COMPARING FIBRATIONS IN CARTESIAN CUBICAL SETS

1. Basic setting

1.1. **The category.** We do some elementary analysis of presheaves over the cartesian cube category $\square$. This is the subcategory of posets with objects of the form $[1]^I$ with I finite and maps $f\colon [1]^J \to [1]^I$ of the form

$$f(x)_i = \begin{cases} x_j & \text{if } u(i) = \tau_2(j), \\ k & \text{if } u(i) = \tau_1(k) \end{cases} \tag{1.1}$$

for $x \in [1]^J$ and $i \in I$, given some $u\colon I \to \{0,1\} + J$. (So it is also the opposite of the Kleisli category of the monad $2 + (-)$ on **Set**.) Note that the inclusion $\square \to$ **Poset** creates finite products.

1.2. **Fibrations.** The notion of fibration in the cartesian cubical model of type theory is as follows. For distinction, we call it *cartesian fibraton.*

Definition 1.1. A *cartesian fibration structure* on a map $Y \to X$ consists of:

- for any $[1]^I \in \square$, $j \in \mathbb{I}([1]^I)$ $\varphi \in \Omega([1]^I)$, $u\colon \square^{I \sqcup \{i\}} \to X$, $v\colon \ulcorner(i = j) \vee \varphi\urcorner \to Y$ making the diagram

$$\begin{tikzcd} \square^I \arrow[r, "v"] \arrow[d, "{[\mathrm{id}, i]}"'] & Y \arrow[d] \\ \square^I \times \mathbb{I} \arrow[r, "u"] \arrow[ur, dotted] & X \end{tikzcd}$$

  commute, a diagonal filler $d_{I,i,\varphi,u,v}$ as indicated.
- for any (I, i, φ, u, v) and $(I', i', \varphi', u', v')$ as above and a map $f\colon I' \to I$ such that $j' = jf$, $\varphi' = \varphi f$, $u' = u(f \times \mathbb{I})$, $v' = v(f \times \mathbb{I})$, coherence of diagonals fillers $d_{I,i,\varphi,u,v}$ and $d_{I',i',\varphi',u',v'}$ as indicated below:

$$\begin{tikzcd} \square^{I'} \arrow[r, "f \times \mathbb{I}"] \arrow[d, "{[\mathrm{id}, i']}"'] & \square^I \arrow[r, "v"] \arrow[d, "{[\mathrm{id}, i]}"'] & Y \arrow[d] \\ \square^{I'} \times \mathbb{I} \arrow[r, "f \times \mathbb{I}"] \arrow[urr, dotted] & \square^I \times \mathbb{I} \arrow[r, "u"] \arrow[ur, dotted] & X. \end{tikzcd}$$

  We call $Y \to X$ a *cartesian fibration* if it admits a fibration structure.

Using the language of extensional type theory in $\widehat{\square}$, we may write Y as a type over X. The above definition of a cartesian fibration structure then arises as the semantic unfolding of the set of terms of the type

$$\begin{gathered} (j : \mathbb{I}) \times (\varphi : \Omega) \times (u : X^{\mathbb{I}}) \times (v : (i : \mathbb{I}) \to \ulcorner(i = j) \vee \varphi\urcorner \to Y(u(i))) \\ \vdash (i : \mathbb{I}) \to ((d : Y(u(i))) \times ((x : \ulcorner(i = j) \vee \varphi\urcorner) \to d = v(i, x))), \end{gathered}$$

which may be seen as justifying the above definition of cartesian fibration structure from the syntactical point of view of cartesian cubical type theory.

After exponential transposition and separation of the equational constraint on d, we see that a term as above corresponds to a diagonal lift as below:

$$\begin{tikzcd}
(j : \mathbb{I}) \times (\varphi : \Omega) \times (u : X^{\mathbb{I}}) \times (v : (i : \mathbb{I}) \to \ulcorner (i = j) \vee \varphi \urcorner \to Y(u(i))) \times (i : \mathbb{I}) \times \ulcorner (i = j) \vee \varphi \urcorner \arrow[r] \arrow[d] & Y \arrow[d] \\
(j : \mathbb{I}) \times (\varphi : \Omega) \times (u : X^{\mathbb{I}}) \times (v : (i : \mathbb{I}) \to \ulcorner (i = j) \vee \varphi \urcorner \to Y(u(i))) \times (i : \mathbb{I}) \arrow[r, "\lambda(j,\varphi,u,vi).u(i)"] \arrow[ur, dotted, "d"] & X.
\end{tikzcd} \tag{1.2}$$

We call this the *universal lifting problem* for the map $Y \to X$. It demonstrates that (as usual in these situations), cartesian fibration structures as defined as a coherent families of diagonal fillers are in fact in bijection with certain diagonal fillers of a single lifting problem of the same shape.

Lemma 1.2. *A map $Y \to X$ is a cartesian fibration exactly if it lifts against all unions*

$$\begin{tikzcd}
B +_A A \times \mathbb{I} \arrow[d] \\
B \times \mathbb{I}
\end{tikzcd}$$

of $[\mathrm{id}, k] \colon B \to B \times \mathbb{I}$ *and* $m \times \mathbb{I}$ *where* $B \in \widehat{\square}$, $k \colon B \to \mathbb{I}$, *and* $m \colon A \to B$ *is mono.*

Proof. For the direction from right to left, observe that the left map of the universal lifting problem (1.2) is the union of $[\mathrm{id}, k] \colon A \to A \times \mathbb{I}$ and $m \times \mathbb{I}$ with

$$\begin{aligned}
A &=_{\mathrm{def}} (j : \mathbb{I}) \qquad\qquad\qquad \times (u : X^{\mathbb{I}}) \times (v : (i : \mathbb{I}) \to \ulcorner (i = j) \qquad \urcorner \to Y(u(i))), \\
B &=_{\mathrm{def}} (j : \mathbb{I}) \times (\varphi : \Omega) \times (u : X^{\mathbb{I}}) \times (v : (i : \mathbb{I}) \to \ulcorner (i = j) \vee \varphi \urcorner \to Y(u(i))),
\end{aligned}$$

$m(j, u, v) =_{\mathrm{def}} (j, \top, u, v)$, and $k(j, \varphi, u, v) =_{\mathrm{def}} j$.

For the direction from left to right, note that any lifting problem

$$\begin{tikzcd}
B +_A A \times \mathbb{I} \arrow[r, "v"] \arrow[d] & Y \arrow[d] \\
B \times \mathbb{I} \arrow[r, "u"] & X
\end{tikzcd}$$

factors through the universal lifting problem (1.2) via the map

$$B \to (j : \mathbb{I}) \times (\varphi : \Omega) \times (u : X^{\mathbb{I}}) \times (v : (i : \mathbb{I}) \to \ulcorner (i = j) \vee \varphi \urcorner \to Y(u(i)))$$

sending b to $(k(b), m^{-1}(b), \lambda i.u(b, i), \lambda i.[\lambda p.v(\tau_1(b)), \lambda x.v(\tau_2(b))])$. $\square$

Let d be the unit of the adjunction $\mathbb{I}_! \dashv \mathbb{I}^*$ at the terminal object, i.e. the diagonal $\mathbb{I} \to \mathbb{I} \times \mathbb{I}$ seen as a map in $\widehat{\square}/\mathbb{I}$ as below:

$$\begin{tikzcd}
\mathbb{I} \arrow[rr, "\langle \mathrm{id}, \mathrm{id} \rangle"] \arrow[dr, "\mathrm{id}"'] & & \mathbb{I} \times \mathbb{I} \arrow[dl, "\pi_1"] \\
& \mathbb{I}. &
\end{tikzcd}$$

Lemma 1.3. *Consider a monomorphism $m \colon A \to B$, a map $k \colon B \to \mathbb{I}$, and a map $p \colon Y \to X$ in $\widehat{\square}$. The following are equivalent:*

(i) the map $B +_A A \times \mathbb{I} \to B \times \mathbb{I}$ of Lemma 1.2 lifts against p,

(ii) the pushout product $m \mathbin{\widehat{\times}} d$ lifts against the image $\mathbb{I}^* p$ of $Y \to X$ in $\widehat{\square}/\mathbb{I}$,

(iii) the mono m lifts against the pullback exponential $\widehat{\exp}(d, \mathbb{I}^* p)$ in $\widehat{\square}/\mathbb{I}$.

Proof. The pushout product $m \mathbin{\widehat{\times}} d$ in $\widehat{\square}/\mathbb{I}$ evaluates to the dotted map in

$$\begin{tikzcd}
(A,k') \arrow[r, "\langle \mathrm{id},k'\rangle"] \arrow[d, "m"'] & (A\times\mathbb{I}, k'\pi_1) \arrow[d] \arrow[ddr, bend left, "m\times A"] & \\
(B,k) \arrow[r] \arrow[drr, bend right, "\langle\mathrm{id},k\rangle"'] & \bullet \arrow[dr, dotted, "m\widehat{\times}d"] & \\
& & (B\times\mathbb{I}, k\pi_1).
\end{tikzcd}$$

By adjunction, lifting problems of $m \mathbin{\widehat{\times}} d$ against I^*p correspond to lifting problems from the underlying map of $m \mathbin{\widehat{\times}} d$ against p. Since $\mathbb{I}_!$ creates pushouts, we see from the above diagram that the underlying map of $m \mathbin{\widehat{\times}} d$ is precisely the map $B +_A A \times \mathbb{I}$ of Lemma 1.2 induced by $m \colon A \to B$ and $k \colon B \to \mathbb{I}$. This shows the equivalence between (i) and (ii).

The equivalence between (ii) and (iii) is standard Leibniz adjunction nonsense. $\square$

A map in $\widehat{\square}$ (and its slices) is called a trivial fibration if it lifts against all monomorphisms.

Corollary 1.4. *A map $Y \to X$ is a cartesian fibration exactly if $\widehat{\exp}(d, \mathbb{I}^*p)$ is a trivial fibration in $\widehat{\square}/\mathbb{I}$.* $\square$

Corollary 1.5. *In the condition of Lemma 1.2, it suffices to restrict to the case where $m \colon A \to B$ is a map $\partial([1]^I)/G \to \square^I/G$ of the cellular model.* $\square$

We write $\mathcal{F}$ for the class of cartesian fibrations.

Corollary 1.6. *The wfs $({}^{\pitchfork}\mathcal{F}, \mathcal{F})$ is cofibrantly generated.* $\square$

1.3. **Model structures.** Let $\mathcal{M}$ denote the class of all monomorphisms. The existence of fibrant dependent products and a fibrant universe for cartesian fibrations gives us a model structure $(\mathcal{M}, \mathcal{W}, \mathcal{F})$ for a certain class of weak equivalences $\mathcal{W}$. By Corollary 1.6, this is a Cisinski model structure. Using Cisinski's methods, we can also generate this model structure from the interval object $\mathbb{I}$ and the class of anodyne extensions ${}^{\pitchfork}\mathcal{F}$; this will give the same cofibrations and fibrant objects, by Joyal's argument hence be the same model structure, also showing it to be complete in Cisinski's sense.

Let $(\mathcal{M}, \mathcal{W}_{\min}, \mathcal{F}_{\min})$ denote the *minimal* Cisinski model structure on the interval object $\mathbb{I}$ (with anodyne extensions generated by $\{\delta_0, \delta_1\} \mathbin{\widehat{\times}} \mathcal{M}$). Note that $\mathcal{F} \subseteq \mathcal{F}_{\min}$, hence $\mathcal{W}_{\min} \subseteq \mathcal{W}$.

Lemma 1.7. *We have $\mathcal{M} \cap \mathcal{W} \subseteq \mathcal{W}_{\min}$.*

Proof. Recall from Lemma 1.3 that $\mathcal{M} \cap \mathcal{W}$ is generated by $\mathbb{I}_!(m \mathbin{\widehat{\times}} d)$ with m mono in $\widehat{\square}/\mathbb{I}$. It will thus suffice to show that (the underlying map of) $m \mathbin{\widehat{\times}} d$ belongs to $\mathcal{W}_{\min}$. This follows from closure of $\mathcal{M} \cap \mathcal{W}_{\min}$ under pushout and 2-out-of-3 for $\mathcal{W}_{\min}$ if we can show that the monomorphism $X \times d$ belongs to $\mathcal{W}_{\min}$ for $X \in \widehat{\square}/\mathbb{I}$. Note that d has a section $\pi_1 \colon (\mathbb{I} \times \mathbb{I}, \pi_1) \to (\mathbb{I}, \mathrm{id})$. By 2-out-of-3, it will suffice to show that $X \times \pi_1 \in \mathcal{W}_{\min}$, i.e. that $\pi_1 \colon X \times \mathbb{I} \to X$ belongs to $\mathcal{W}_{\min}$. This is the case as it has the anydone extension $X \times \delta_0$ as section. $\square$

Corollary 1.8. *We have $(\mathcal{M}, \mathcal{W}_{\min}, \mathcal{F}_{\min}) = (\mathcal{M}, \mathcal{W}, \mathcal{F})$.*

Proof. Since $\mathcal{M} \cap \mathcal{W} \subseteq \mathcal{M} \cap \mathcal{W}_{\min}$, we have $\mathcal{F}_{\min} \subseteq \mathcal{F}$, i.e. $\mathcal{F}_{\min} = \mathcal{F}$. Since the classes of cofibrations and fibrations coincide, the model structures must be equal. $\square$

In particular, the minimal Cisinski model structure on the interval object $\mathbb{I}$ is complete in Cisinski's sense.

References

[BR13] J. Bergner and C. Rezk. Reedy categories and the Θ-construction. *Mathematische Zeitschrift*, 274(1):499–514, 2013.