $\Delta^{(N)}$ of $\mathcal{G}^{(N)}$ are all operator-valued random variables. Note that the entries of the adjacency matrix are just the $\{0,1\}$-valued random variables we introduced for each edge: We have

$$\left(A^{(N)}\right)_{ij} = g_{i,j}$$

for $1 \leq i,j \leq N$. Also, the degree of a vertex, appearing in the diagonal of the degree matrix, may be written as the sum of corresponding edge variables: For each $i \in \{1,\ldots,N\}$, we have

$$\left(D^{(N)}\right)_{ii} = \deg(i) = \sum_{\substack{1\leq j\leq N,\\ j\neq i}} g_{i,j} = \sum_{j=1}^{N} g_{i,j}.$$

The eigenvalues

$$0 = \lambda_1^{(N)} \leq \ldots \leq \lambda_N^{(N)}$$

of the Laplacian $\Delta^{(N)}$, counted with multiplicity, are of course also random variables by themselves. We define the *expected normalized eigenvalue counting function* $\sigma^{(N)} : \mathbb{R} \to [0,1]$ as

$$\begin{aligned}\sigma^{(N)}(E) :&= \frac{1}{N}\cdot\mathbb{E}^{(N)}\left(\gamma_{\Delta^{(N)}}(E)\right)\\ &= \frac{1}{N}\cdot\mathbb{E}^{(N)}\left(\left|\left\{i\in\{1,\ldots,N\}\ \middle|\ \lambda_i^{(N)}\leq N\right\}\right|\right),\end{aligned}$$

where $\mathbb{E}^{(N)}$ denotes the expectation with respect to $\mathbb{P}^{(N)}$. Note that $\sigma^{(N)}(E)$ is monotonically increasing and $\sigma^{(N)}(E) = 0$ for $E < 0$ as $\Delta^{(N)}$ is non-negative definite.

We can now define the *integrated density of states* of the Laplacian for Erdős–Rényi random graphs, which is the unique right-continuous distribution function $\sigma : \mathbb{R} \to [0, 1]$ such that

$$\sigma(E) = \lim_{N\to\infty} \sigma^{(N)}(E)$$

for all but at most countably many discontinuity points $E \in \mathbb{R}$. The existence and uniqueness of the integrated density of states is proved in [19] using standard moment methods. Note, as follows from the corresponding properties of $\sigma^{(N)}(E)$, that σ is monotonically increasing and $\sigma(E) = 0$ for $E < 0$.

The main result of this work is the following theorem about the nature of the high-end asymptotics of the density of states:

Theorem 1 (Thesis result)**.** *We have*

$$\lim_{E\to\infty} \frac{-\ln(1 - \sigma(E))}{E \cdot \ln(E)} = 1.$$

This will be proved by finding suitable bounding functions $f_{\text{low}}, f_{\text{high}} : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f_{\text{low}}(E) \leq 1 - \sigma(E) \leq f_{\text{high}}(E)$$

for all $E > 0$ at which σ is continuous and then showing that

$$\lim_{E\to\infty} \frac{-\ln(f_{\text{low}}(E))}{E \cdot \ln(E)} = \frac{-\ln(f_{\text{high}}(E))}{E \cdot \ln(E)} = 1.$$[5]

The establishment of the lower and upper bound will be the content of theorems 2 and 3, respectively, of the following sections. Since the complement of a countable set is still dense in $\mathbb{R}$, the above bounds on the density of states σ hold for a dense set of choices of E in $\mathbb{R}$. By monotonicity of σ, the final result about the large-value asymptoptics of σ follows.

[5]Here, the limits do require $f_{\text{low}}(E)$ and $f_{\text{high}}(E)$ to attain positive values (as required for well-definedness of the inner expression) only for sufficiently large values of E.

3 The lower bound

For this section, let $E \geq 1$ and let $\tilde{E} := \lfloor E \rfloor + 1$ be the smallest integer larger than E. Here and in the following, $\lfloor x \rfloor$ denotes the largest integer not exceeding x for $x \in \mathbb{R}$. For the rest this section, we assume $N \geq \tilde{E} + 1 \geq 3$.

Let

$$X := \left\{ 1 \leq i \leq N \mid \deg(i) = \tilde{E} \right\}$$

be the set of vertices of $\mathcal{G}^{(N)}$ with degree $\tilde{E}$ and let $\mathcal{H}^{(N)}$ be the subgraph of $\mathcal{G}^{(N)}$ induced by the vertex set X. Consider also the set $Y \subseteq X$ of *isolated* vertices of $\mathcal{H}^{(N)}$, i.e.

$$Y := \{ v \in X \mid \deg_{\mathcal{H}^{(N)}}(v) = 0 \} .$$

Denoting $U := \langle e_y \mid y \in Y \rangle$ the subspace of $\mathbb{C}^N$ generated by the vectors of the canonical basis corresponding to the vertices in Y, the significance of U is that the Rayleigh quotients of non-zero elements ϕ of U with respect to $D^{(N)}$, $A^{(N)}$ and, consequently, $\Delta^{(N)}$ can readily be computed:

Lemma 3.1. *For all non-zero $\phi \in U := \langle e_y \mid y \in Y \rangle \subseteq \mathbb{C}^N$, we have $R_{\Delta^{(N)}}(\phi) = \tilde{E}$.*

Proof. For the degree matrix, we have

$$\begin{aligned} R_{D^{(N)}}(\phi) &= \frac{\left\langle \phi, D^{(D)} \phi \right\rangle}{\langle \phi, \phi \rangle} \\ &= \frac{\sum_{i=1}^{N} \deg(i) \cdot |\phi_i|^2}{\sum_{i=1}^{N} |\phi_i|^2} \end{aligned}$$

Since ϕ_i is zero for $i \notin Y$:

$$(\text{cont.}) = \frac{\sum_{y \in Y} \deg(y) \cdot |\phi_y|^2}{\sum_{y \in Y} |\phi_y|^2}$$

Since $\deg(y) = \tilde{E}$ for all $y \in Y \subseteq X$ by construction of X:

$$(\text{cont.}) = \frac{\sum_{y \in Y} \tilde{E} \cdot |\phi_y|^2}{\sum_{y \in Y} |\phi_y|^2} = \tilde{E}.$$

For the adjacency matrix, we have

$$\begin{aligned} R_{A^{(N)}}(\phi) &= \frac{\left\langle \phi, D^{(N)} \phi \right\rangle}{\langle \phi, \phi \rangle} \\ &= \frac{\sum_{1 \leq i,j \leq N} g_{i,j} \cdot \overline{\phi_i} \phi_j}{\sum_{i=1}^{N} |\phi_i|^2} \end{aligned}$$

Since ϕ_i is zero for $i \notin Y$:

$$\text{(cont.)} = \frac{\sum_{u,v \in Y} g_{u,v} \cdot \overline{\phi_u} \phi_v}{\sum_{y \in Y} |\phi_y|^2}$$

Since distinct $u, v \in Y$ are isolated in $\mathcal{H}^{(N)}$ by construction of Y as the set of isolated vertices of $\mathcal{H}^{(N)}$:

$$\text{(cont.)} = \frac{\sum_{u,v \in Y} 0 \cdot \overline{\phi_u} \phi_v}{\sum_{y \in Y} |\phi_y|^2} = 0.$$

Using linearity of the Rayleigh quotient in the matrix argument, we conclude

$$\begin{aligned}
R_{\Delta^{(N)}}(\phi) &= \frac{\left\langle \phi, \Delta^{(N)} \phi \right\rangle}{\langle \phi, \phi \rangle} \\
&= \frac{\left\langle \phi, \left(D^{(N)} - A^{(N)}\right) \phi \right\rangle}{\langle \phi, \phi \rangle} \\
&= \frac{\left\langle \phi, D^{(N)} \phi \right\rangle - \left\langle \phi, A^{(N)} \phi \right\rangle}{\langle \phi, \phi \rangle} \\
&= \frac{\left\langle \phi, D^{(N)} \phi \right\rangle}{\langle \phi, \phi \rangle} - \frac{\left\langle \phi, A^{(N)} \phi \right\rangle}{\langle \phi, \phi \rangle} \\
&= R_{D^{(N)}}(\phi) - R_{A^{(N)}}(\phi) \\
&= \tilde{E} - 0 = \tilde{E}.
\end{aligned}$$

□

Lemma 3.2. *We have*

$$1 - \sigma^{(N)}(E) \geq \frac{1}{N} \cdot \mathbb{E}^{(N)}\left(|Y|\right).$$

Proof. Applying the min-max principle as formulated in the second equation in the corresponding lemma A.1 in the appendix, for Y non-empty we find

$$\lambda^{(N)}_{N-|Y|+1} = \sup_{\substack{V \lhd \mathbb{C}^n, \\ \dim(V) \geq |Y|}} \inf_{0 \neq \phi \in V} R_{\Delta^{(N)}}(\phi)$$

Considering, in particular, the above subspace U of dimension $|Y|$:

$$\text{(cont.)} \geq \inf_{0 \neq \phi \in U} R_{\Delta^{(N)}}(\phi)$$

And using the previous lemma 3.1:

$$\text{(cont.)} = \inf_{0 \neq \phi \in U} \tilde{E} = \tilde{E} > E.$$

Since $E < \lambda_{N-|Y|+1}^{(N)} \leq \ldots \leq \lambda_N^{(N)}$, we hence have found $|Y|$ eigenvalues of $\Delta^{(N)}$ that are larger than E. Note that this final conclusion also holds for the case $Y = \emptyset$.

Finishing the proof of the lemma,

$$\begin{aligned} 1 - \sigma^{(N)}(E) &= 1 - \frac{1}{N} \cdot \mathbb{E}^{(N)} \left(|\{i \in \{1, \ldots, N\} \mid \lambda_i \leq E\}| \right) \\ &= \frac{1}{N} \cdot \mathbb{E}^{(N)} \left(N - |\{i \in \{1, \ldots, N\} \mid \lambda_i \leq E\}| \right) \\ &= \frac{1}{N} \cdot \mathbb{E}^{(N)} \left(|\{i \in \{1, \ldots, N\} \mid \lambda_i > E\}| \right) \end{aligned}$$

Plugging in our newly found lower bound for the size of the outer set:

$$\text{(cont.)} \geq \frac{1}{N} \cdot \mathbb{E}^{(N)} \left(|Y| \right)$$

□

What remains is to suitably estimate the size of Y.

Lemma 3.3. *We have*

$$\mathbb{E}^{(N)} \left(|X| \right) = N \cdot B_{N-1, \frac{p}{N}} \left(\tilde{E} \right).$$

Proof. We have, being particularly explicit,

$$\begin{aligned} \mathbb{E}^{(N)} \left(|X| \right) &= \mathbb{E}^{(N)} \left(\sum_{i=1}^{N} \begin{cases} 1 & \text{if } \deg(i) = \tilde{E}, \\ 0 & \text{else} \end{cases} \right) \\ &= \sum_{i=1}^{N} \mathbb{E}^{(N)} \left(\begin{cases} 1 & \text{if } \deg(i) = \tilde{E}, \\ 0 & \text{else} \end{cases} \right) \\ &= \sum_{i=1}^{N} \mathbb{P}^{(N)} \left(\deg(i) = \tilde{E} \right) \end{aligned}$$

Applying the automorphism of $\Omega^{(N)}$ corresponding to the automorphism of $K^{(N)}$ swapping vertices 1 and i, i. e. exploiting symmetry:

$$\begin{aligned}(\text{cont.}) &= N \cdot \mathbb{P}^{(N)}\left(\deg(1) = \tilde{E}\right) \\ &= N \cdot \mathbb{P}^{(N)}\left(\sum_{i=2}^{N} g_{1,i} = \tilde{E}\right)\end{aligned}$$

Recognizing that, with respect to $\mathbb{P}^{(N)}$, the sum $\sum_{i=2}^{N} g_{1,i}$ of $N-1$ independent Bernoulli distributed random variables with parameter $\frac{p}{N}$ is binomially distributed with parameters $N-1$ and $\frac{p}{N}$:

$$(\text{cont.}) = N \cdot B_{N-1,\frac{p}{N}}\left(\tilde{E}\right).$$

□

Let $\mathcal{F}^{(N)}$ denote the set of edges of $\mathcal{H}^{(N)}$.

Lemma 3.4. *We have*

$$\mathbb{E}^{(N)}\left(\left|\mathcal{F}^{(N)}\right|\right) \leq N \cdot \frac{p}{2} \cdot B_{N-2,\frac{p}{N}}\left(\tilde{E}-1\right)^2.$$

Proof. We have

$$\mathbb{E}^{(N)}\left(\left|\mathcal{F}^{(N)}\right|\right) = \sum_{1 \leq u < v \leq N} \mathbb{P}^{(N)}\left(\deg(u) = \deg(v) = \tilde{E}, [u,v] \in \mathcal{E}^{(N)}\right)$$

Using the automorphism of $\Omega^{(N)}$ corresponding to the automorphism of $K^{(N)}$ swapping vertices u and v with 1 and 2, respectively:

$$\begin{aligned}(\text{cont.}) &= \binom{N}{2} \cdot \mathbb{P}^{(N)}\left(\deg(1) = \deg(2) = \tilde{E}, g_{1,2} \in \mathcal{E}^{(N)}\right) \\ &= \binom{N}{2} \cdot \mathbb{P}^{(N)}\left(\sum_{\substack{1 \leq i \leq N, \\ i \neq 1}} g_{1,i} = \sum_{\substack{1 \leq i \leq N, \\ i \neq 2}} g_{2,i} = \tilde{E}, g_{1,2} = 1\right) \\ &= \binom{N}{2} \cdot \mathbb{P}^{(N)}\left(g_{1,2} = 1, \sum_{i=3}^{N} g_{1,i} = \tilde{E}-1, \sum_{i=3}^{N} g_{2,i} = \tilde{E}-1\right)\end{aligned}$$

Utilizing the independence of $g_{i,j}$ for $1 \leq i < j \leq N$:

$$\begin{aligned}(\text{cont.}) = &\binom{N}{2} \cdot \mathbb{P}^{(N)}\left(g_{1,2} = 1\right) \\ &\cdot \mathbb{P}^{(N)}\left(\sum_{i=3}^{N} g_{1,i} = \tilde{E} - 1\right) \cdot \mathbb{P}^{(N)}\left(\sum_{i=3}^{N} g_{2,i} = \tilde{E} - 1\right)\end{aligned}$$

again using the automorphism of $K^{(N)}$ swapping vertices 1 and 2:

$$(\text{cont.}) = \binom{N}{2} \cdot \mathbb{P}^{(N)}\left(g_{1,2} = 1\right) \cdot \mathbb{P}^{(N)}\left(\sum_{i=3}^{N} g_{1,i} = \tilde{E} - 1\right)^2$$

Noting that $\sum_{i=3}^{N} g_{1,i}$ is binomially distributed with parameters $N-2$ and $\frac{p}{N}$:

$$\begin{aligned}(\text{cont.}) &= \frac{N(N-1)}{2} \cdot \frac{p}{N} \cdot B_{N-2,\frac{p}{N}}\left(\tilde{E} - 1\right)^2 \\ &\leq N \cdot \frac{p}{2} \cdot B_{N-2,\frac{p}{N}}\left(\tilde{E} - 1\right)^2.\end{aligned}$$

□

The previous two lemmata give us the means to conclude:

Lemma 3.5. *We have*

$$\mathbb{E}^{(N)}\left(|Y|\right) \geq N \cdot \left(B_{N-1,\frac{p}{N}}\left(\tilde{E}\right) - p \cdot B_{N-2,\frac{p}{N}}\left(\tilde{E} - 1\right)^2\right).$$

Proof. Since every vertex in $X \backslash Y$ must be incident to at least one edge and every edge is incident to exactly two vertices, we know that $|X \backslash Y| \leq 2\left|\mathcal{F}^{(N)}\right|$. Hence,

$$\begin{aligned}\mathbb{E}^{(N)}\left(|Y|\right) &= \mathbb{E}^{(N)}\left(|X| - |X \backslash Y|\right) \\ &= \mathbb{E}^{(N)}\left(|X|\right) - \mathbb{E}^{(N)}\left(|X \backslash Y|\right) \\ &\geq \mathbb{E}^{(N)}\left(|X|\right) - 2 \cdot \mathbb{E}^{(N)}\left(\left|\mathcal{F}^{(N)}\right|\right)\end{aligned}$$

Applying lemmata 3.3 and 3.4:

$$\begin{aligned}(\text{cont.}) &\geq N \cdot B_{N-1,\frac{p}{N}}\left(\tilde{E}\right) - 2 \cdot N \cdot \frac{p}{2} \cdot B_{N-2,\frac{p}{N}}\left(\tilde{E} - 1\right)^2 \\ &= N \cdot \left(B_{N-1,\frac{p}{N}}\left(\tilde{E}\right) - p \cdot B_{N-2,\frac{p}{N}}\left(\tilde{E} - 1\right)^2\right).\end{aligned}$$

□

Combining the previous result with lemma 3.2 yields the lower bound we need:

Lemma 3.6. *We have* $1 - \sigma^{(N)}(E) \geq f_{\text{low}}^{(N)}(E)$ *with*

$$f_{\text{low}}^{(N)}(E) := B_{N-1,\frac{p}{N}}\left(\tilde{E}\right) - p \cdot B_{N-2,\frac{p}{N}}\left(\tilde{E}-1\right)^2.$$

We can now observe what happens when taking the limit $N \to \infty$. Assuming E to be a continuity point of σ, we have $\lim_N \sigma^{(N)}(E) = \sigma(N)$. Corollary A.6 in the appendix shows that

$$\lim_N B_{N-1,\frac{p}{N}}\left(\tilde{E}\right) = \pi_p\left(\tilde{E}\right)$$

and

$$\begin{aligned}\lim_N B_{N-2,\frac{p}{N}}\left(\tilde{E}-1\right) = \pi_p\left(\tilde{E}-1\right) &= \frac{p^{\tilde{E}-1}}{\left(\tilde{E}-1\right)!} \cdot e^{-p} \\ &= \frac{\tilde{E}}{p} \cdot \frac{p^{\tilde{E}}}{\tilde{E}!} \cdot e^{-p} = \frac{\tilde{E}}{p} \cdot \pi_p\left(\tilde{E}\right).\end{aligned}$$

We hence finally deduce

$$\begin{aligned}1 - \sigma(E) &= \lim_N 1 - \sigma^{(N)}(E) \\ &\geq \lim_N B_{N-1,\frac{p}{N}}\left(\tilde{E}\right) - p \cdot B_{N-2,\frac{p}{N}}\left(\tilde{E}-1\right)^2 \\ &= \pi_p\left(\tilde{E}\right) - p \cdot \left(\frac{\tilde{E}}{p} \cdot \pi_p\left(\tilde{E}\right)\right)^2 \\ &= \pi_p\left(\tilde{E}\right) - \frac{\tilde{E}^2}{p} \cdot \pi_p\left(\tilde{E}\right)^2 \\ &= \pi_p\left(\tilde{E}\right) \cdot \left(1 - \frac{\tilde{E}^2}{p} \cdot \pi_p\left(\tilde{E}\right)\right).\end{aligned}$$

Here, as usual, the existence of the respective limits follows from a backwards reading.

We can now state

Theorem 2. *For* $E > 0$ *a continuitiy point of* σ*, we have*

$$1 - \sigma(E) \geq f_{\text{low}}(E)$$

with

$$f_{\text{low}}(E) := \pi_p\left(\tilde{E}\right) \cdot \left(1 - \frac{\tilde{E}^2}{p} \cdot \pi_p\left(\tilde{E}\right)\right),$$

where $\tilde{E} := \lfloor E \rfloor + 1$, *for* $E \geq 1$ *and* $f_{\text{low}}(E) := 1 - \sigma(E)$ *for* $0 < E < 1$. *Also,*

$$\lim_{E\to\infty} \frac{-\ln\left(f_{\text{low}}(E)\right)}{E \cdot \ln(E)} = 1.$$

Proof. It remains to prove the last statement. First note that, for $E > 2$, i.e. $\tilde{E} \geq 3$,

$$\frac{\tilde{E}^2}{p} \cdot \pi_p\left(\tilde{E}\right) = \frac{\tilde{E}^2}{p} \cdot \frac{p^{\tilde{E}}}{\tilde{E}!} \cdot e^{-p} \leq \frac{5p^2}{\tilde{E}} \cdot \frac{p^{\left(\tilde{E}-3\right)}}{\left(\tilde{E}-3\right)!} \cdot e^{-p} = \frac{5p^2}{\tilde{E}} \cdot \underbrace{\pi_p\left(\tilde{E}-3\right)}_{\leq 1}$$

since π_p is a discrete probability density function, and hence

$$\lim_{E\to\infty} \frac{\tilde{E}^2}{p} \cdot \pi_p\left(\tilde{E}\right) = 0,$$

i. e.

$$\lim_{E\to\infty} \ln\left(1 - \frac{\tilde{E}^2}{p} \cdot \pi_p\left(\tilde{E}\right)\right) = 0. \tag{3.1}$$

Next,

$$\begin{aligned}\lim_{E\to\infty} \frac{\ln\left(\pi_p\left(\tilde{E}\right)\right)}{E \cdot \ln(E)} &= \lim_{E\to\infty} \frac{\ln\left(\frac{p^{\tilde{E}}}{\tilde{E}!} \cdot e^{-p}\right)}{E \cdot \ln(E)} \\ &= \lim_{E\to\infty} \frac{-p + \tilde{E} \cdot \ln(p) - \ln\left(\tilde{E}!\right)}{E \cdot \ln(E)} = -1\end{aligned} \tag{3.2}$$

by corollary A.10.

Now,

$$\lim_{E\to\infty} \frac{-\ln\left(f_{\text{low}}(E)\right)}{E \cdot \ln(E)} = \lim_{E\to\infty} \frac{-\ln\left(\pi_p\left(\tilde{E}\right) \cdot \left(1 - \frac{\tilde{E}^2}{p} \cdot \pi_p\left(\tilde{E}\right)\right)\right)}{E \cdot \ln(E)}$$

By (3.1) and (3.2):

$$\begin{aligned}(\text{cont.}) &= \lim_{E\to\infty} -\frac{\ln\left(\pi_p\left(\tilde{E}\right)\right)}{E \cdot \ln(E)} - \frac{\ln\left(1 - \frac{\tilde{E}^2}{p} \cdot \pi_p\left(\tilde{E}\right)\right)}{E \cdot \ln(E)} \\ &= -(-1) - 0 = 1.\end{aligned}$$

□

4 The upper bound

4.1 The general idea

For this section, let $E \geq 5$ be a continuity point of σ. Our way of finding a bound to the number of eigenvalues of the Laplacian $\Delta^{(N)}$ of $\mathcal{G}^{(N)}$ larger than E will be the following: We will modify $\mathcal{G}^{(N)}$ in such a way that the resulting graph will have a Laplacian with no eigenvalue exceeding E. The amount of modification, measured as the number of edges inserted or removed, will then serve as an upper bound to the number of eigenvalues of the original Laplacian $\Delta^{(N)}$ larger than E. Specifically, we have the following

Lemma 4.1. *Let G and H be subgraphs of $K^{(n)}$ for some $n \in \mathbb{N}$, denoting their sets of edges by E and F, respectively. For sets X, Y, let $X\Delta Y := X\backslash Y \cup Y\backslash X$ denote the symmetric difference of X and Y. Letting γ_G and γ_H denote the eigenvalue counting functions of the Laplacians Δ_G and Δ_H of G and H, respectively, we have $|\gamma_G - \gamma_H|_\infty \leq |E\Delta F|$, that is,*

$$|\gamma_G(E) - \gamma_H(E)| \leq |E\Delta F|$$

for all $E \in \mathbb{R}$.

Proof. For $u, v \in \{1, \ldots, n\}$ with $u \neq v$, let $B^{(u,v)} \in \mathbb{C}^{n\times n}$ denote the matrix given by

$$\begin{aligned} B^{(u,v)}_{uu} &= B^{(u,v)}_{vv} = 1, \\ B^{(u,v)}_{uv} &= B^{(u,v)}_{vu} = -1, \end{aligned}$$

and $B^{(u,v)}_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$ distinct from u and v. Note that, alternatively, $B^{(u,v)}$ may be characterized by

$$\begin{aligned} \left\langle \psi, B^{(u,v)}\phi \right\rangle &= 1 \cdot \overline{\psi_u}\phi_u + 1 \cdot \overline{\psi_v}\psi_v + (-1) \cdot \overline{\psi_u}\phi_v + (-1) \cdot \overline{\psi_v}\phi_u \\ &= \overline{(\psi_u - \psi_v)}(\phi_u - \phi_v) \\ &= \langle \psi, e_u - e_v\rangle \cdot \langle e_u - e_v, \phi\rangle \end{aligned}$$

for all $\phi, \psi \in \mathbb{C}^n$, and is hence of rank 1.

Note that, by our earlier characterization of the Laplacian, we have

$$\begin{aligned} \langle \psi, \Delta_G\phi\rangle &= \sum_{[u,v]\in E} \overline{(\psi_u - \psi_v)}(\phi_u - \phi_v) \\ &= \sum_{[u,v]\in E} \left\langle \psi, B^{(u,v)}\phi \right\rangle \end{aligned}$$

for $\phi, \psi \in \mathbb{C}^n$, and hence

$$\Delta_G = \sum_{[u,v]\in E} B^{(u,v)}.$$

Similarly,

$$\Delta_H = \sum_{[u,v]\in F} B^{(u,v)}.$$

Further,

$$\begin{aligned}\Delta_G - \Delta_H &= \left(\sum_{[u,v]\in E} B^{(u,v)}\right) - \left(\sum_{[u,v]\in F} B^{(u,v)}\right) \\ &= \left(\sum_{[u,v]\in E\setminus F} B^{(u,v)}\right) - \left(\sum_{[u,v]\in F\setminus E} B^{(u,v)}\right).\end{aligned}$$

It follows that

$$\begin{aligned}\text{rnk}\left(\Delta_G - \Delta_H\right) &= \text{rnk}\left(\left(\sum_{[u,v]\in E\setminus F} B^{(u,v)}\right) - \left(\sum_{[u,v]\in F\setminus E} B^{(u,v)}\right)\right) \\ &\leq \text{rnk}\left(\sum_{[u,v]\in E\setminus F} B^{(u,v)}\right) + \text{rnk}\left(\sum_{[u,v]\in F\setminus E} B^{(u,v)}\right) \\ &\leq \left(\sum_{[u,v]\in E\setminus F} \text{rnk}\left(B^{(u,v)}\right)\right) + \left(\sum_{[u,v]\in F\setminus E} \text{rnk}\left(B^{(u,v)}\right)\right) \\ &= \left(\sum_{[u,v]\in E\setminus F} 1\right) + \left(\sum_{[u,v]\in F\setminus E} 1\right) \\ &= |E\setminus F| + |F\setminus E| \\ &= |E\Delta F|,\end{aligned}$$

where $\text{rnk}(T)$ denotes the rank of the matrix T.

The matrices Δ_G and Δ_H hence differ by a matrix of rank at most $|E\Delta F|$. We can now apply lemma $A.2$ of the appendix and conclude $|\gamma_G - \gamma_H| \leq |E\Delta F|$, i.e.

$$|\gamma_G(E) - \gamma_H(E)| \leq |E\Delta F|$$

for all $E \in \mathbb{R}$. □

But in what way must we modify $\mathcal{G}^{(N)}$ such that all of its eigenvalues do not exceed a predetermined bound? In other words, what is an effective bound to the eigenvalues of the Laplacian of a graph, and how many single-edge surgeries do we have to perform to arrive at a graph fulfilling the corresponding bound? There are many inequalities for estimating the largest eigenvalue of the Laplacian, differing in their approximation qualities. It turns out that a result of Merris [26] is appropriate for our purposes, which is stated in the appendix as lemma A.4. Phrased in a way more directly related to our sketch of proof, we can reformulate it as follows:

Corollary 4.2. *Let $x > 0$ be given. Let $n \in \mathbb{N}$ and G be a graph with vertex set $\{1, \ldots, n\}$ and edge set E. If, for all vertices v of G, we have*

$$\deg(v) + \mathrm{mdeg}(v) \leq x,$$

where

$$\mathrm{mdeg}(v) = \begin{cases} \frac{1}{\deg(v)} \cdot \sum_{\substack{1 \leq u \leq n, \\ [u,v] \in E}} \deg(u) & \text{if } \deg(v) > 0, \\ 0 & \text{else} \end{cases}$$

for $1 \leq v \leq n$, then the Laplacian Δ_G has no eigenvalues larger than x.

We could naively try to iteratively remove edges from vertices v violating the inequality $\deg(v) + \mathrm{mdeg}(v) \leq E$, knowing that at the end we must arrive at a suitably modified graph. This is because there are only finitely many edges to be removed and the graph with no edges, i.e. only isolated vertices, certainly fulfills the eigenvalue bound.

But the problem lies in establishing a fitting estimate on the number of single-edge surgeries having to be performed. Surely, being given a vertex v such that the above inequality is violated, we could remove $\lceil \deg(v) + \mathrm{mdeg}(v) - E \rceil$ edges incident to v, where $\lceil x \rceil$ denotes the smallest integer at least as large as x for $x \in \mathbb{R}$. Iteratively choosing the edge with the vertex on the other side having maximal degree of all neighboring vertices would ensure that the average degree of neighboring vertices of v would not increase. At the end, the inequality $\deg(v) + \mathrm{mdeg}(v) \leq E$ would be satisfied for the new graph and we would have a good bound on the number of single-edge surgeries performed for the vertex v.

But what about those vertices u which were adjacent to v prior to the surgery, but have as neighbor v no longer? Since the set of neighbors of u has been altered without paying attention to u, the value of $\mathrm{mdeg}(u)$, and hence of $\lceil \deg(u) + \mathrm{mdeg}(u) - E \rceil$, the sign of which is directly controlling the validity of $\deg(u) + \mathrm{mdeg}(u) \leq E$, could have changed arbitrarily.

So we must find a way to perform our single-edge surgeries when working on decreasing the value of $\deg(v) + \mathrm{mdeg}(v)$ for a given vertex v while at the same time making sure that the value of $\deg(u) + \mathrm{mdeg}(u)$ for other vertices u does not increase either. A simple approach goes as follows:

As a first step, we add a large enough supply of new isolated vertices to the graph. This does only increase the multiplicity of the eigenvalue zero of the Laplacian. We follow the outlined approach in that for a chosen vertex v with $\deg(v) + \mathrm{mdeg}(v)$ larger than E, we iteratively remove the edge connecting v to another vertex u of maximal degree of all neighbors. But following each such step, we add a new edge connecting u to one of our previously added isolated vertices. Though this increases $\deg(u)$ by one, it has previously been decreased by one. Considering $\mathrm{mdeg}(u)$, the set of neighbors of u has changed in that v has been removed and a new vertex of (now) degree one has been added. Surely, this can only decrease the value of $\mathrm{mdeg}(u)$. In effect, using $2\lceil \deg(v) + \mathrm{mdeg}(v) - E\rceil$ single-edge surgeries, we modified the graph such that $\deg(v) + \mathrm{mdeg}(v) \leq E$ is now valid, while ensuring that the same operation for each other vertex will not require more single-edge surgeries than if we had chosen this vertex to begin with. Iterating over all vertices from 1 to N, we arrive at a graph fulfilling the eigenvalue bound with a total of

$$2 \cdot \sum_{v=1}^{N} \Xi\left(\lceil \deg(v) + \mathrm{mdeg}(v) - E\rceil\right)$$

single-edge surgeries performed, where

$$\Xi(x) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{else} \end{cases}$$

for $x \in \mathbb{R}$.

4.2 The exact construction

Formally, for each vertex $1 \leq v \leq N$, we define a set F_v of edges of $\mathcal{G}^{(N)}$ incident to v to be removed as follows: If $\deg(v) + \mathrm{mdeg}(v) \leq E$, set

$$F_v := \emptyset.$$

Otherwise, the number $k := \lceil \deg(v) + \mathrm{mdeg}(v) - E\rceil$ is positive. Let $d := \deg(v)$ and consider the neighboring vertices $w_1, \ldots, w_d$ of v, ordered such that $\deg(w_1) \geq \ldots \geq \deg(w_d)$. Set

$$F_v := \left\{[v, w_1], \ldots, \left[v, w_{\min(k,d)}\right]\right\}.$$

Note that, alternatively, we may have defined

$$F_v := \{[v, w_1], \ldots, [v, w_m]\} \tag{4.1}$$

with

$$m := \min\left(\Xi\left(\lceil \deg(v) + \mathrm{mdeg}(v) - E \rceil\right), \deg(v)\right) = |F_v| \tag{4.2}$$

and no extra case distinction.

Now consider the union $F := \bigcup_{i=1}^{N} F_v$ of all edges to be removed. For each vertex $1 \leq v \leq N$, we have to compensate for the edges

$$G_v := \{[v,w] \in F \mid 1 \leq w \leq N, [v,w] \notin F_v\} \tag{4.3}$$

of the removal set F incident to v not coming from F_v. Introducing our supply of isolated vertices, we construct a modified graph $\tilde{\mathcal{G}}^{(N)}$ with vertices

$$1, \ldots, N, N+1, \ldots, \underbrace{N + \sum_{v=1}^{N} |G_v|}_{=:M},$$

but the same set $\mathcal{E}^{(N)}$ of edges. In effect, we have added $\sum_{v=1}^{N} |G_v|$ clusters of size one. By what was said in the introduction of graph operators, the Laplacian $\tilde{\Delta}^{(N)}$ of $\tilde{\mathcal{G}}^{(N)}$ decomposes into the sum of the original Laplacian $\Delta^{(N)}$ and $\sum_{v=1}^{N} |G_v|$ times the Laplacian of a graph with one vertex and no edges, which is just the zero operator. The non-zero eigenvalues of $\tilde{\Delta}^{(N)}$ are hence the same as those of $\Delta^{(N)}$.

Let $z_1 := N$ and $z_{v+1} := z_v + |G_v|$ for $1 \leq v \leq N$ (note that $z_{N+1} = M$). To compensate for the forthcoming loss of the edges G_v, where $1 \leq v \leq N$, which might increase $\mathrm{mdeg}(v)$ in unintended ways, we will connect v to correspondingly many isolated vertices of our newly added supply. Let

$$H_v := \{[v, z_v + j] \mid 1 \leq j \leq |G_v|\} \tag{4.4}$$

be the set of edges which will connect v to previously isolated vertices. When adding the union $H := \bigcup_{v=1}^{N} H_v$ to the edge set of $\tilde{\mathcal{G}}^{(N)}$, this will ensure each of the supply vertices will gain only one neighbor.

Finally, let $\mathcal{H}^{(N)}$ be the graph with vertices $1, \ldots, M$ and edge set

$$\mathcal{F}^{(N)} := \left(\mathcal{E}^{(N)} \backslash F\right) \cup H$$

(note that $F \subseteq \mathcal{E}^{(N)}$ and $\mathcal{E}^{(N)}, H$ are disjoint). Let us check that $\mathcal{H}^{(N)}$ indeed fulfills the eigenvalue bound for E, i.e. that $\deg_{\mathcal{H}^{(N)}}(v) + \mathrm{mdeg}_{\mathcal{H}^{(N)}}(v) \leq E$ for all $1 \leq v \leq M$. Here and in the following series of lemmata, we will use subscripts to indicate relative to which graphs our notions are to be understood.

Lemma 4.3. *For $N+1 \leq v \leq M$, we have $\deg_{\mathcal{H}^{(N)}}(v) = 1$ and $[u,v] \in \mathcal{F}^{(N)}$ for some unique $u \in \{1, \ldots, N\}$.*

Proof. Just note that, since $\mathcal{F}^{(N)} = \left(\mathcal{E}^{(N)} \setminus F\right) \cup H$ and $\mathcal{E}^{(N)}$ does not contain edges incident to vertices from $\{N+1, \ldots, M\}$,

$$\begin{aligned}\left\{[u,v] \in \mathcal{F}^{(N)} \mid 1 \leq u \leq M\right\} &= \{[u,v] \in H \mid 1 \leq u \leq M\} \\ &= \bigcup_{u=1}^{N} \{[u,v] \in H_u \mid z_v < v \leq z_{v+1}\}.\end{aligned}$$

Since $N = z_1 \leq \ldots \leq z_{N+1} = M$, there is a unique $1 \leq u \leq N$ such that $z_u < v \leq z_{u+1}$. This means that the above union contains exactly element, namely $[u,v]$, and it follows that $\deg_{\mathcal{H}^{(N)}}(v) = 1$. □

Lemma 4.4. *For $1 \leq v \leq N$, we have*

$$\deg_{\mathcal{H}^{(N)}}(v) \leq \deg_{\mathcal{G}^{(N)}}(v).$$

Specifically, $\deg_{\mathcal{H}^{(N)}}(v) = \deg_{\mathcal{G}^{(N)}}(v) - |F_v|$.

Proof. Since

$$\begin{aligned}\{[v,w] \in F \mid 1 \leq w \leq M\} &= \{[v,w] \in F \mid 1 \leq w \leq N, [v,w] \in F_v\} \\ &\quad \cup \{[v,w] \in F \mid 1 \leq w \leq N, [v,w] \notin F_v\} \\ &= F_v \cup G_v,\end{aligned}$$

where all unions are disjoint, we have removed $|F_v| + |G_v|$ edges incident to v. The edges incident to v we added are given by

$$\begin{aligned}\{[v,w] \in H \mid 1 \leq w \leq M\} &= H_v \\ &= \{[v, z_v + j] \mid 1 \leq j \leq |G_v|\}\end{aligned}$$

and have a count of $|G_v|$. In total, the degree of v has changed by

$$-\left(|F_v| + |G_v|\right) + |G_v| = -\left|F_v\right|,$$

i.e. $\deg_{\mathcal{H}^{(N)}}(v) = \deg_{\mathcal{G}^{(N)}}(v) - |F_v|$. □

Lemma 4.5. *For $1 \leq v \leq N$, we have*

$$\operatorname{mdeg}_{\mathcal{H}^{(N)}}(v) \leq \operatorname{mdeg}_{\mathcal{G}^{(N)}}(v).$$

Proof. First note that for $\deg_{\mathcal{G}^{(N)}}(v) = 0$ we must also have $\deg_{\mathcal{H}^{(N)}}(v) = 0$ by the previous lemma, and hence $\operatorname{mdeg}_{\mathcal{H}^{(N)}}(v) = \operatorname{mdeg}_{\mathcal{G}^{(N)}}(v) = 0$.

Let $d := \deg_{\mathcal{G}^{(N)}}(v) > 0$ and let $w_1, \ldots, w_d$ be the neighbors of v in $\mathcal{G}^{(N)}$, ordered such that $\deg_{\mathcal{G}^{(N)}}(w_1) \geq \ldots \geq \deg_{\mathcal{G}^{(N)}}(w_d)$. Recall that

$$F_v = \{[v, w_1], \ldots, [v, w_m]\}$$

with

$$m = \min\left(\Xi\left(\lceil \deg_{\mathcal{G}^{(N)}}(v) + \mathrm{mdeg}_{\mathcal{G}^{(N)}}(v) - E\rceil\right), d\right).$$

Let $k := |G_v|$ and $1 \leq u_1, \ldots, u_k \leq N$ such that $G_v = \{[v, u_1], \ldots, [v, u_k]\}$. Since $\mathcal{F}^{(N)} = \left(\mathcal{E}^{(N)} \backslash F\right) \cup H$ and by the proof of the previous lemma, the neighbors of v in $\mathcal{H}^{(N)}$ are now given by

$$N_{\mathcal{H}^{(N)}}(v) = \left(\{w_{m+1}, \ldots, w_d\} \setminus \{u_1, \ldots, u_k\}\right) \cup \{z_v + 1, \ldots, z_v + k\}, \quad (4.5)$$

where $\{u_1, \ldots, u_k\} \subseteq \{w_{m+1}, \ldots, w_d\}$ since F_v and G_v are disjoint.

Note that if $m = d$, we must have $k = 0$. Then, again, $N_{\mathcal{H}^{(N)}}(v) = \emptyset$, i.e. $\deg_{\mathcal{H}^{(N)}}(v) = 0$ and $\mathrm{mdeg}_{\mathcal{H}^{(N)}}(v) = 0 \leq \mathrm{mdeg}_{\mathcal{G}^{(N)}}(v)$.

Now assume $m < d$. First note that by the degree ordering of $w_1, \ldots, w_d$, we have $\deg_{\mathcal{G}^{(N)}}(w_i) > \deg_{\mathcal{G}^{(N)}}(w_j)$ for $1 \leq i \leq m$ and $m + 1 \leq j \leq d$. Summing over all such i and j, we get

$$\begin{aligned} & (d - m)\left(\deg_{\mathcal{G}^{(N)}}(w_1) + \ldots + \deg_{\mathcal{G}^{(N)}}(w_m)\right) \\ \geq \quad & m\left(\deg_{\mathcal{G}^{(N)}}(w_{m+1}) + \ldots + \deg_{\mathcal{G}^{(N)}}(w_d)\right). \end{aligned}$$

Adding $(d - m)\left(\deg_{\mathcal{G}^{(N)}}(w_{m+1}) + \ldots + \deg_{\mathcal{G}^{(N)}}(w_d)\right)$ to both sides, we derive

$$\begin{aligned} & (d - m)\left(\deg_{\mathcal{G}^{(N)}}(w_1) + \ldots + \deg_{\mathcal{G}^{(N)}}(w_d)\right) \\ \geq \quad & d\left(\deg_{\mathcal{G}^{(N)}}(w_{m+1}) + \ldots + \deg_{\mathcal{G}^{(N)}}(w_d)\right). \end{aligned}$$

Dividing by $(d - m)d > 0$ now gives

$$\begin{aligned} \mathrm{mdeg}_{\mathcal{G}^{(N)}}(v) &= \frac{\deg_{\mathcal{G}^{(N)}}(w_1) + \ldots + \deg_{\mathcal{G}^{(N)}}(w_d)}{d} \\ &\geq \frac{\deg_{\mathcal{G}^{(N)}}(w_{m+1}) + \ldots + \deg_{\mathcal{G}^{(N)}}(w_d)}{d - m} \end{aligned}$$

Since $\deg_{\mathcal{H}^{(N)}}(z_v + i) = 1 \leq \deg_{\mathcal{G}^{(N)}}(u_i)$ for $i \in \{1, \ldots, k\}$ by lemma 4.3 and noting that $v \in N_{\mathcal{G}^{(N)}}(u_i)$:

$$\begin{aligned} \text{(cont.)} &\geq \frac{1}{d - m} \cdot \left(\left(\sum_{i=m+1}^{d} \deg_{\mathcal{G}^{(N)}}(w_i)\right)\right. \\ &\qquad \left. + \left(\sum_{i=1}^{k} \left(\deg_{\mathcal{H}^{(N)}}(z_v + i) - \deg_{\mathcal{G}^{(N)}}(u_i)\right)\right)\right) \\ &= \frac{1}{d - m} \cdot \left(\left(\sum_{r \in \{w_{m+1}, \ldots, w_d\} \setminus \{u_1, \ldots, u_k\}} \deg_{\mathcal{G}^{(N)}}(r)\right)\right. \\ &\qquad \left. + \left(\sum_{i=1}^{k} \deg_{\mathcal{H}^{(N)}}(z_v + i)\right)\right) \end{aligned}$$

And by the previous lemma 4.4:

$$\text{(cont.)} \geq \frac{1}{d-m} \cdot \left(\left(\sum_{r \in \{w_{m+1},\ldots,w_d\} \setminus \{u_1,\ldots,u_k\}} \deg_{\mathcal{H}^{(N)}}(r)\right)\right.$$
$$\left. + \left(\sum_{i=1}^{k} \deg_{\mathcal{H}^{(N)}}(z_v + i)\right)\right)$$

By (4.5):

$$\text{(cont.)} = \frac{1}{d-m} \cdot \sum_{r \in N_{\mathcal{H}^{(N)}}(v)} \deg_{\mathcal{H}^{(N)}}(r)$$

Noting that $d - m = \deg_{\mathcal{G}^{(N)}}(v) - |F_v| = \deg_{\mathcal{H}^{(N)}}(v)$, for example by lemma 4.4:

$$\text{(cont.)} = \operatorname{mdeg}_{\mathcal{H}^{(N)}}(v).$$

□

Lemma 4.6. *For $1 \leq v \leq N$, we have*

$$\deg_{\mathcal{H}^{(N)}}(v) + \operatorname{mdeg}_{\mathcal{H}^{(N)}}(v) \leq E.$$

Proof. Recall that

$$|F_v| = \min\left(\Xi\left(\lceil \deg_{\mathcal{G}^{(N)}}(v) + \operatorname{mdeg}_{\mathcal{G}^{(N)}}(v) - E \rceil\right), \deg_{\mathcal{G}^{(N)}}(v)\right)$$

and, by lemma 4.4,

$$\deg_{\mathcal{H}^{(N)}}(v) = \deg_{\mathcal{G}^{(N)}}(v) - |F_v|.$$

If $|F_v| = \deg_{\mathcal{G}^{(N)}}(v)$, then $\deg_{\mathcal{H}^{(N)}}(v) = 0$ and hence

$$\deg_{\mathcal{H}^{(N)}}(v) + \operatorname{mdeg}_{\mathcal{H}^{(N)}}(v) = 0 \leq E.$$

Otherwise,

$$\begin{aligned} |F_v| &= \Xi\left(\lceil \deg_{\mathcal{G}^{(N)}}(v) + \operatorname{mdeg}_{\mathcal{G}^{(N)}}(v) - E \rceil\right) \\ &\geq \lceil \deg_{\mathcal{G}^{(N)}}(v) + \operatorname{mdeg}_{\mathcal{G}^{(N)}}(v) - E \rceil \\ &\geq \deg_{\mathcal{G}^{(N)}}(v) + \operatorname{mdeg}_{\mathcal{G}^{(N)}}(v) - E, \end{aligned}$$

so

$$\begin{aligned}\deg_{\mathcal{H}^{(N)}}(v) &= \deg_{\mathcal{G}^{(N)}}(v) - |F_v| \\ &\leq \deg_{\mathcal{G}^{(N)}}(v) - (\deg_{\mathcal{G}^{(N)}}(v) + \operatorname{mdeg}_{\mathcal{G}^{(N)}}(v) - E) \\ &= E - \operatorname{mdeg}_{\mathcal{G}^{(N)}}(v) \\ &\leq E - \operatorname{mdeg}_{\mathcal{H}^{(N)}}(v),\end{aligned}$$

where the last inequality follows from lemma 4.5. We conclude

$$\deg_{\mathcal{H}^{(N)}}(v) + \operatorname{mdeg}_{\mathcal{H}^{(N)}}(v) \leq (E - \operatorname{mdeg}_{\mathcal{H}^{(N)}}(v)) + \operatorname{mdeg}_{\mathcal{H}^{(N)}}(v) = E.$$

□

Lemma 4.7. *For $N+1 \leq v \leq M$, we have*

$$\deg_{\mathcal{H}^{(N)}}(v) + \operatorname{mdeg}_{\mathcal{H}^{(N)}}(v) \leq E.$$

Proof. By lemma 4.3, $\deg_{\mathcal{H}^{(N)}}(v) = 1$ and there is a unique $u \in \{1, \dots, N\}$ such that $[u, v] \in \mathcal{F}^{(N)}$. This means

$$\operatorname{mdeg}_{\mathcal{H}^{(N)}}(v) = \frac{\deg_{\mathcal{H}^{(N)}}(u)}{1} = \deg_{\mathcal{H}^{(N)}}(u).$$

Since $v \in N_{\mathcal{H}^{(N)}}(u)$, we have $\deg_{\mathcal{H}^{(N)}}(u) \geq 1$ and

$$\begin{aligned}\operatorname{mdeg}_{\mathcal{H}^{(N)}}(u) &= \frac{\sum_{w \in N(u)} \deg_{\mathcal{H}^{(N)}}(w)}{\deg_{\mathcal{H}^{(N)}}(u)} \\ &\geq \frac{\sum_{w \in N(u)} 1}{\deg_{\mathcal{H}^{(N)}}(u)} = 1\end{aligned}$$

because $u \in N_{\mathcal{H}^{(N)}}(w)$ and thus $\deg_{\mathcal{H}^{(N)}}(w) \geq 1$ for all $w \in N_{\mathcal{H}^{(N)}}(u)$. It follows that

$$\begin{aligned}\deg_{\mathcal{H}^{(N)}}(v) + \operatorname{mdeg}_{\mathcal{H}^{(N)}}(v) &= 1 + \deg_{\mathcal{H}^{(N)}}(u) \\ &\leq \operatorname{mdeg}_{\mathcal{H}^{(N)}}(u) + \deg_{\mathcal{H}^{(N)}}(u) \leq E\end{aligned}$$

by lemma 4.6. □

Lemmata 4.7 and 4.6 prove that $\mathcal{H}^{(N)}$ indeed fulfills the conditions of corollary 4.2, so we can conclude:

Lemma 4.8. *The Laplacian $\Delta_H^{(N)}$ of $\mathcal{H}^{(H)}$ has no eigenvalues larger than E, i.e.*

$$\gamma_{\mathcal{H}^{(H)}}(E) = M,$$

where $\gamma_{\mathcal{H}^{(H)}}$ denotes the eigenvalue counting function of $\Delta_H^{(N)}$.

We can now turn to the application of lemma 4.1 to the graphs $\tilde{\mathcal{G}}^{(N)}$ and $\mathcal{H}^{(N)}$. The symmetric difference of the edge sets of $\tilde{\mathcal{G}}^{(N)}$ and $\mathcal{H}^{(N)}$, i.e. of the sets $\mathcal{E}^{(N)}$ and $\left(\mathcal{E}^{(N)}\backslash F\right) \cup H$, is just $F \cup H$ (recall that $F \subseteq \mathcal{E}^{(N)}$ and $H \cap \mathcal{E}^{(N)} = \emptyset$). Its size can readily be estimated:

First note that for distinct vertices $u, v \in \{1, \ldots, N\}$, the sets G_u and G_v are disjoint: Since all edges in G_u are incident to u and all edges in G_v are incident to v, we would otherwise have $[u, v] \in G_u, G_v$. By definition (4.3) of G_u and G_v, this would mean $[u, v] \notin F_u, F_v$ as well as $[u, v] \in F$. Since $F = \bigcup_{w=1}^N F_w$, this would mean $[u, v] \in F_w$ for a vertex $1 \leq w \leq N$ distinct from u and v. Since all edges in F_w are incident to w, this is a contradition.

Now, since

$$
\begin{aligned}
|G| &= \left|\bigcup_{v=1}^N G_v\right| = \sum_{v=1}^N |G_v|\,, \\
|H| &= \left|\bigcup_{v=1}^N H_v\right| = \sum_{v=1}^N |H_v|
\end{aligned}
$$

and $|G_v| = |H_v|$ for all $1 \leq v \leq N$ by (4.4), we have $|H| = |G|$. Since G is a subset of F, this means $|H| \leq |F|$. We conclude

$$
\left|\mathcal{E}^{(N)} \Delta \mathcal{F}^{(N)}\right| = |F \cup H| \leq 2\,|F|\,.
$$

Furthermore, we have

$$
\begin{aligned}
|F| &= \left|\bigcup_{v=1}^N F_v\right| \leq \sum_{v=1}^N |F_v| \\
&\stackrel{(4.2)}{=} \sum_{v=1}^N \min\left(\Xi\left(\lceil \deg(v) + \mathrm{mdeg}(v) - E \rceil\right), \deg(v)\right) \\
&\leq \sum_{v=1}^N \Xi\left(\lceil \deg(v) + \mathrm{mdeg}(v) - E \rceil\right).
\end{aligned}
$$

Recalling the notions of the eigenvalue counting functions $\gamma_{\mathcal{G}^{(N)}}, \gamma_{\tilde{\mathcal{G}}^{(N)}}$ and $\gamma_{\mathcal{H}^{(N)}}$ of the Laplacians of $\mathcal{G}^{(N)}, \tilde{\mathcal{G}}^{(N)}$ and $\mathcal{H}^{(N)}$, respectively, we already argued that the eigenvalues of the Laplacians $\Delta^{(N)}$ and $\tilde{\Delta}^{(N)}$ of $\mathcal{G}^{(N)}$ and $\tilde{\mathcal{G}}^{(N)}$, respectively, differ only in their multiplicity of the eigenvalue zero. This means that

$$
N - \gamma_{\mathcal{G}^{(N)}}(E) = M - \gamma_{\tilde{\mathcal{G}}^{(N)}}(E)
$$

since $E > 0$. The previously mentioned lemma 4.1 tells us that

$$
\left|\gamma_{\mathcal{H}^{(N)}}(E) - \gamma_{\tilde{\mathcal{G}}^{(N)}}(E)\right| \leq \left|\mathcal{E}^{(N)} \Delta \mathcal{F}^{(N)}\right| \leq 2\,|F|\,.
$$

But $\gamma_{\mathcal{H}^{(N)}}(E) = M$, by lemma 4.8, and $\gamma_{\tilde{\mathcal{G}}^{(N)}}(E) \leq M$, so

$$M - \gamma_{\tilde{\mathcal{G}}^{(N)}}(E) \leq 2\,|F|\,.$$

We conclude

$$N - \gamma_{\mathcal{G}^{(N)}}(E) \leq 2\,|F|\,.$$

Considering now the expected normalized eigenvalue counting function $\sigma^{(N)}$ of the Laplacian $\Delta^{(N)}$ of $\mathcal{G}^{(N)}$, we have

$$\begin{aligned} 1 - \sigma^{(N)}(E) &= \frac{1}{N} \cdot \mathbb{E}^{(N)}\left(N - \gamma_{\mathcal{G}^{(N)}}(E)\right) \\ &\leq \frac{1}{N} \cdot \mathbb{E}^{(N)}\left(2\,|F|\right) \\ &\leq \frac{1}{N} \cdot \mathbb{E}^{(N)}\left(2 \cdot \sum_{v=1}^{N} \Xi\left(\lceil \deg(v) + \mathrm{mdeg}(v) - E \rceil\right)\right) \end{aligned}$$

Invoking the symmetry corresponding to transposition of the vertices 1 and v:

$$(\text{cont.}) = 2 \cdot \mathbb{E}^{(N)}\left(\Xi\left(\lceil \deg(1) + \mathrm{mdeg}(1) - E \rceil\right)\right).$$

4.3 Establishing the bounding value

Conditioning over the degree of vertex 1, we continue

$$\begin{aligned} 1 - \sigma^{(N)}(E) &\leq 2 \cdot \mathbb{E}^{(N)}\left(\Xi\left(\lceil \deg(1) + \mathrm{mdeg}(1) - E \rceil\right)\right) \\ &= 2 \cdot \sum_{k=0}^{N-1} \mathbb{P}^{(N)}\left(\deg(1) = k\right) \\ &\quad \cdot \mathbb{E}^{(N)}\left(\Xi\left(\lceil k + \mathrm{mdeg}(1) - E \rceil\right) \mid \deg(1) = k\right) \\ &\leq 2 \cdot \sum_{k=1}^{N-1} B_{N-1,\frac{p}{N}}(k) \\ &\quad \cdot \mathbb{E}^{(N)}\left(\Xi\left(\mathrm{mdeg}(1) + (k - \lfloor E \rfloor + 1)\right) \mid \deg(1) = k\right), \end{aligned} \tag{4.6}$$

where we used that $\deg(1) = \sum_{v=2}^{N} g_{1,v}$ has binomial distribution with parameters $N-1$ and $\frac{p}{N}$.

Focussing on the expectation part, we further condition over the neighbors of vertex 1 and write

$$\begin{aligned} &\mathbb{E}^{(N)}\left(\Xi\left(\mathrm{mdeg}(1) + (k - \lfloor E \rfloor + 1)\right) \mid \deg(1) = k\right) \\ &= \mathbb{E}^{(N)}\left(\Xi\left(\mathrm{mdeg}(1) + (k - \lfloor E \rfloor + 1)\right) \mid N(1) = \{2, \ldots, k+1\}\right), \end{aligned} \tag{4.7}$$

where we used the symmetry induced by the permutation of vertices swapping $v_1, \dots, v_k$ with $2, \dots, k+1$, respectively, noting that each particular set of neighboring vertices of cardinality k is, by symmetry, equally likely, and that the inner term, in particular mdeg(1), of the expectation is invariant under this symmetry.

Let us now turn our attention to mdeg(1). Using our edge variable notation, we can express mdeg(1) under the condition that $N(1) = \{2, \dots, k+1\}$ as

$$\begin{aligned}\mathrm{mdeg}(1) =&\frac{1}{k} \cdot \sum_{u=2}^{k+1} \deg(u) = \frac{1}{k} \cdot \sum_{u=2}^{k+1} \sum_{v=1}^{N} g_{u,v} \\ =&\frac{1}{k} \cdot \sum_{u=2}^{k+1} \left(1 + \sum_{v=2}^{N} g_{u,v} \right) = 1 + \frac{1}{k} \cdot \sum_{u=2}^{k+1} \sum_{v=2}^{N} g_{u,v} \\ =&1 + \frac{1}{k} \cdot \left(2 \underbrace{\left(\sum_{2 \le u < v \le k+1} g_{u,v} \right)}_{=:X} + \underbrace{\left(\sum_{2 \le u \le k+1 < v \le N} g_{u,v} \right)}_{=:Y} \right).\end{aligned} \tag{4.8}$$

By our introductory remarks, $X, Y, N(1)$ are independent as random variables. As the sum of independent random variables of Bernoulli distribution with parameter $\frac{p}{N}$, we see that X has binomial distribution with parameters $\binom{k}{2}$ and $\frac{p}{N}$ while Y has binomial distribution with parameters $k(N-k-1)$ and $\frac{p}{N}$.

Using (4.7) and (4.8), we can now write the initial inequality (4.6) of this subsection as

$$\begin{aligned}&1 - \sigma^{(N)}(E) \\ &\le 2 \cdot \sum_{k=1}^{N-1} B_{N-1, \frac{p}{N}}(k) \cdot \mathbb{E}^{(N)} \left(\Xi \left(\frac{2X+Y}{k} + (k - \lfloor E \rfloor + 2) \right) \right).\end{aligned} \tag{4.9}$$

We would like to take limits $N \to \infty$ at this point, but to do so in a meaningful way, we need to be able to exchange summation and taking of the limit on the right-hand side. For this to be valid, we need to prove that the sum is uniform in N, i.e. give for each summand a bound independent of N, with the sum of the bounds still convergent.

Lemma 4.9. *The sum*

$$\sum_{k=1}^{N-1} B_{N-1, \frac{p}{N}}(k) \cdot \mathbb{E}^{(N)} \left(\Xi \left(\frac{2X+Y}{k} + (k - \lfloor E \rfloor + 2) \right) \right)$$

is uniform in N.

Proof. For $1 \leq k \leq N-1$, we have

$$\begin{aligned} B_{N-1,\frac{p}{N}}(k) &= \binom{N-1}{k} \cdot \left(\frac{p}{N}\right)^k \cdot \left(1 - \frac{p}{N}\right)^{N-1-k} \\ &\leq \frac{N^k}{k!} \cdot \frac{p^k}{N^k} = \frac{p^k}{k!} \end{aligned}$$

and, since $E \geq 2$,

$$\begin{aligned} &\mathbb{E}^{(N)}\left(\Xi\left(\frac{2X+Y}{k} + (k - \lfloor E \rfloor + 2)\right)\right) \\ &\leq \mathbb{E}^{(N)}\left(\Xi\left(2X+Y+k\right)\right) \\ &= \mathbb{E}^{(N)}\left(2X+Y+k\right) \\ &= 2 \cdot \mathbb{E}^{(N)}(X) + \mathbb{E}^{(N)}(Y) + k \end{aligned}$$

Knowing the expectation of a random variable of binomial distribution with parameters n and p is just np:

$$\begin{aligned} \text{(cont.)} &= 2 \cdot \frac{p}{N} \cdot \binom{k}{2} + \frac{p}{N} \cdot k(N-k-1) + k \\ &\leq pk^2 + pk + k \leq 2(1+p)k^2. \end{aligned}$$

We use this to establish an N-independent bound b_k for each summand, writing

$$\begin{aligned} &B_{N-1,\frac{p}{N}}(k) \cdot \mathbb{E}^{(N)}\left(\Xi\left(2X+Y+(k-\lfloor E \rfloor + 2)\right)\right) \\ &\leq \frac{p^k}{k!} \cdot \left(2(1+p)k^2\right) \leq \begin{cases} 2p(1+p) & \text{if } k = 1, \\ \frac{p^{k-2}}{(k-2)!} \cdot 4(1+p)p^2 & \text{else} \end{cases} =: b_k. \end{aligned}$$

The sum of the bounds does still converge:

$$\begin{aligned} \sum_{k=1}^{\infty} b_k &= b_1 + 4(1+p)p^2 \cdot \sum_{k=2}^{\infty} \frac{p^{k-2}}{(k-2)!} \\ &= b_1 + 4(1+p)p^2 \cdot \sum_{k=0}^{\infty} \frac{p^k}{k!} = 2p(1+p) + 4(1+p)p^2 \cdot e^p \end{aligned}$$

□

Now we can write, recalling that we assumed E to be continuity point of σ,

$$\begin{aligned}
1-\sigma(E) &= \lim_N 1-\sigma^{(N)}(E) \\
&\overset{(4.9)}{\leq} \lim_N 2\cdot\sum_{k=1}^{N-1} B_{N-1,\frac{p}{N}}(k) \\
&\qquad \cdot \mathbb{E}^{(N)}\left(\Xi\left(\frac{2X+Y}{k}+(k-\lfloor E\rfloor+2)\right)\right)
\end{aligned} \tag{4.10}$$

By the previous lemma:

$$\begin{aligned}
\text{(cont.)} &= 2\cdot\sum_{k=1}^{\infty}\lim_{\substack{N,\\ N>k}} B_{N-1,\frac{p}{N}}(k) \\
&\qquad \cdot \mathbb{E}^{(N)}\left(\Xi\left(\frac{2X+Y}{k}+(k-\lfloor E\rfloor+2)\right)\right) \\
&= 2\cdot\sum_{k=1}^{\infty}\left(\lim_{\substack{N,\\ N>k}} B_{N-1,\frac{p}{N}}(k)\right) \\
&\qquad \cdot\left(\lim_{\substack{N,\\ N>k}} \mathbb{E}^{(N)}\left(\Xi\left(\frac{2X+Y}{k}+(k-\lfloor E\rfloor+2)\right)\right)\right).
\end{aligned}$$

Of course, the validity of these claims still hinges on the fact that all of the mentioned limits exist. As usual, this will follow from a backwards reading. By corollary A.6 in the appendix, we have

$$\lim_{\substack{N,\\ N>k}} B_{N-1,\frac{p}{N}}(k)=\pi_p(k). \tag{4.11}$$

As for the other limit,

$$\begin{aligned}
&\mathbb{E}^{(N)}\left(\Xi\left(\frac{2X+Y}{k}+(k-\lfloor E\rfloor+2)\right)\right) \\
&= \sum_{x,y\geq 0} B_{\binom{k}{2},\frac{p}{N}}(x)\cdot B_{k(N-k-1),\frac{p}{N}}(y)\cdot\left(\Xi\left(\frac{2x+y}{k}+(k-\lfloor E\rfloor+2)\right)\right).
\end{aligned} \tag{4.12}$$

Lemma 4.10. *The sum*

$$\sum_{x,y\geq 0} B_{\binom{k}{2},\frac{p}{N}}(x)\cdot B_{k(N-k-1),\frac{p}{N}}(y)\cdot\left(\Xi\left(\frac{2x+y}{k}+(k-\lfloor E\rfloor+2)\right)\right)$$

is uniform in $N>k$.

Proof. For $x \geq 0$, we have

$$\begin{aligned} B_{\binom{k}{2},\frac{p}{N}}(x) &= \binom{\frac{k(k-1)}{2}}{x} \cdot \left(\frac{p}{N}\right)^x \cdot \left(1-\frac{p}{N}\right)^{\frac{k(k-1)}{2}-x} \\ &\leq \frac{1}{x!} \cdot \left(\frac{k(k-1)}{2}\right)^x \cdot \frac{p^x}{N^x} \leq \frac{1}{x!} \cdot \left(\frac{pk^2}{2}\right)^x. \end{aligned}$$

For $y \geq 0$, we have

$$\begin{aligned} B_{k(N-k-1),\frac{p}{N}}(y) &= \binom{k(N-k-1)}{y} \cdot \left(\frac{p}{N}\right)^y \cdot \left(1-\frac{p}{N}\right)^{k(N-k-1)-y} \\ &\leq \frac{(kN)^y}{y!} \cdot \frac{p^y}{N^y} \leq \frac{1}{y!} \cdot (pk)^y. \end{aligned}$$

In total, for $x, y \geq 0$, we have

$$\begin{aligned} & B_{\binom{k}{2},\frac{p}{N}}(x) \cdot B_{k(N-k-1),\frac{p}{N}}(y) \cdot \left(\Xi\left(\frac{2x+y}{k} + (k - \lfloor E \rfloor + 2)\right)\right) \\ \leq & \left(\frac{1}{x!} \cdot \left(\frac{pk^2}{2}\right)^x\right) \cdot \left(\frac{1}{y!} \cdot (pk)^y\right) \cdot (2x+y+k) =: b_{x,y}. \end{aligned}$$

The sum of the bounds does still converge:

$$\begin{aligned} \sum_{x,y\geq 0} b_{x,y} &= 2 \cdot \left(\sum_{x=0}^{\infty} \frac{1}{x!} \cdot \left(\frac{pk^2}{2}\right)^x \cdot x\right) \cdot \left(\sum_{y=0}^{\infty} \frac{1}{y!} \cdot (pk)^y\right) \\ &\quad + \left(\sum_{x=0}^{\infty} \frac{1}{x!} \cdot \left(\frac{pk^2}{2}\right)^x\right) \cdot \left(\sum_{y=0}^{\infty} \frac{1}{y!} \cdot (pk)^y \cdot y\right) \\ &\quad + k \cdot \left(\sum_{x=0}^{\infty} \frac{1}{x!} \cdot \left(\frac{pk^2}{2}\right)^x\right) \cdot \left(\sum_{y=0}^{\infty} \frac{1}{y!} \cdot (pk)^y\right) \\ &= pk^2 \cdot \left(\sum_{x=0}^{\infty} \frac{1}{x!} \cdot \left(\frac{pk^2}{2}\right)^x\right) \cdot \left(\sum_{y=0}^{\infty} \frac{1}{y!} \cdot (pk)^y\right) \\ &\quad + pk \cdot \left(\sum_{x=0}^{\infty} \frac{1}{x!} \cdot \left(\frac{pk^2}{2}\right)^x\right) \cdot \left(\sum_{y=0}^{\infty} \frac{1}{y!} \cdot (pk)^y\right) \\ &\quad + k \cdot \left(\sum_{x=0}^{\infty} \frac{1}{x!} \cdot \left(\frac{pk^2}{2}\right)^x\right) \cdot \left(\sum_{y=0}^{\infty} \frac{1}{y!} \cdot (pk)^y\right) \\ &\leq 3(1+p)k^2 \cdot e^{\frac{pk^2}{2}} \cdot e^{pk}. \end{aligned}$$

□

We can continue (4.12), writing

$$
\begin{aligned}
&\lim_{\substack{N,\\ N>k}} \mathbb{E}^{(N)} \left(\Xi \left(\frac{2x+y}{k} + (k - \lfloor E \rfloor + 2) \right) \right) \qquad (4.13) \\
&= \lim_{\substack{N,\\ N>k}} \sum_{x,y\geq 0} B_{\binom{k}{2},\frac{p}{N}}(x) \cdot B_{k(N-k-1),\frac{p}{N}}(y) \\
&\qquad \cdot \Xi \left(\frac{2x+y}{k} + (k - \lfloor E \rfloor + 2) \right) \\
&= \sum_{x,y\geq 0} \lim_{\substack{N,\\ N>k}} B_{\binom{k}{2},\frac{p}{N}}(x) \cdot B_{k(N-k-1),\frac{p}{N}}(y) \\
&\qquad \cdot \Xi \left(\frac{2x+y}{k} + (k - \lfloor E \rfloor + 2) \right) \\
&= \sum_{x,y\geq 0} \left(\lim_{\substack{N,\\ N>k}} B_{\binom{k}{2},\frac{p}{N}}(x) \right) \cdot \left(\lim_{\substack{N,\\ N>k}} B_{k(N-k-1),\frac{p}{N}}(y) \right) \\
&\qquad \cdot \Xi \left(\frac{2x+y}{k} + (k - \lfloor E \rfloor + 2) \right)
\end{aligned}
$$

Applying lemma $A.5$, noting that $\lim_{\substack{N,\\ N>k}} \frac{p}{N} \cdot \binom{k}{2} = 0$ and $\lim_{\substack{N,\\ N>k}} \frac{p}{N} \cdot k(N-k-1) = kp$:

$$
\begin{aligned}
\text{(cont.)} &= \sum_{x,y\geq 0} \pi_0(x) \cdot \pi_{kp}(y) \cdot \Xi \left(\frac{2x+y}{k} + (k - \lfloor E \rfloor + 2) \right) \\
&= \sum_{y=0}^{\infty} \pi_{kp}(y) \cdot \Xi \left(\frac{y}{k} + (k - \lfloor E \rfloor + 2) \right)
\end{aligned}
$$

since $\pi_0(x) = \delta_{x0}$.

We incorporate (4.11) and (4.13) into (4.10), totalling

$$
1 - \sigma(E) \leq 2 \cdot \sum_{k=1}^{\infty} \pi_p(k) \cdot \sum_{y=0}^{\infty} \pi_{kp}(y) \cdot \Xi \left(\frac{y}{k} + (k - \lfloor E \rfloor + 2) \right). \qquad (4.14)
$$

4.4 Evaluating the bounding value

We simplify this bound in a series of steps. Our first goal is to get rid of the Ξ-term:

$$
1 - \sigma(E) \qquad (4.15)
$$

$$\leq 2 \cdot \sum_{k=1}^{\infty} \pi_p(k) \cdot \sum_{y=0}^{\infty} \pi_{kp}(y) \cdot \Xi \left(\frac{y}{k} + (k - \lfloor E \rfloor + 2) \right)$$
$$= 2 \cdot \sum_{k=1}^{\infty} \pi_p(k) \cdot \sum_{y=k(\lfloor E \rfloor - 2 - k)+1}^{\infty} \pi_{kp}(y) \cdot \left(\frac{y}{k} + (k - \lfloor E \rfloor + 2) \right)$$

Noting that $\frac{y}{k} \leq y$ and $k - \lfloor E \rfloor + 2 \leq k$:

$$\text{(cont.)} \leq 2 \cdot \sum_{k=1}^{\infty} \pi_p(k) \cdot \sum_{y=k(\lfloor E \rfloor - 2 - k)+1}^{\infty} \pi_{kp}(y) \cdot (y + k)$$
$$= 2 \cdot \sum_{k=1}^{\infty} \pi_p(k) \cdot \sum_{y=k(\lfloor E \rfloor - 2 - k)+1}^{\infty} \pi_{kp}(y) \cdot k \left(\frac{y}{k} + 1 \right)$$
$$= 2 \cdot \sum_{k=1}^{\infty} (k \cdot \pi_p(k)) \cdot \sum_{y=k(\lfloor E \rfloor - 2 - k)+1}^{N} \left(\left(\frac{y}{k} \cdot \pi_{kp}(y) \right) + \pi_{kp}(y) \right)$$

We relieve us of the factors k and $\frac{y}{k}$ as follows: For the first parenthesized term, we have

$$k \cdot \pi_p(k) = k \cdot \frac{p^k}{k!} \cdot e^{-p} = p \cdot \frac{p^{k-1}}{(k-1)!} \cdot e^{-p} = p \cdot \pi_p(k-1)$$

(note that $k \geq 1$), and for the second, we have

$$\frac{y}{k} \cdot \pi_{kp}(y) = \frac{y}{k} \cdot \frac{(kp)^y}{y!} \cdot e^{-kp} = p \cdot \frac{(kp)^{y-1}}{(y-1)!} \cdot e^{-kp} = p \cdot \pi_{kp}(y-1)$$

if $y \geq 1$ and $\frac{y}{k} \cdot \pi_{kp}(y) = 0 = p \cdot \pi_{kp}(y-1)$ if $y \leq 0$. With this, we can continue our calculation (4.15) and simplify

$$\begin{aligned} &1 - \sigma(E) \\ &\leq 2 \cdot \sum_{k=1}^{\infty} (p \cdot \pi_p(k-1)) \cdot \sum_{y=k(\lfloor E \rfloor - 2 - k)+1}^{\infty} ((p \cdot \pi_{kp}(y-1)) + \pi_{kp}(y)) \, . \end{aligned} \tag{4.16}$$

The second sum can be bounded in a straightforward way writing

$$\begin{aligned}
&\sum_{y=k(\lfloor E\rfloor-2-k)+1}^{\infty} (p\cdot\pi_{kp}(y-1)+\pi_{kp}(y)) \\
&= p\cdot\left(\sum_{y=k(\lfloor E\rfloor-2-k)+1}^{\infty} \pi_{kp}(y-1)\right) + \left(\sum_{y=k(\lfloor E\rfloor-2-k)+1}^{\infty} \pi_{kp}(y)\right) \\
&\leq p\cdot\left(\sum_{y=k(\lfloor E\rfloor-2-k)}^{\infty} \pi_{kp}(y)\right) + \left(\sum_{y=k(\lfloor E\rfloor-2-k)}^{\infty} \pi_{kp}(y)\right) \\
&= (1+p)\cdot \sum_{y=k(\lfloor E\rfloor-2-k)}^{\infty} \pi_{kp}(y).
\end{aligned}$$

Using this, the bound (4.16) turns into

$$\begin{aligned}
&1-\sigma(E) \\
&\leq 2p(1+p)\cdot\sum_{k=1}^{\infty}\pi_p(k-1)\cdot \sum_{y=k(\lfloor E\rfloor-2-k)}^{\infty} \pi_{kp}(y) \\
&= 2p(1+p)\cdot\sum_{k=0}^{\infty}\pi_p(k)\cdot \sum_{y=(k+1)(\lfloor E\rfloor-3-k)}^{\infty} \pi_{(k+1)p}(y).
\end{aligned} \tag{4.17}$$

Let us take a step back and think about how we should proceed bounding this term. Recalling how we arrived at this term from our starting point

$$1-\sigma(E)\leq 2\cdot\mathbb{E}^{(N)}\left(\Xi\left(\lceil \deg(1)+\mathrm{mdeg}(1)-E\rceil\right)\right),$$

the outer sum and the factor $\pi_p(k)$ roughly correspond to vertex 1 having degree $1+k$ while the the inner sum and the term $\pi_{(k+1)p}(y)$ roughly correspond to the average degree mdeg(1) of the neighboring vertices being $1+\frac{y}{k}$.

For k near E, corresponding to dominating deg(1) in the sum deg(1) + mdeg(1), the factor in the outer sum alone is enough to guarantee a bound of logarithmical asymptotics

$$\pi_p(\lfloor E\rfloor)=\frac{p^{\lfloor E\rfloor}}{\lfloor E\rfloor!}\cdot e^{-p}\sim\frac{1}{\lfloor E\rfloor!}\sim\frac{1}{E\cdot\ln(E)}.$$

If the difference $E-k$ is substantial with respect to E, we have to take into account, or even rely on, the inner sum being small enough, which corresponds to larger values of mdeg(1). Even if there are just more then a few, say of the order of $\ln(E)$, many neighbors of 1, larger values of mdeg(1) become extremely unlikely: The degree of a neighbor minus one being asymptotically Poisson distributed with mean p, their mean is a random variable having as limit distribution a downscaled Poisson distribution with parameter $\deg(1) \cdot p$ (note that the mean is still p). Lastly, if there are only a few, say $\deg(1) < \ln(E)$, neighbors of 1, then mdeg(1) would have to attain values near E in order for $\deg(1) + \mathrm{mdeg}(1) \leq E$ to be violated. But this, even for just one neighbor, is as unlikely as deg(1) itself being that large.

This suggests the following decomposition of the outer sum (note that $\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 3 \geq 0$ since $E \geq 4$) in (4.17):

$$\begin{aligned} &\sum_{k=0}^{\infty} \pi_p(k) \cdot \sum_{y=(k+1)(\lfloor E \rfloor - 3 - k)}^{\infty} \pi_{(k+1)p}(y) \\ &= \left(\sum_{k=0}^{\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 4} \pi_p(k) \cdot \sum_{y=(k+1)(\lfloor E \rfloor - 3 - k)}^{\infty} \pi_{(k+1)p}(y) \right) \\ &\qquad + \left(\sum_{k=\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 3}^{\infty} \pi_p(k) \cdot \sum_{y=(k+1)(\lfloor E \rfloor - 3 - k)}^{\infty} \pi_{(k+1)p}(y) \right) \end{aligned}$$

In the first summand, we do away with the factor $\pi_p(k) \leq 1$, while in the second summand, we disregard the entire second sum, noting that, $\pi_{(k+1)p}$ being a discrete probability density function, it is bounded by 1 from above:

$$\begin{aligned} (\text{cont.}) \leq &\left(\sum_{k=0}^{\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 4} \sum_{y=(k+1)(\lfloor E \rfloor - 3 - k)}^{\infty} \pi_{(k+1)p}(y) \right) \\ &\qquad + \left(\sum_{k=\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 3}^{\infty} \pi_p(k) \right) \end{aligned}$$

Splitting the first summand further (note that $0 < \lfloor \ln(E) \rfloor \leq \lfloor E \rfloor - \lfloor \ln(E) \rfloor - 3$ since $E \geq 5$):

$$\begin{aligned} (\text{cont.}) = &\underbrace{\sum_{k=0}^{\lfloor \ln(E) \rfloor - 1} \sum_{y=(k+1)(\lfloor E \rfloor - 3 - k)}^{\infty} \pi_{(k+1)p}(y)}_{=:P} \\ &\qquad + \underbrace{\sum_{k=\lfloor \ln(E) \rfloor}^{\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 4} \sum_{y=(k+1)(\lfloor E \rfloor - 3 - k)}^{\infty} \pi_{(k+1)p}(y)}_{=:Q} \end{aligned}$$

$$+ \underbrace{\sum_{k=\lfloor E\rfloor-\lfloor \ln(E)\rfloor-3}^{\infty} \pi_p(k)}_{=:R},$$

i.e.

$$1-\sigma(E) \leq 2p(1+p)\left(P+Q+R\right). \tag{4.18}$$

In the following, we will make heavy use of some well-known helpful facts, which for convenience are proved in the appendix and summarized here:

Lemma A.7. *For $\lambda \geq 0$ and $u \in \mathbb{N}$, we have*

$$\sum_{v=u}^{\infty} \pi_\lambda(v) \leq \frac{\lambda^u}{u!}.$$

Lemma A.8. *For $u \in \mathbb{N}$, we have*

$$u! \geq e^{u\cdot(\ln(u)-1)}.$$

We proceed bounding each of P, Q, R separately:

Lemma 4.11. *There is a function f_P in E depending only on p such that*

$$P \leq e^{-E\cdot\ln(E)\cdot(1+f_P(E))}$$

for all $E \geq 4$ and $\lim_{E\to\infty} f_P(E) = 0$.

Proof. Observe that, for $0 \leq k \leq \lfloor \ln(E)\rfloor - 1$, we have

$$\begin{aligned}(k+1)\left(\lfloor E\rfloor-3-k\right) &= k\left(\lfloor E\rfloor-3-k\right)+\left(\lfloor E\rfloor-3-k\right)\\ &= k\left(\lfloor E\rfloor-4-k\right)+\left(\lfloor E\rfloor-3\right)\\ &\geq \lfloor E\rfloor-3\end{aligned}$$

since $\lfloor E\rfloor - 4 - k \geq \lfloor E\rfloor - 3 - \lfloor \ln(E)\rfloor \geq 0$. It follows that

$$\begin{aligned}P &= \sum_{k=0}^{\lfloor \ln(E)\rfloor-1} \sum_{y=(k+1)(\lfloor E\rfloor-3-k)}^{\infty} \pi_{(k+1)p}(y)\\ &\leq \sum_{k=0}^{\lfloor \ln(E)\rfloor-1} \sum_{y=\lfloor E\rfloor-3}^{\infty} \pi_{(k+1)p}(y)\end{aligned}$$

By lemma A.7:

$$
\begin{aligned}
&\leq \sum_{k=0}^{\lfloor \ln(E) \rfloor -1} \frac{((k+1)p)^{\lfloor E \rfloor -3}}{(\lfloor E \rfloor - 3)!} \\
&\leq \sum_{k=0}^{\lfloor \ln(E) \rfloor -1} \frac{(p \cdot \ln(E))^{\lfloor E \rfloor -3}}{(\lfloor E \rfloor - 3)!} \\
&= \lfloor \ln(E) \rfloor \cdot \frac{(p \cdot \ln(E))^{\lfloor E \rfloor -3}}{(\lfloor E \rfloor - 3)!} \\
&\leq (\lfloor E \rfloor + 1)^4 \cdot ((1+p) \cdot \ln(E))^E \cdot \frac{1}{(\lfloor E \rfloor + 1)!}
\end{aligned}
$$

Applying lemma A.8:

$$
\begin{aligned}
\text{(cont.)} &\leq (\lfloor E \rfloor + 1)^4 \cdot ((1+p) \cdot \ln(E))^E \cdot e^{-(\lfloor E \rfloor + 1) \cdot (\ln(\lfloor E \rfloor + 1) - 1)} \\
&\leq 16E^4 \cdot ((1+p) \cdot \ln(E))^E \cdot e^{E+1} \cdot e^{-E \cdot \ln(E)} \\
&= \exp\left(\ln(16) + 4 \cdot \ln(E) + (E+1)\right. \\
&\qquad \left. + E \cdot (\ln(1+p) + \ln(\ln(E))) - E \cdot \ln(E)\right) \\
&= \exp\left(-E \cdot \ln(E) \cdot (1 + f_P(E))\right)
\end{aligned}
$$

with

$$
f_P(E) := -\frac{\ln(16) + 4 \cdot \ln(E) + (E+1) + E \cdot \ln(1+p) + E \cdot \ln(\ln(E))}{E \cdot \ln(E)}.
$$

Now note that $\lim_{E \to \infty} f_P(E) = 0$. □

Lemma 4.12. *There is a function f_Q in E depending only on p such that*

$$
Q \leq e^{-E \cdot \ln(E) \cdot \left(1 + f_Q(E)\right)}
$$

for all $E \geq 5$ and $\lim_{E \to \infty} f_Q(E) = 0$.

Proof. Consider arbitrary $\lceil \ln(E) \rceil \leq k \leq \lfloor E \rfloor - \lfloor \ln(E) \rfloor - 4$ and set $x := k - \lfloor \ln(E) \rfloor$. Observe that $0 \leq x \leq \lfloor E \rfloor - 2 \lfloor \ln(E) \rfloor - 4$. We have

$$
\begin{aligned}
&\quad (k+1)\left(\lfloor E \rfloor - 3 - k\right) \\
&= (\lfloor \ln(E) \rfloor + (1+x))(\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 2 - (1+x)) \\
&= \lfloor \ln(E) \rfloor \cdot (\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 2) + \underbrace{(1+x)}_{\geq 1} \cdot \underbrace{(\lfloor E \rfloor - 2 \lfloor \ln(E) \rfloor - 3 - x)}_{\geq 1} \\
&\geq \lfloor \ln(E) \rfloor \cdot (\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 2) =: v.
\end{aligned}
$$

Note that $v \geq \ln(E) \geq 1$ since $E \geq 5$. It follows that

$$\begin{aligned} Q &= \sum_{k=\lfloor \ln(E) \rfloor}^{\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 4} \sum_{y=(k+1)(\lfloor E \rfloor - 3 - k)}^{\infty} \pi_{(k+1)p}(y) \\ &\leq \sum_{k=\lfloor \ln(E) \rfloor}^{\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 4} \sum_{y=v}^{\infty} \pi_{(k+1)p}(y) \end{aligned}$$

By lemma A.7:

$$\begin{aligned} \text{(cont.)} &\leq \sum_{k=\lfloor \ln(E) \rfloor}^{\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 4} \frac{((k+1)p)^v}{v!} \\ &\leq \sum_{k=\lfloor \ln(E) \rfloor}^{\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 4} \frac{(Ep)^v}{v!} \leq E \cdot \frac{(Ep)^v}{v!} \end{aligned}$$

Applying lemma A.8:

$$\begin{aligned} \text{(cont.)} &\leq E \cdot (Ep)^v \cdot e^{-v \cdot (\ln(v) - 1)} \\ &= \exp\left(\underbrace{\ln(E)}_{\leq v} - v \cdot (\ln(v) - \ln(E) - \ln(p) - 1) \right). \\ &\leq \exp\left(-v \cdot (\ln(v) - \ln(E) - \ln(p) - 2) \right). \\ &\leq \exp\left(-E \cdot \ln(E) \cdot (1 + f_Q(E)) \right). \end{aligned}$$

with

$$f_Q(E) := -1 + \frac{v \cdot \min\left(1, \ln(v) - \ln(E) - \ln(p) - 2\right)}{E \cdot \ln(E)}.$$

Now note that

$$\ln(v) = \ln(\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 2) + \ln(\lfloor \ln(E) \rfloor) \geq \ln(E) + \ln(p) + 3$$

for sufficiently large E because

$$\begin{aligned} \lim_{E \to \infty} \ln\left(\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 2 \right) - \ln(E) &= \lim_{E \to \infty} \ln\left(\frac{\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 2}{E} \right) \\ &= \ln\left(\lim_{E \to \infty} \frac{\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 2}{E} \right) \\ &= \ln(1) = 0, \end{aligned}$$

hence

$$\lim_{E\to\infty} \min\left(1, \ln(v) - \ln(E) - \ln(p) - 2\right) = 1.$$

Also note that

$$\begin{aligned}
\lim_{E\to\infty} \frac{v}{E\cdot\ln(E)} &= \lim_{E\to\infty} \frac{\lfloor\ln(E)\rfloor\cdot(\lfloor E\rfloor - \lfloor\ln(E)\rfloor - 2)}{E\cdot\ln(E)} \\
&= \left(\lim_{E\to\infty} \frac{\lfloor E\rfloor - \lfloor\ln(E)\rfloor - 2}{E}\right)\cdot\left(\lim_{E\to\infty}\frac{\lfloor\ln(E)\rfloor}{\ln(E)}\right) \\
&= 1\cdot 1 = 1,
\end{aligned}$$

and therefore

$$\begin{aligned}
\lim_{E\to\infty} f_Q(E) &= \lim_{E\to\infty} -1 + \frac{v\cdot\min\left(1, \ln(v) - \ln(E) - \ln(p) - 2\right)}{E\cdot\ln(E)} \\
&= -1 + \left(\lim_{E\to\infty}\frac{v}{E\cdot\ln(E)}\right) \\
&\quad\cdot\left(\lim_{E\to\infty}\min\left(1, \ln(v) - \ln(E) - \ln(p) - 2\right)\right) \\
&= -1 + 1\cdot 1 = 0.
\end{aligned}$$

□

Lemma 4.13. *There is a function f_R in E depending only on p such that*

$$R \le e^{-E\cdot\ln(E)\cdot(1+f_R(E))}$$

for all $E \ge 4$ and $\lim_{E\to\infty} f_R(E) = 0$.

Proof. Abbreviate $u := \lfloor E\rfloor - \lfloor\ln(E)\rfloor - 3 \ge 0$. We have

$$Q = \sum_{k=\lfloor E\rfloor - \lfloor\ln(E)\rfloor - 3}^{\infty} \pi_p(k) = \sum_{k=u}^{\infty}\pi_p(k)$$

By lemma A.7:

$$(\text{cont.}) \le \frac{p^u}{u!}$$

Applying lemma A.8:

$$\begin{aligned}
(\text{cont.}) &\le p^u\cdot e^{-u\cdot(\ln(u)-1)} \\
&= \exp\left(-u\cdot(\ln(u) - \ln(p) - 1)\right) \\
&= \exp\left(-E\cdot\ln(E)\cdot(1 + f_R(E))\right)
\end{aligned}$$

with

$$f_R(E) := -1 + \frac{u \cdot (\ln(u) - \ln(p) - 1)}{E \cdot \ln(E)}.$$

Now note that

$$\lim_{E\to\infty} \frac{u}{E} = \lim_{E\to\infty} \frac{\lfloor E \rfloor - \lfloor \ln(E) \rfloor - 3}{E} = 1,$$

hence also

$$\lim_{E\to\infty} \ln(u) - \ln(E) = \lim_{E\to\infty} \ln\left(\frac{u}{E}\right) = \ln\left(\lim_{E\to\infty} \frac{u}{E}\right) = \ln(1) = 0$$

and

$$\lim_{E\to\infty} \frac{\ln(u)}{\ln(E)} = \lim_{E\to\infty} 1 + \frac{\ln(u) - \ln(E)}{\ln(E)} = 1.$$

Therefore,

$$\begin{aligned}
\lim_{E\to\infty} f_R(E) &= \lim_{E\to\infty} -1 + \frac{u \cdot (\ln(u) - \ln(p) - 1)}{E \cdot \ln(E)} \\
&= -1 + \left(\lim_{E\to\infty} \frac{u}{E}\right) \cdot \left(\left(\lim_{E\to\infty} \frac{\ln(u)}{\ln(E)}\right) + \left(\lim_{E\to\infty} \frac{\ln(p) - 1}{\ln(E)}\right)\right) \\
&= -1 + 1 \cdot (1 + 0) = 0.
\end{aligned}$$

□

Let $f := \min(f_P, f_Q, f_R)$, i.e.

$$f(E) := \min(f_P(E), f_Q(E), f_R(E))$$

pointwise for all $E \geq 5$. Of course, we still have $\lim_{E\to\infty} f(E) = 0$. Continuing where we left off in (4.18),

$$\begin{aligned}
1 - \sigma(E) &\leq 2p(p+1)(P + Q + R) \\
&\leq 2p(p+1) \cdot \big(e^{-E\cdot\ln(E)\cdot(1+f_P(E))} \\
&\quad + e^{-E\cdot\ln(E)\cdot(1+f_Q(E))} + e^{-E\cdot\ln(E)\cdot(1+f_R(E))}\big) \\
&\leq 2p(p+1) \cdot \big(e^{-E\cdot\ln(E)\cdot(1+f(E))} \\
&\quad + e^{-E\cdot\ln(E)\cdot(1+f(E))} + e^{-E\cdot\ln(E)\cdot(1+f(E))}\big) \\
&= 6p(p+1) \cdot e^{-E\cdot\ln(E)\cdot(1+f(E))}.
\end{aligned}$$

This finally gives us

Theorem 3. *For $E > 0$ a continuity point of σ, we have*

$$1 - \sigma(E) \leq f_{\text{high}}(E)$$

with $f_{\text{high}} : \mathbb{R}^+ \to \mathbb{R}$ defined as

$$f_{\text{high}}(E) := 6p(p+1) \cdot e^{-E \cdot \ln(E) \cdot (1+f(E))},$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is a certain function such that $\lim_{E\to\infty} f(E) = 0$, for $E \geq 5$ and $f_{\text{high}}(E) := 1 - \sigma(E)$ for $0 < E < 5$. Also,

$$\lim_{E\to\infty} \frac{-\ln\left(f_{\text{high}}(E)\right)}{E \cdot \ln(E)} = 1.$$

Proof. We only need to prove the last part. But this is just a reformulation of the asymptotic properties of f_{high}:

$$\begin{aligned}
\lim_{E\to\infty} \frac{-\ln\left(f_{\text{high}}(E)\right)}{E \cdot \ln(E)} &= \lim_{E\to\infty} \frac{-\ln\left(6p(p+1) \cdot e^{-E \cdot \ln(E) \cdot (1+f(E))}\right)}{E \cdot \ln(E)} \\
&= \lim_{E\to\infty} \frac{-\ln(6p(p+1)) + E \cdot \ln(E) \cdot (1 + f(E))}{E \cdot \ln(E)} \\
&= 1 + \left(\lim_{E\to\infty} \frac{-\ln(6p(p+1))}{E \cdot \ln(E)}\right) + \left(\lim_{E\to\infty} f(E)\right) \\
&= 1.
\end{aligned}$$

□

A Appendix

All of the lemmata in this section are well-known results. For convenience and completeness of the thesis, and since in most cases we only need a simplified version of the original result, we prove them anyway. For details, generalizations and/or further information, we provide individual references.

For generalizations of the following lemma and its consequence lemma A.2 to the infinite-dimensional case, see Reed-Simon [27].

Lemma A.1 (Min-max principle, variational characterization of eigenvalues)**.** *Given $n \in \mathbb{N}$ and a self-adjoint matrix $M \in \mathbb{C}^{n \times n}$, let $\lambda_1 \leq \ldots \leq \lambda_n$ denote the eigenvalues of M, counted with multiplicity. For $k = 1, \ldots, n$, we have the following variational eigenvalue characterizations:*

$$we\lambda_k = \inf_{\substack{U \lhd \mathbb{C}^n, \\ \dim(U) \geq k}} \sup_{0 \neq \phi \in U} R_M(\phi)$$

$$\lambda_{n-k+1} = \sup_{\substack{U \lhd \mathbb{C}^n, \\ \dim(U) \geq k}} \inf_{0 \neq \phi \in U} R_M(\phi)$$

Here, $U \lhd \mathbb{C}^n$ denotes U as a vector subspace of $\mathbb{C}^n$ and

$$R_M(\phi) := \frac{\langle \phi, M\phi \rangle}{\langle \phi, \phi \rangle}$$

is the Rayleigh quotient *of $0 \neq \phi \in \mathbb{C}$ with respect to M.*

Proof. First note that the claim of the lemma corresponding to the second equation is equivalent to the claim corresponding to the first equation after replacing M by $-M$ and multiplying the equation by -1. By symmetry, we hence only have to prove the claim corresponding to the first equation.

Since M is self-adjoint, there is an orthonormal base $\psi_1, \ldots, \psi_n$ where ψ_i is eigenvector with eigenvalue λ_i. For all $1 \leq u \leq v \leq n$ and non-zero $\phi \in \langle \psi_u, \ldots, \psi_v \rangle$, the subspace of $\mathbb{C}^n$ generated by $\psi_u, \ldots, \psi_v$, there are $\alpha_u, \ldots, \alpha_v \in \mathbb{C}$ not all zero such that $\phi = \alpha_u \psi_u + \ldots + \alpha_v \psi_v$. By orthonormality of $\psi_1, \ldots, \psi_n$, we have

$$R_M(\phi) = \frac{|\alpha_u|^2 \lambda_u + \ldots + |\alpha_v|^2 \lambda_v}{|\alpha_u|^2 + \ldots + |\alpha_v|^2}$$

and, since $\lambda_u \leq \ldots \leq \lambda_v$,

$$R_M(\phi) \geq \frac{|\alpha_u|^2 \lambda_u + \ldots + |\alpha_v|^2 \lambda_u}{|\alpha_u|^2 + \ldots + |\alpha_v|^2} = \lambda_u, \tag{A.1}$$

$$R_M(\phi) \leq \frac{|\alpha_u|^2 \lambda_v + \ldots + |\alpha_v|^2 \lambda_v}{|\alpha_u|^2 + \ldots + |\alpha_v|^2} = \lambda_v. \tag{A.2}$$

Let $k \in \mathbb{N}$ be given. Note that for any $U \lhd \mathbb{C}^n$ with $\dim(U) \geq k$, we have

$$\begin{aligned}\dim\left(U \cap \langle \psi_k, \ldots, \psi_n \rangle\right) &= \dim(U) + \dim\left(\langle \psi_k, \ldots, \psi_n \rangle\right) \\ &\quad - \dim\left(U + \langle \psi_k, \ldots, \psi_n \rangle\right) \\ &\geq k + (n - k + 1) - n = 1.\end{aligned}$$

There hence must be a non-zero $\phi \in U$ which also lies in $\langle \psi_k, \ldots, \psi_n \rangle$. By (A.1), $R_M(\phi) \geq \lambda_k$. This proves

$$\lambda_k \leq \inf_{\substack{U \lhd \mathbb{C}^n, \\ \dim(U) \geq k}} \sup_{0 \neq \phi \in U} R_M(\phi).$$

Let $U := \langle \psi_1, \ldots, \psi_k \rangle$. Then, by (A.2), all non-zero $\phi \in U$ satisfy $R_M(\phi) \leq \lambda_k$. But this means

$$\lambda_k \geq \inf_{\substack{U \lhd \mathbb{C}^n, \\ \dim(U) \geq k}} \sup_{0 \neq \phi \in U} R_M(\phi).$$

□

Lemma A.2. *Let $n \in \mathbb{N}$ and A, B be symmetric matrices in $\mathbb{C}^{n \times n}$. Let $\lambda_1 \leq \ldots \leq \lambda_n$ and $\mu_1 \leq \ldots \leq \mu_n$ denote the eigenvalues of A and $A + B$, respectively. Let $\gamma_A, \gamma_{A+B} : \mathbb{R} \to [0, 1]$ denote the eigenvalue counting functions for A and $A + B$, given by*

$$\begin{aligned}\gamma_A(E) &= \left|\{i \in \{1, \ldots, n\} \mid \lambda_i \leq E\}\right|, \\ \gamma_{A+B}(E) &= \left|\{i \in \{1, \ldots, n\} \mid \mu_i \leq E\}\right|,\end{aligned}$$

for $E \in \mathbb{R}$. Let finally $k :=$ rnk(B) denote the rank *of B, that is, the dimension of im(B). Then, $|\sigma_A - \sigma_{A+B}|_\infty \leq k$, that is,*

$$|\gamma_A(E) - \gamma_{A+B}(E)| \leq k$$

for all $E \in \mathbb{R}$.

Proof. Fix $E \in \mathbb{R}$. Note that it is sufficient to prove $\gamma_A(E) - \gamma_{A+B}(E) \leq k$ as the inequality $\gamma_{A+B}(E) - \gamma_A(E) \leq k$ follows from this by replacing A by $A + B$ and B by $-B$ after noting that $\text{rnk}(B) = \text{rnk}(-B)$.

We have $\dim(\ker(B)) = n - \dim(\text{im}(B)) = n - k$. Let $i \in \{1, \ldots, n - k\}$.

By the variational characterization of eigenvalues (see the previous lemma),

$$\begin{aligned}
\lambda_{i+k} &= \inf_{\substack{U \triangleleft \mathbb{C}^n, \\ \dim(U) \geq i+k}} \sup_{0 \neq \phi \in U} R_A(\phi) \\
&\geq \inf_{\substack{U \triangleleft \mathbb{C}^n, \\ \dim(U) \geq i+k}} \sup_{0 \neq \phi \in U \cap \ker(B)} R_A(\phi) \\
&= \inf_{\substack{U \triangleleft \mathbb{C}^n, \\ \dim(U) \geq i+k}} \sup_{0 \neq \phi \in U \cap \ker(B)} R_{A+B}(\phi) \\
&\geq \inf_{\substack{U \triangleleft \mathbb{C}^n, \\ \dim(U \cap \ker(B)) \geq i}} \sup_{0 \neq \phi \in U \cap \ker(B)} R_{A+B}(\phi) \\
&\geq \inf_{\substack{V \triangleleft \mathbb{C}^n, \\ \dim(V) \geq i}} \sup_{0 \neq \phi \in V} R_{A+B}(\phi) = \mu_i,
\end{aligned} \tag{A.3}$$

where the third line follows from the fact that R_A and R_{A+B} are identical on $\ker(B)$ and the fourth line holds because

$$\begin{aligned}
\dim(U) - \dim\left(U \cap \ker(B)\right) &= \dim(U + \ker(B)) - \dim(\ker(B)) \\
&\leq n - (n-k) = k.
\end{aligned}$$

Also note that the supremum over non-zero elements of $\dim(U \cap \ker(B))$ in the above is well-defined since

$$\begin{aligned}
\dim\left(U \cap \ker(B)\right) &= \dim(U) + \dim\left(\ker(B)\right) - \dim\left(U + \ker(B)\right) \\
&\geq (k+1) + (n-k) - n = 1.
\end{aligned}$$

We now have

$$\begin{aligned}
\gamma_A(E) &= \left|\{i \in \{1, \ldots, n\} \mid \lambda_i\} \leq E\right| \\
&\leq k + \left|\{i \in \{1, \ldots, n-k\} \mid \lambda_{i+k}\} \leq E\right| \\
&\overset{(A.3)}{\leq} k + \left|\{i \in \{1, \ldots, n-k\} \mid \mu_i\} \leq E\right| \\
&\leq k + \left|\{i \in \{1, \ldots, n\} \mid \mu_i\} \leq E\right| \\
&= k + \gamma_{A+B}(E),
\end{aligned}$$

proving $\gamma_A(E) - \gamma_{A+B}(E) \leq k$. □

The next lemma and its general forms were first established by Gershgorin [15]. We only need it to prove lemma A.4.

Lemma A.3 (Gershgorin circle theorem)**.** *Let $n \in \mathbb{N}$ and $M \in \mathbb{C}^{n \times n}$ be an arbitrary matrix. Let $\lambda_{\max} \in \mathbb{R}$ denote the largest eigenvalue of M. Then,*

$$\lambda_{\max} \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left|M_{ij}\right|.$$

Proof. Let $0 \neq \psi \in \mathbb{C}^n$ be an eigenvector of M corresponding to the eigenvalue $\lambda_{\max}$. Choose $k \in \{1, \ldots, n\}$ such that $|\psi_k|$ is maximal, that is, $|\psi_j| \leq |\psi_k|$ for all $j \in \{1, \ldots, n\}$. Note that $\psi_k \neq 0$. We have $M\psi = \lambda_{\max}\psi$. In particular,

$$\lambda_{\max}\psi_k = \sum_{j=1}^{n} M_{kj}\psi_j,$$

and hence

$$\begin{aligned}\lambda_{\max} &= \sum_{j=1}^{n} M_{kj} \cdot \frac{\psi_j}{\psi_k} \leq \sum_{j=1}^{n} |M_{kj}| \cdot \underbrace{\frac{|\psi_j|}{|\psi_k|}}_{\leq 1} \\ &\leq \sum_{j=1}^{n} |M_{kj}| \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |M_{ij}| .\end{aligned}$$

□

This result was discovered by Merris [26]:

Lemma A.4 (Upper bound to Laplacian eigenvalues)**.** *Let $n \in \mathbb{N}$ and G be a subgraph of $K^{(n)}$ with edge set E. Let $D \in \mathbb{C}^{n \times n}$ denote the degree matrix, $A \in \mathbb{C}^{n \times n}$ the adjacency matrix, and $\Delta = D - A$ the Laplacian of G. For $1 \leq i \leq n$, let further* $\deg(i)$ *denote the degree of vertex i in G and* $\mathrm{mdeg}(i)$ *denote the mean degree of all neighbouring vertices of i in G if there are any, and zero otherwise:*

$$\mathrm{mdeg}(i) = \begin{cases} \frac{1}{\deg(i)} \cdot \sum_{\substack{1 \leq j \leq n, \\ [i,j] \in E}} \deg(j) & \textit{if } \deg(i) > 0, \\ 0 & \textit{else.} \end{cases}$$

Let $\lambda_{\max}$ denote the maximal eigenvalue of Δ. Then,

$$\lambda_{max} \leq \max_{1 \leq i \leq n} \deg(i) + \mathrm{mdeg}(i).$$

Proof. Consider the diagonal matrix $\tilde{D} \in \mathbb{C}^{n \times n}$ with diagonal elements

$$\tilde{D}_{ii} = \begin{cases} \deg(i) & \text{if } \deg(i) > 0, \\ 1 & \text{else,} \end{cases}$$

for $i = 1, \ldots, n$, which differs from D in only those entries corresponding to isolated vertices. Note that this makes $\tilde{D}$ invertible:

$$\tilde{D}^{-1}_{ii} = \begin{cases} \frac{1}{\deg(i)} & \text{if } \deg(i) > 0, \\ 1 & \text{else.} \end{cases}$$

Now apply the previous lemma to the matrix $\tilde{D}^{-1}\Delta\tilde{D}$, which is conjugated to Δ and hence has the same eigenvalues:

$$
\begin{aligned}
\lambda_{max} &\leq \max_{1\leq i\leq n} \sum_{j=1}^{n} \left| \left(\tilde{D}^{-1}\Delta\tilde{D} \right) \right|_{ij} \\
&= \max_{1\leq i\leq n} \sum_{j=1}^{n} \left| \sum_{1\leq r,s\leq n} \tilde{D}^{-1}_{ir} \cdot \Delta_{rs} \cdot \tilde{D}_{sj} \right| \\
&= \max_{1\leq i\leq n} \begin{cases} \sum_{j=1}^{n} \left| \frac{1}{\deg(i)} \cdot \Delta_{ij} \cdot \deg(j) \right| & \text{if } \deg(i) > 0, \\ \sum_{j=1}^{n} \left| 1 \cdot \Delta_{ij} \cdot 1 \right| & \text{else,} \end{cases} \\
&= \max_{1\leq i\leq n} \begin{cases} \frac{1}{\deg(i)} \cdot \sum_{j=1}^{n} \left| \Delta_{ij} \cdot \deg(j) \right| & \text{if } \deg(i) > 0, \\ \sum_{j=1}^{n} \Delta_{ij} & \text{else,} \end{cases} \\
&= \max_{1\leq i\leq n} \begin{cases} \frac{1}{\deg(i)} \cdot \Big(\left| \deg(i) \cdot \deg(i) \right| & \text{if } \deg(i) > 0, \\ \quad + \sum_{\substack{1\leq j\leq n, \\ [i,j]\in E}} \left| (-1) \cdot \deg(j) \right| \Big) & \\ 0 & \text{else,} \end{cases} \\
&\leq \max_{1\leq i\leq n} \begin{cases} \deg(i) + \frac{1}{\deg(i)} \cdot \left| \sum_{\substack{1\leq j\leq n, \\ [i,j]\in E}} \deg(j) \right| & \text{if } \deg(i) > 0, \\ 0 & \text{else,} \end{cases} \\
&= \max_{1\leq i\leq n} \deg(i) + \operatorname{mdeg}(i).
\end{aligned}
$$

□

The next propositions, lemma A.5 and corollary A.6, are basic stochastic results. They should be found in most introductory textbooks, e.g. [14].

Lemma A.5. *For $p \in [0,1]$ and $n \in \mathbb{N}$, let*

$$B_{n,p}(k) := \binom{n}{k} p^k (1-p)^{n-k}$$

for $k \in \mathbb{N}_0$ and $B_{n,p}(k) := 0$ for $k \in \mathbb{Z}\backslash\mathbb{N}_0$ denote the binomial distribution with parameters n and p (note that $\binom{n}{k} = 0$ for $k > n$). For $\lambda \geq 0$ real, let

$$\pi_\lambda(k) := \frac{\lambda^k}{k!} \cdot e^{-\lambda}$$

for $k \in \mathbb{N}_0$ and $\pi_\lambda(k) := 0$ for $k \in \mathbb{Z}\backslash N_0$ denote the Poisson distribution with parameter λ. For $\lambda \geq 0$ and any sequence $(p_n)_{n\in\mathbb{N}} \in]0,1]^{\mathbb{N}}$ such that $\lim_n np_n = \lambda$, we then have

$$\lim_n B_{n,p_n}(k) = \pi_\lambda(k)$$

for every $k \in \mathbb{Z}$*.*

Proof. Fix $k \in \mathbb{N}$. Regarding the existence of the respective limits, the following string of equations is to be read backwards:

$$\begin{aligned}
\lim_n B_{n,p_n}(k) &= \lim_n \binom{n}{k} p_n^k \left(1 - p_n\right)^{n-k} \\
&= \lim_n \frac{\prod_{i=0}^{k-1}(n-i)}{k!} \cdot p_n^k \left(1 - p_n\right)^{n-k} \\
&= \lim_n \frac{1}{k!} \cdot (np_n)^k \cdot (1-p_n)^n \cdot \left(\prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right)\right) \cdot (1-p_n)^{-k} \\
&= \frac{1}{k!} \cdot (\lim_n np_n)^k \cdot \left(\lim_n (1-p_n)^n\right) \\
&\quad \cdot \left(\prod_{i=0}^{k-1} \lim_n 1 - \frac{i}{n}\right) \cdot \left(\lim_n 1 - p_n\right)^{-k} \\
&\overset{(A.4)}{=} \frac{1}{k!} \cdot \lambda^k \cdot e^{-\lambda} \cdot \left(\prod_{i=0}^{k-1} 1\right) \cdot 1^{-k} \\
&= \frac{\lambda^k}{k!} \cdot e^{-\lambda} = \pi_\lambda(k).
\end{aligned}$$

Here we have used that $\lim_n np_n = \lambda$ implies $\lim_n p_n = 0$. For $n \to \infty$ we hence also have $p_n^{-1} \to \infty$. The non-trivial limes of $(1-p_n)^n$ can then be evaluated by noting that

$$\begin{aligned}
\lim_n (1-p_n)^n &= \lim_n \left(\left(1 - \frac{1}{p_n^{-1}}\right)^{p_n^{-1}} \right)^{np_n} \\
&= \left(\lim_n \left(1 - \frac{1}{p_n^{-1}}\right)^{p_n^{-1}} \right)^{\lim_n np_n} \\
&= \left(\frac{1}{e}\right)^{\lambda} = e^{-\lambda}
\end{aligned} \tag{A.4}$$

since $\lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^x = \frac{1}{e}$. □

From this, we immediately deduce

Corollary A.6. *For any* $b \in \mathbb{Z}$ *and* $c > 0$*, we have*

$$\lim_{\substack{n\to\infty,\\ n>c}} B_{n+b,\frac{c}{n}}(k) = \pi_c(k)$$

for all $k \in \mathbb{Z}$*.*

Proof. Just note that

$$\lim_{\substack{n\to\infty,\\ n>c}} B_{n+b,\frac{c}{n}}(k) = \lim_{\substack{n\to\infty,\\ n>c+b}} B_{n,\frac{c}{n-b}} = \pi_c(k),$$

where the last equation follows from the previous lemma (with finitely many starting terms of the sequence missing not affecting the limit behaviour): Setting $p_n := \frac{c}{n-b}$, we have

$$\lim_{\substack{n\to\infty,\\ n>c+b}} np_n = \lim_{\substack{n\to\infty,\\ n>c+b}} n \cdot \frac{c}{n-b} = \lim_{\substack{n\to\infty,\\ n>c+b}} c \cdot \left(1 + \frac{b}{n-b}\right) = c.$$

□

The next lemma is a plain basic estimate which should need no extra reference.

Lemma A.7. *For $\lambda \geq 0$ and $u \in \mathbb{N}$, we have*

$$\sum_{v=u}^{\infty} \pi_\lambda(v) \leq \frac{\lambda^u}{u!}.$$

Proof. We have

$$\begin{aligned}
\sum_{v=u}^{\infty} \pi_\lambda(v) &= \sum_{v=u}^{\infty} \frac{\lambda^v}{v!} \cdot e^{-\lambda} \\
&= \frac{\lambda^u}{u!} \cdot e^{-\lambda} \cdot \sum_{v=u}^{\infty} \frac{\lambda^{v-u}}{\prod_{k=u+1}^{v} k} \\
&= \frac{\lambda^u}{u!} \cdot e^{-\lambda} \cdot \sum_{v=0}^{\infty} \frac{\lambda^v}{\prod_{k=1}^{v} (k+u)} \\
&\leq \frac{\lambda^u}{u!} \cdot e^{-\lambda} \cdot \sum_{v=0}^{\infty} \frac{\lambda^v}{v!} \\
&= \frac{\lambda^u}{u!} \cdot e^{-\lambda} \cdot e^{\lambda} = \frac{\lambda^u}{u!}.
\end{aligned}$$

□

Finally, the remaining lemmata A.8, A.9 and corollary A.10 constitute a simplified version of the general Stirling's formula, see e.g. [1, p. 257].

Lemma A.8. *For $u \in \mathbb{N}$, we have $u! \geq e^{u\cdot(\ln(u)-1)}$.*

Proof. We have

$$\begin{aligned}
\ln(u!) &= \sum_{v=1}^{u} \ln(v) = \sum_{v=2}^{u} \ln(v) \\
&= \int_{x=1}^{u} \ln(\lceil x \rceil) dx \geq \int_{x=1}^{u} \ln(x) dx \\
&= [x \cdot (\ln(x) - 1)]_1^u \\
&= (u \cdot (\ln(u) - 1)) - (1 \cdot (\ln(1) - 1)) \geq u \cdot (\ln(u) - 1),
\end{aligned}$$

hence $u! \geq e^{u \cdot (\ln(u) - 1)}$ by monotonicity of exp. □

Lemma A.9. *For $u \in \mathbb{N}$, we have $u! \leq e^{(u+1)\cdot(\ln(u+1)-1)+1}$.*

Proof. In complete anology with the previous proof, we have

$$\begin{aligned}
\ln(u!) &= \sum_{v=1}^{u} \ln(v) = \int_{x=1}^{u+1} \ln(\lfloor x \rfloor) dx \\
&\leq \int_{x=1}^{u+1} \ln(x) dx = [x \cdot (\ln(x) - 1)]_1^{u+1} \\
&= ((u+1) \cdot (\ln(u+1) - 1)) - (1 \cdot (\ln(1) - 1)) \\
&= (u+1) \cdot (\ln(u+1) - 1) + 1,
\end{aligned}$$

hence $u! \leq e^{(u+1)\cdot(\ln(u+1)-1)+1}$ by monotonicity of exp. □

Corollary A.10. *We have*

$$\lim_u \frac{\ln(u!)}{u \cdot \ln(u)} = 1.$$

Proof. By the previous two lemmata,

$$\frac{u \cdot (\ln(u) - 1)}{u \cdot \ln(u)} \leq \frac{\ln(u!)}{u \cdot \ln(u)} \leq \frac{(u+1) \cdot (\ln(u+1) - 1) + 1}{u \cdot \ln(u)}.$$

But

$$\lim_u \frac{u}{u} = \lim_u \frac{u+1}{u} = 1$$

and

$$\lim_u \frac{\ln(u) - 1}{\ln(u)} = \lim_u \frac{\ln(u+1) - 1}{\ln(u)} = 1$$

as well as

$$\lim_u \frac{1}{u \cdot \ln(u)} = 0,$$

so

$$\lim_u \frac{\ln(u!)}{u \cdot \ln(u)} = 1.$$

□

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