COALGEBRA MODELS OF TYPE THEORY

The goal of this note is to present a type-theoretic analogue of the well-known construction of the topos of coalgebras for a left exact comonad on a topos. As a particular instance, we recover the recent presheaf translation [Péd20] of Pédrot.

We take *categories with families* (cwfs) as our notion of model of type theory. We take their basic theory for granted.

1. The model construction

Definition 1.1. A *comonad pseudomorphism* on a cwf $\mathcal{E}$ is a comonad N on the category $\mathcal{E}$ whose underlying functor extends to a pseudomorphism of cwfs. □

The notion of pseudomorphism is an approximation to the notion of left exact functor. We only have preservation of pullbacks along context projections. Thus, the notion of comonad pseudomorphism is an approximation to the notion of left exact comonad.

Definition 1.2 (Cwf of coalgebras)**.** Let N be a comonad pseudomorphism on a cwf $\mathcal{E}$. The category $\mathrm{Coalg}(N)$ of coalgebras of N extends to a cwa as follows:

- The terminal context is given by the unique coalgebra on 1, using that N preserves terminal objects.
- A type over $u\colon X \to NX$ is given by a "coalgebra fibration", i.e. a type $Y \in \mathsf{Ty}(X)$ together with a coalgebra structure $v\colon X.Y \to N(X.Y)$ on the context extension $X.Y$ such that the context projection $p\colon X.Y \to X$ forms a morphism of coalgebras:

$$\begin{tikzcd} X.Y \arrow[r, "v"] \arrow[d, two heads, "p"] & N(X.Y) \arrow[d, two heads, "Np"] \\ X \arrow[r, "u"] & NX. \end{tikzcd}$$

  The extension of this type is given by the coalgebra $v\colon X.Y \to N(X.Y)$.
- Substitution of types is given by substitution of types in E, using that N preserves pullbacks along context projections. The resulting morphism of comprehensions (context extensions) in $\mathrm{Coalg}(N)$ is cartesian as required.

Using the equivalence between cwas and cwfs, we also have $\mathrm{Coalg}(N)$ as a cwf. □

Remark 1.3. It is possible to give Definition 1.2 more directly in the style of cwfs using the action of the comonad (N, ε, ν) on types and terms.

- A type over $u\colon X \to NX$ consists of $Y \in \mathsf{Ty}(X)$ with a map $v\colon Y \to NY[u]$ in the fibrant slice over X such that $\varepsilon_Y[u] \circ v = \mathrm{id}$ and $\nu_Y[u] \circ v = Nv[u] \circ v$. Here, $\varepsilon_Y\colon NY \to Y[\varepsilon_X]$ and $\nu_Y\colon NY \to N^2Y[\nu_X]$ denote the "relative counit" and "relative comultiplication" at Y in the fibrant slice over NX, obtained from $\varepsilon_{X.Y}$ and $\nu_{X.V}$ over ε_X and ν_X, respectively.
- A term of this type consists of $t \in \mathsf{Tm}(X, Y)$ such that $v \circ t = Nt[u]$ (here, we see terms as maps in the fibrant slice with source the empty telescope).

All the operations can be developed in this style (all fairly elementary). □

Definition 1.4 (Cwf structures on $\mathcal{D} \downarrow F$)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a pseudomorphism of cwfs. Then there a two important cwf structures on the comma category $\mathcal{D} \downarrow F$, with different notions of types over a context $(X_0 \in \mathcal{C}, X_1 \in \mathcal{D}, u\colon X_1 \to FX_0)$:

- the *levelwise* structure $(\mathcal{E} \downarrow N)_{\mathsf{lvl}}$, in which a type is a triple (Y_0, Y_1, v) of types $Y_0 \in \mathsf{Ty}(X_0)$ and $Y_1 \in \mathsf{Ty}(X_1)$ and a map $v\colon X_1.Y_1 \to FX_0.FY_0$ over u (i.e. a map $Y_1 \to FY_0[u]$ in the fibrant slice over X_1),
- the *Reedy* structure $(\mathcal{E} \downarrow N)_{\mathsf{rd}}$, in which a type is a pair (Y_0, Y_1) of types $Y_0 \in \mathsf{Ty}(X)$ and $Y_1 \in \mathsf{Ty}(X.NY_0[u])$.

We have a pseudomorphism of cwf structures $(\mathcal{E} \downarrow N)_{\mathsf{rd}} \to (\mathcal{E} \downarrow N)_{\mathsf{lvl}}$. This is an actual morphism if we use telescopes as types in $\mathcal{D}$. It is essentially surjective on types up to identity types in $\mathcal{D}$, stably under substitution.

The Reedy structure corresponds to the gluing construction [KHS19], there presented with a slightly different category of contexts. The Reedy structure is well adapted to interpret type formers found in $\mathcal{C}$ and $\mathcal{D}$, the levelwise structure is generally not. Of note, we have dependent products of Reedy types with levelwise domain type. □

Lemma 1.5. *Let N be a comonad pseudomorphism on a cwf $\mathcal{E}$. We have a fully faithful functor $\mathrm{Coalg}(N) \to \mathcal{E} \downarrow N$. This extends to a cwf morphism that is bijective on terms. Here, we use the levelwise cwf structure on $\mathcal{E} \downarrow N$.*

Proof. The first statement is folklore and probably more known in the dual situation of monads. The functor sends a coalgebra $(X, u\colon X \to NX)$ to the object $(X, X, u\colon X \to NX)$ of the comma category. It is easy to see that this is fully faithful.

Let us extend this functor to a cwf morphism. We send a type (Y, v) over (X, u) to the type (Y, Y, v) over $(X, X, u\colon X \to NX)$. This is evidently stable under substitution and it is clear that context extension is preserved strictly and naturally.

Under the equivalence of cwas and cwfs, this suffices to construct the desired cwf morphism. Alternatively, we could have given just as directly the action on terms and associated operations. Finally, the cwf morphism is bijective on terms because the underlying functor is fully faithful. □

Definition 1.6 (Right adjoint on types)**.** Let $L\colon \mathcal{C} \to \mathcal{D}$ be a pseudomorphism of cwfs. A *right adjoint on types* R consists of the following operations, natural in $\Gamma \in \mathcal{C}$:

- for $A \in \mathsf{Ty}_{\mathcal{D}}(L\Gamma)$, a type $RA \in \mathsf{Ty}_{\mathcal{C}}(\Gamma)$,
- for $t \in \mathsf{Tm}_{\mathcal{D}}(L\Gamma, A)$, a term $Rt \in \mathsf{Tm}_{\mathcal{C}}(\Gamma, RA)$,

such that the action on terms is invertible. □

Recall the "stacking" monoidal structure on cwf structures on a category $\mathcal{C}$. Given two such cwf structures S and T, their monoidal product $S \bullet T$ has types $\mathsf{Ty}_{S \bullet T}(X)$ given by pairs (A_S, A_T) where $A_S \in \mathsf{Ty}_S(X)$ and $A_T \in \mathsf{Ty}_T(X.A_S)$. The extension of X by (A_S, A_T) is given by $X.A_S.A_T$. Let $\top$ denote the unit of this monoidal structure.

Lemma 1.7. *Let $L\colon \mathcal{C} \to \mathcal{D}$ be a functor.*

(i) The pseudomorphism $L\colon (\mathcal{C}, \top) \to (\mathcal{D}, \top)$ of cwfs has a right adjoint on types.

(ii) Given extensions of L to pseudomorphisms $L\colon (\mathcal{C}, S_{\mathcal{C}}) \to (\mathcal{C}, S_{\mathcal{D}})$ and $L\colon (\mathcal{C}, S_{\mathcal{C}}) \to (\mathcal{C}, S_{\mathcal{D}})$ of cwfs with right adjoints on types, the induced pseudomorphism $L\colon (\mathcal{C}, S_{\mathcal{C}} \bullet T_{\mathcal{C}}) \to (\mathcal{D}, S_{\mathcal{D}} \bullet T_{\mathcal{D}})$ of cwfs has a right adjoint on types.

Furthermore, these constructions are suitably functorial.

Proof. The right adjoint on types in item (i) is uniquely determined.

For item (ii), consider $X \in \mathcal{C}$ with $(B_S, B_T) \in \mathsf{Ty}_{S\bullet T}(LX)$, i.e. $B_S \in \mathsf{Ty}_S(LX)$ and $B_T \in \mathsf{Ty}_T(LX.B_S)$. We have $RB_S \in \mathsf{Ty}_S(X)$. In context $X.RB_1$, we have a term of type $RB_1[p] = R(B_1[Lp])$. Using the dependent right adjoint, this gives a term of type $B_1[Lp]$ in context $L(X.RB_1)$. This induces a morphism $f\colon L(X.RB_1) \to LX.B_1$ over LX, which we may think of as the relative counit at B_1 over LX. This gives us $R(B_T[f]) \in \mathsf{Ty}(X.RB_1)$. We now define $R(B_S, B_T) = (RB_S, R(B_T[f]))$.

Let (b_S, b_T) be a term of (B_S, B_T). This means $b_S \in \mathsf{Tm}(LX, B_S)$ and $b_T \in \mathsf{Tm}(LX, B_t[\mathrm{id}_X, b_S])$.

We get $Rb_S \in \mathsf{Tm}(X, RB_S)$ and $Rb_T \in \mathsf{Tm}(X, R(B_t[\mathrm{id}_X, b_S]))$. We need $\mathsf{Tm}(X, R(B_T[f])[\mathrm{id}, Rb_S])$. But $R(B_T[f])[\mathrm{id}, Rb_S] = R(B_T[f][L(\mathrm{id}, Rb_S)])$. □

Given a cwf $(\mathcal{C}, \mathsf{Ty})$, we have the induced cwf structure Ty^* of telescpes on $\mathcal{C}$. This is the free monoid on Ty in the monoidal category of cwf structures of $\mathcal{C}$, with monoidal product given by "stacking" of cwf structures. Telescope formation forms a monad on cwfs that lives over the identity monad on categories. It also forms a 2-monad on the 2-category of cwfs (where the 1-arrows are pseudomorphisms).

Lemma 1.8. *Telescope formation extends to a monad on the category of pseudomorphisms of cwfs with right adjoint.*

Proof. □

Let $L\colon (\mathcal{C}, \mathsf{Ty}_{\mathcal{C}}) \to (\mathcal{D}, \mathsf{Ty}_{\mathcal{D}})$ be a pseudomorphism of cwfs with a right adjoint on types R. Consider the induced pseudomorphism $L^*\colon (\mathcal{C}, \mathsf{Ty}^*_{\mathcal{C}}) \to (\mathcal{D}, \mathsf{Ty}^*_{\mathcal{D}})$. We have to define a right adjoint on types R^* of L^*.

Let $X \in \mathcal{C}$ and $[B_1, ..., B_n] \in \mathsf{Ty}^*_{\mathcal{D}}(LX)$.

Lemma 1.9. *Let $L\colon \mathcal{C} \to \mathcal{D}$ be a pseudomorphism of cwfs with a right adjoint on types R. Then the induced functor $L\colon \mathcal{C}_{\mathsf{fib}} \to \mathcal{D}_{\mathsf{fib}}$ has a right adjoint.*

Proof. We have to show that $L \downarrow [B_1, ..., B_n]$ has an initial object.

□

Definition 1.10 (Cwf structure created by right adjoint on types)**.** Let $L\colon \mathcal{C} \to \mathcal{D}$ be a pseudomorphism of cwfs with a right adjoint on types R. Then the cwf structure $\mathcal{C}'$ *created by* L on $\mathcal{C}$ is defined as follows. We again view cwfs under the cwa lens to simplify the exposition.

The types of $\mathcal{C}'$ are defined as created by L. To define the cwf structure on $\mathcal{C}'$ and make $\mathcal{C}' \to \mathcal{C}$ into a morphism of cwf structures is equivalent to just give the action of types of $\mathcal{C}' \to \mathcal{C}$. Given $\Gamma \in \mathcal{C}$, we send $A \in \mathsf{Ty}(L\Gamma)$ to $RA \in \mathsf{Ty}(\Gamma)$. This is evidently stable under substitution.

Almost by construction, F lifts to an oplax cwf morphism $\mathcal{C}' \to \mathcal{D}$ bijective on types. The oplax action on context extension comes from the comparison map $L(\Gamma.RA) \to L\Gamma.A$ over $L\Gamma$. □

Lemma 1.11. *Let $L\colon \mathcal{C} \to \mathcal{D}$ be a pseudomorphism of cwfs with a right adjoint on types R. Then the cwf structure $\mathcal{C}'$ created by L has function types.*

Proof. The input data for a function type in $\mathcal{C}'$ consists of $\Gamma \in \mathcal{C}$ and $A, B \in \mathsf{Ty}(L\Gamma)$. Then $RA \in \mathsf{Ty}(\Gamma)$ and $LRA \in \mathsf{Ty}(L\Gamma)$. We define the function type $\hom(A, B) \in \mathsf{Ty}(\Gamma)$ in $\mathcal{C}'$ as the function type $\hom(LRA, B)$ in context $L\Gamma$ in $\mathcal{D}$. Terms in $\mathcal{C}'$ of this function type are by definition terms of $R(\hom(LRA, B))$ in $\boldsymbol{\mathcal{C}}$, which correspond to terms of $\hom(LRA, B)$ in $\mathcal{D}$, which in turn correspond to terms of $B[p]$ in context $L\Gamma.LRA$ in $\mathcal{D}$.

Want: terms of type $RB[p]$ in context $\Gamma.RA$. Want: terms of type $B[p]$ in context $L(\Gamma.RA)$.

The input data for a dependent product in $\mathcal{C}'$ consists of $\Gamma \in \mathcal{C}$, $A \in \mathsf{Ty}(L\Gamma)$, and $B \in \mathsf{Ty}(L(\Gamma.RA))$.

LΓ.LRA → L(Γ.RA)?

Let $B' \in \mathsf{Ty}(L\Gamma.LRA)$ denote the substitution of B along the coherence isomorphism for context extension. We define the dependent product $\Pi(A, B) \in \mathsf{Ty}(\Gamma)$ in $\mathcal{C}'$ as the dependent product $\Pi(LRA, B')$ in context $L\Gamma$ in $\mathcal{D}$. Terms in $\mathcal{C}'$ of this dependent product □

Lemma 1.12. *Let N be a good comonad pseudomorphism on a cwf $\mathcal{E}$ in the sense of* **??**. *Then the cwf structure* $\mathrm{Coalg}(N)_{\mathsf{rd}}$ *created by L has dependent products.*

Proof. Fix a context

□

2. Example: Pédrot's presheaf translation

Definition 2.1. Given a cwf $\mathcal{C}$ and context $\Gamma \in \mathcal{C}$, we have the cwf $[\Gamma, \mathcal{C}]$ of Γ-indexed types. It is simply the fibrant slice over Γ, so another notation for it would be $(\mathcal{C}/\Gamma)_{\mathsf{fib}}$. Its underlying category is the full subcategory of $\mathcal{C}/\Gamma$ on fibrant objects, i.e. types over Γ. □

If one wants to be generous and work around dependent sums in $\mathcal{C}$, one could also let the contexts be telescopes over Γ.

Definition 2.2. Let $\mathcal{C}$ be a cwf. A *Reedy fibrant internal category* in $\mathcal{C}$ is a category object whose underlying graph is Reedy fibrant, i.e. presented by $C_0 \in \mathsf{Ty}(1)$ and $C_1 \in \mathsf{Ty}(1.C_0.C_0)$. □

We also write $s, t \colon C_1 \to C_0$ for the category object. Here, C_0 stands for $1.C_0$ and C_1 stands for $1.C_0.C_0.C_1$. Source and target maps are given by projection to the first and second component, respectively.

Remark 2.3. Recall that categories may be characterized as monads in the bicategory of spans. The internal variant of this says that category objects in a category $\mathcal{C}$ with pullbacks are monads in the bicategory of spans in $\mathcal{C}$. For a cwf $\mathcal{C}$, we only have pullbacks along context projections of types, so we compensate by adding Reedy fibrancy (technically, the weaker requirement of the source and target maps being fibrations, i.e. presented by types, seems sufficient). Then a Reedy fibrant internal category is a monad in the bicategory of Reedy fibrancy spans in $\mathcal{C}$. □

Lemma 2.4. *Let C be a Reedy fibrant internal category in a cwf $\mathcal{C}$. We have an induced comonad (N, ε, ν) on $[C_0, \mathcal{C}]$ that forms a pseudomorphism of cwfs.*

Proof. Any Reedy fibrant span

$$\begin{tikzcd} & S \arrow[dl, two heads, "f"'] \arrow[dr, two heads, "g"] & \\ A & & B \end{tikzcd}$$

induces a pseudomorphism

$$[A, \mathcal{C}] \xrightarrow{\Delta_f} [S, \mathcal{C}] \xrightarrow{\Pi_g} [B, \mathcal{C}].$$

This construction is contravariantly functorial: a morphism

$$\begin{tikzcd} & S_0 \arrow[dl, two heads, "f_0"'] \arrow[dr, two heads, "g_0"] \arrow[dd, "h"] & \\ A & & B \\ & S_1 \arrow[ul, two heads, "f_1"] \arrow[ur, two heads, "g_1"'] & \end{tikzcd}$$

of spans over A and B induces a natural transformation

$$\Pi_{g_1}\Delta_{f_1} \longrightarrow \Pi_{g_0}\Delta_{f_0}.$$

The desired pseudomorphism N is given by this construction applied to the span of the category object:

$$[C_0, \mathcal{C}] \xrightarrow{\Delta_t} [C_1, \mathcal{C}] \xrightarrow{\Pi_s} [C_0, \mathcal{C}].$$

The monad structure on C in the bicategory of Reedy fibrant spans induces the comonad structure on N. □

Remark 2.5. Note that the comonad N constructed in Lemma 2.4 is right adjoint to the monad pseudomorphism T given by the composite

$$[C_0, \mathcal{C}] \xrightarrow{\Delta_s} [C_1, \mathcal{C}] \xrightarrow{\Sigma_t} [C_0, \mathcal{C}].$$

This is a cartesian monad. In particular, the diagram

$$\begin{tikzcd} \mathrm{Id} \arrow[r, "\eta"] & T \arrow[r, shift left=2, "T\eta"] \arrow[r, shift right=2, "\eta T"'] & T^2 \arrow[l, "\mu" description] \end{tikzcd}$$

is a coreflexive equalizer (this is equivalent to

$$\begin{tikzcd} \mathrm{Id} \arrow[r, "\eta"] \arrow[d, "\eta"'] & T \arrow[d, "T\eta"] \\ T \arrow[r, "\eta T"] & T^2 \end{tikzcd} \tag{2.1}$$

being a pullback). □

Lemma 2.6. *The pseudomorphism* $L\colon \mathrm{Coalg}(N) \to (\mathcal{E} \downarrow N)_{\mathsf{rd}}$ *has a right adjoint on types.*

Proof. Let $(X, u\colon X \to NX)$ be a coalgebra. Let (Y_0, Y_1) be a type over $(X, X, u\colon X \to NX)$ in the Reedy cwf structure on $\mathcal{E} \downarrow N$. Recall this means $Y_0 \in \mathsf{Ty}(X)$ and $Y_1 \in \mathsf{Ty}(X.NY_0[u])$. We define $R(Y_0, R_1) = (Y', v\colon X.Y \to N(X.Y))$ as follows. Let $Y' \in \mathsf{Ty}(X)$ be the dependent sum of $NY_0[u]$ and the base change of NY_1 along a coherence isomorphism, u, and the local comultiplication at Y_0 (over the given coalgebra) such that it becomes a type over $X.NY_0[u]$. Then v can be defined in a canonical way. This is stable under substitution.

Given a term (t_0, t_1) of type (Y_0, Y_1), we define $R(t_0, t_1)$ as the pairing of $Nt_0[u]$ and $Nt_1[u]$. This is also stable under substitution.

It is mechanical to verify the properties of a right adjoint on types. □

Proof. Consider the following endofunctor K on $\mathcal{E} \downarrow N$. An object $Z = (X_0, X_1, f\colon X_1 \to NX_0)$ is sent to the object $KZ = (X_1, X_2, g\colon X_2 \to NX_1)$ where

$$\begin{tikzcd} X_2 \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & NX_1 \arrow[d, "Nf"] \\ NX_0 \arrow[r, "\nu_{X_0}"] & N^2X_0. \end{tikzcd}$$

The functorial action is given by naturality of ν and functoriality of pullbacks. This endofunctor has a canonical copointing c, with $c_Z = (\varepsilon_{X_0} \circ f, \varepsilon_{X_1} \circ g)$. For this, note that the diagram

$$\begin{tikzcd}
X_2 \arrow[r, "g"] \arrow[d, "g"'] & NX_1 \arrow[r, "\varepsilon_{X_1}"] & X_1 \arrow[d, "f"] \\
NX_1 \arrow[r, "Nf"] & N^2X_0 \arrow[r, "N\varepsilon_{X_0}"] & NX_0
\end{tikzcd}$$

commutes: both composites reduce to q. Naturality of the copointing follows from naturality of ε.

Is this functor wellcopointed? For this, we need $c_{KZ} = Kc_Z$. Write $K^2Z = (X_2, X_3, h\colon X_3 \to NX_2)$ where

$$\begin{tikzcd}
X_3 \arrow[r, "h"] \arrow[d, "r"'] \arrow[dr, phantom, very near start, "\lrcorner"] & NX_2 \arrow[d, "Ng"] \\
NX_1 \arrow[r, "\nu_{X_1}"] & N^2X_1.
\end{tikzcd}$$

Then $c_{KZ} = (\varepsilon_{X_1} \circ g, \varepsilon_{X_2} \circ h)$ and $Kc_Z = (\varepsilon_{X_1} \circ g, s)$ where s is the map induced as follows:

$$\begin{tikzcd}
X_3 \arrow[rr, "h"] \arrow[dr, dotted, "s"] \arrow[dd, "r"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & NX_2 \arrow[dr, "N(\varepsilon_{X_1}\circ g)"] \arrow[dd, "Ng", near start] & \\
& X_2 \arrow[rr, "g", near start] \arrow[dd, "q", near start] \arrow[dr, phantom, very near start, "\lrcorner"] & & NX_1 \arrow[dd, "Nf"] \\
NX_1 \arrow[rr, "\nu_{X_1}", near start] \arrow[dr, "N(\varepsilon_{X_0}\circ f)"'] & & N^2X_1 \arrow[dr, "N^2(\varepsilon_{X_0}\circ f)"'] & \\
& NX_0 \arrow[rr, "\nu_{X_0}"'] & & N^2X_0.
\end{tikzcd}$$

For $c_{KZ} = Kc_Z$, we need $s = \varepsilon_{X_2} \circ h$. This means:

$$N(\varepsilon_{X_1} \circ g) \circ h = g \circ \varepsilon_{X_2} \circ h$$
$$N(\varepsilon_{X_0} \circ f) \circ r = q \circ \varepsilon_{X_2} \circ h.$$

For the first equation, we have

$$\begin{aligned}
N(\varepsilon_{X_1} \circ g) \circ h &= N\varepsilon_{X_1} \circ Ng \circ h \\
&= N\varepsilon_{X_1} \circ \nu_{X_1} \circ r \\
&= r
\end{aligned}$$

and

$$\begin{aligned}
g \circ \varepsilon_{X_2} \circ h &= \varepsilon_{NX_1} \circ Ng \circ h \\
&= \varepsilon_{NX_1} \circ \nu_{X_1} \circ r \\
&= r.
\end{aligned}$$

For the second equation, we have

$$\begin{aligned}
N(\varepsilon_{X_0} \circ f) \circ r &= N\varepsilon_{X_0} \circ Nf \circ r \\
&= N\varepsilon_{X_0} \circ Nf \circ \varepsilon_{NX_1} \circ \nu_{X_1} \circ r \\
&= N\varepsilon_{X_0} \circ Nf \circ \varepsilon_{NX_1} \circ Ng \circ h \\
&= N\varepsilon_{X_0} \circ Nf \circ g \circ \varepsilon_{X_2} \circ h \\
&= N\varepsilon_{X_0} \circ \nu_{X_0} \circ q \circ \varepsilon_{X_2} \circ h \\
&= q \circ \varepsilon_{X_2} \circ h.
\end{aligned}$$

Since K is wellpointed, an algebra for it is simply an object Z as before such c_Z is invertible. This means that $\varepsilon_{X_0} \circ f$ and $\varepsilon_{X_1} \circ g$ are invertible.

□

Proof. Let $(X_0, X_1, f \colon X_1 \to NX_0)$ be an object of $\mathcal{E} \downarrow N$. We claim that its coreflection in $\mathrm{Coalg}(N)$ is given by (Y, u) where Y is given by the pullback

$$\begin{tikzcd}
Y \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & NX_1 \arrow[d, "Nf"] \\
NX_0 \arrow[r, "\nu_{X_0}"] & N^2X_0
\end{tikzcd}$$

and $u \colon Y \to NY$ is the induced map in the diagram

$$\begin{tikzcd}
Y \arrow[rr, "p"] \arrow[dr, dotted] \arrow[dd, "q"'] & & NX_1 \arrow[dr, "\nu_{X_1}"] \arrow[dd, "Nf" near start] & \\
& NY \arrow[rr] \arrow[dd] & & N^2X_1 \arrow[dd, "N^2 f"] \\
NX_0 \arrow[rr, "\nu_{X_0}" near start] \arrow[dr, "\nu_{X_0}"'] & & N^2X_0 \arrow[dr, "\nu_{NX_0}"'] & \\
& N^2X_0 \arrow[rr, "N\nu_{X_0}"'] & & N^3X_0.
\end{tikzcd}$$

□

Proof. This is a consequence of the properties examined in Remark 2.5. In particular, from (2.1) it follows that we have the following pullback:

$$\begin{tikzcd}
N \arrow[r, "\nu"] \arrow[d, "\nu"'] & N^2 \arrow[d, "N\nu"] \\
N^2 \arrow[r, "\nu N"] & N^3.
\end{tikzcd} \tag{2.2}$$

[The following is the categorical argument. Relativize so that it works with types.] Via Lemma 1.5, we may see coalgebras of N as fully faithfully embedded in $\mathcal{E} \downarrow N$ via L. Our goal is to construct a cofree coalgebra on any object of $\mathcal{E} \downarrow N$. For this, we follow the well-known categorical construction [Kel80]. Coalgebras for N may be identified with copointed endofunctor coalgebras for the well-copointed endofunctor K on $\mathcal{E} \downarrow N$ sending $(X_0, X_1, f_0 \colon X_1 \to NX_0)$ to $(X_1, X_2, f_1 \colon X_2 \to$

NX_1) where

$$\begin{tikzcd} X_2 \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & NX_1 \arrow[d, "Nf_0"] \\ NX_0 \arrow[r, "\nu_{X_0}"] & N^2X_0. \end{tikzcd}$$

Thus, our goal becomes to construct cofree coalgebras for K. Since it is well-copointed, these are given by iteratively applying K and taking cosequential limits at limit ordinals. However, from (2.2) being a pullback, it can be seen that this process stabilizes already after two steps. Thus, no infinite limits are required.

Let $Z_n = (X_n, X_{n+1}, f_n \colon X_{n+1} \to NX_n)$ be the result of n-many times applying K to the starting object $(X_0, X_1, f_0 \colon X_1 \to NX_0)$. We argue that the transition map $Z_3 \to Z_2$ is invertible.

We use that N preserves pullbacks to illuminate the construction of X_3:

$$\begin{tikzcd} X_3 \arrow[r, "f_2"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & NX_2 \arrow[r] \arrow[d, "Nf_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & N^2X_0 \arrow[d, "N\nu_{X_0}"] \\ NX_1 \arrow[r, "\nu_{X_1}"] & N^2X_1 \arrow[r, "N^2f_0"] & N^3X_0. \end{tikzcd}$$

Rewriting the bottom map using naturality of ν and using the pullback square (2.2), we have

$$\begin{tikzcd} X_3 \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & NX_0 \arrow[r, "\nu_{X_0}"] \arrow[d, "\nu_{X_0}"] \arrow[dr, phantom, very near start, "\lrcorner"] & N^2X_0 \arrow[d, "N\nu_{X_0}"] \\ NX_1 \arrow[r, "Nf_0"] & N^2X_0 \arrow[r, "\nu_{NX_0}"] & N^3X_0. \end{tikzcd}$$

This is equivalent to just $\varepsilon_{X_2} \circ f_2$ being invertible (it then follows that also $\varepsilon_{X_3} \circ f_3$ is invertible; these two maps make up $Z_3 \to Z_2$). □

From $\mathcal{C}$, we may build the intermediate cwf $[C_0, \mathcal{C}]$ of C_0-indexed types. Its underlying category is the full subcategory of $\mathcal{C}/1.C_0$ on fibrant objects, i.e. types over $1.C_0$. The rest of its structure is completely inherited from $\mathcal{C}$.

Lemma 2.7. *Let (N, ε, ν) be a comonad. Then the following is an (absolute?) pullback square:*

$$\begin{tikzcd} N \arrow[r, "\nu"] \arrow[d, "\nu"'] & N^2 \arrow[d, "N\nu"] \\ N^2 \arrow[r, "\nu N"] & N^3. \end{tikzcd} \tag{2.3}$$

Proof. We may equivalently show that

$$\begin{tikzcd} N \arrow[r, "\nu"] & N^2 \arrow[r, shift left=2, "N\nu"] \arrow[r, shift right=2, "\nu N"'] & N^3 \arrow[l, "N\varepsilon N" description] \end{tikzcd}$$

is a coreflexive equalizer. □

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