FROM FIBRATIONS IN SEMISIMPLICIAL SETS TO UNIFORM FIBRATIONS IN SYMMETRIC SIMPLICIAL SETS

Abstract. In this constructive note, we show that a (trivial) Kan fibration between Kan fibrant objects with levelwise decidable equality in semisimplicial sets gives rise via right Kan extension to a uniform (trivial) fibration in (symmetric) simplicial sets.

1. Introduction

For a constructive proof of homotopy canonicity of homotopy type theory, we desire a glueing functor valued in a constructive model. Examples of constructive models are provided by the CCHM model construction, developed in [CCHM18] for the particular case of de Morgan cubical sets, but having instances more generally in presheaf categories with an interval object with connections and an object of cofibrant propositions satisfying the axioms of [OP17] in which product with the interval preserves representables. Note that simplicial sets fail to validate these assumptions for the standard choice of Δ^1 as the interval, but only because the simplex category is not closed under product with [1]. We remedy this by switching to the symmetric simplex category, and obtain a constructive model of homotopy type theory in *symmetric simplicial sets* (with the CCHM parameters of the interval given by Δ^1 and the cofibrant propositions given by decidable sieves).[1] Interestingly, the homotopy theory of this model does not give standard homotopy types, but this does not appear to be an obstacle to the use of the model for homotopy canonicity.

It remains to establish the glueing functor. As in the classical proof of homotopy canonicity, the source model will be homotopical semisimplicial diagrams in the initial model $\mathcal{M}$, and the first factor of the glueing functor will be the semisimplicial global sections functor Γ, which sends (contractible) context projections to (trivial) Kan fibrations in semisimplicial sets. Also as before, $\mathcal{M}$ is contextual, so Γ is valued in Kan complexes. By a normalization argument (itself proceeding via a glueing construction), equality of morphisms in $\mathcal{M}$ is decidable. This implies that Γ is valued in semisimplicial sets with levelwise decidable equality. With these

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[1]This is equivalent to the CCHM model in Boolean cubical sets.

observations, we construct the remaining part of the glueing functor as follows:

(trivial) fibration between fibrant objects with levelwise decidable equality	Semisimplicial sets
↓ Propositions 3.3 and 3.4	↓ Ran
degeneracy-uniform (trivial) fibration	Simplicial sets
↓ Propositions 3.5 and 3.7	‖
uniform (trivial) fibration	Simplicial sets
↓ Propositions 3.8 and 3.9	↓ Ran
uniform (trivial) fibration	Symmetric simplicial sets

The main subtlety lies in the first step. Intuitively, a degeneracy-uniform fibration in simplicial sets is a map with specified lifts against horn inclusions that are degenerate whenever the lifting problem allows it. As already used in the classical homotopy canonicity proof, a fibration $Y \to X$ between fibrant objects in semisimplicial sets gives rise to a fibration $i_* Y \to i_* X$ in simplicial sets via right Kan extension along the inclusion $i : \Delta_+ \to \Delta$ from the semisimplex to the simplex category. In order to convert this to a degeneracy-uniform fibration, one would like to perform case distinction in a given lifting problem of $\Lambda^n_k \to \Delta^n$ against $i_* Y \to i_* X$ on whether a degenerate lift is possible and use it if the answer is positive. This involves testing a finite number of elements of $i_* X$ and $i_* Y$ for degeneracy. However, transposing this to semisimplicial sets, an n-simplex of $i_* X$ consists of a countably infinite family of elements of X indexed over $\Delta_+ \downarrow [n]$, and even with the assumption that X has levelwise decidable equality we cannot in general test if such a family factors via $\Delta_+ \downarrow [n] \to \Delta_+ \downarrow [k]$ for a degeneracy map $[n] \to [k]$.

The solution lies in the realization that the transposed diagonal filler $i^* \Delta^n \to Y$ is similarly a countably infinite family of elements of Y. Decomposing $i^* \Lambda^n_k \to i^* \Delta^n$ into a countably infinite sequence of maps with finite complement that each lift against fibrations between fibrant objects, we construct the diagonal filler step by step, at each stage making a case distinction testing if the truncation to some finite dimension of the original lifting problem in semisimplicial sets admits a degenerate lift, and if so, use it. In this fashion, the degeneracy check (which in total involves infinitely many equality checks) is interleaved in a corecursive fashion with the construction of the diagonal filler.

2. Preliminaries

An injection $A \to B$ is *decidable* if it is complemented, i.e. there is $\overline{A} \to B$ such that $A + \overline{A} \to B$ is iso. A set A has *decidable equality* if the diagonal inclusion $A \to A \times A$ is decidable.

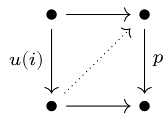
Two objects in a category with maps back and forth between them are *logically equivalent*.

Given a category $\mathcal{C}$, we write $\widehat{\mathcal{C}}$ for the category of presheaves over $\mathcal{C}$. Given a functor $F : \mathcal{C} \to \mathcal{D}$, we have an adjoint triple

$$\widehat{\mathcal{C}} \underset{F_*}{\overset{F_!}{\rightleftarrows}} \widehat{\mathcal{D}}, \qquad F_! \dashv F^* \dashv F_*, \quad F^* : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$$

where F^* denotes restriction along F.

We use the algebraic lifting notation of [Gar09]. Given a category $\mathcal{C}$ with a category $u \colon I \to \mathcal{C}^{\to}$ of arrows in $\mathcal{C}$, we write $I^{\pitchfork}$ for the category of maps $p \in \mathcal{C}^{\to}$ equipped with a right lifting operation $I \pitchfork p$, i.e. a family of lifts



against arrows given by $i \in I$ such that the provided lifts are coherent with respect to the morphisms of I. In the special case that I is discrete, we obtain a category $I^{\pitchfork}$ of maps in $\mathcal{C}$ equipped with lifts against the family of arrows I. We stress that, in contrast to the classical use in homotopy theory of $I^{\pitchfork}$ as a subclass of maps of $\mathcal{C}$ with a lifting property, the objects of $I^{\pitchfork}$ include the lifts as data. All our notions of fibrations are defined using the algebraic right lifting closure operation, hence for us being a fibration is always structure on the underlying map rather than just a property. Whenever we write that some map is a fibration, we mean that we have a fibration structure on that map. With this convention, there will never be any need for non-constructive choice in our development.

2.1. **Simplicial sets.** We write Δ for the simplex category, a skeleton of the category of non-empty finite total orders. The objects of Δ are written $[n] = \{0 < ... < n\}$ for $n \geq 0$. The category of simplicial sets is $\widehat{\Delta}$. The representable on $[n]$ is written Δ^n. We have the familiar Reedy factorization system of face maps and degeneracy maps on Δ; this makes Δ into an elegant Reedy category [BR13].

Given $n \in \{0, 1, ..., \infty\}$, we write Sk^n for the skeleton idempotent comonad, taken to be the identity for $n = \infty$. We also use this notation for the skeleton in semisimplicial sets and symmetric simplicial sets below.

2.1.1. *(Trivial) fibrations.* Let I be the set of boundary inclusions $\partial\Delta^n \to \Delta^n$. A *trivial fibration* is an element of $I^{\pitchfork}$. Let J be the set of horn inclusions $\Lambda^n_k \to \Delta^n$. A *fibration* is an element of $J^{\pitchfork}$.

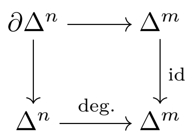
2.1.2. *Uniform (trivial) fibrations.* Let I_{uniform} be the subcategory of arrows whose objects are levelwise decidable inclusions into a representable and whose morphisms are pullback squares. A *uniform trivial fibration* is an element of $I^{\pitchfork}_{\mathsf{uniform}}$. Let

$$J_{\mathsf{uniform}} = \{\{k\} \hookrightarrow \Delta^1 \mid k = 0, 1\} \mathbin{\widehat{\times}} I_{\mathsf{uniform}}.$$

A *uniform fibration* is an element of $J^{\pitchfork}_{\mathsf{uniform}}$.

Let $\mathcal{M}_{\mathsf{dec}}$ be the subcategory of levelwise decidable inclusions. Then the set of uniform trivial fibration structures on a map p is logically equivalent to $\mathcal{M}_{\mathsf{dec}} \pitchfork p$, and similarly the set of uniform fibration structures on p is logically equivalent to $\{\{k\} \hookrightarrow \Delta^1 \mid k = 0, 1\} \mathbin{\widehat{\times}} \mathcal{M}_{\mathsf{dec}} \pitchfork p$ (see for example [GS17]).

2.1.3. *Degeneracy-uniform (trivial) fibrations.* Let I_{deg} be the subcategory of arrows whose objects are boundary inclusions $\partial\Delta^n \to \Delta^n$ and identities $\Delta^n \to \Delta^n$ and whose non-identity morphisms are of the form



where the bottom map is a non-invertible degeneracy map. A *degeneracy-uniform trivial fibration* is an element of $I^{\pitchfork}_{\mathsf{deg}}$. Let J_{deg} be the subcategory of arrows whose objects are horn inclusions $\Lambda^n_k \to \Delta^n$ and identities $\Delta^n \to \Delta^n$ and whose non-identity morphisms are of the form

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \Delta^m \\ \big\downarrow & & \big\downarrow \mathrm{id} \\ \Delta^n & \xrightarrow{\text{deg.}} & \Delta^m \end{array}$$

where the bottom map is a non-invertible degeneracy map. A *degeneracy-uniform fibration* is an element of $J^{\pitchfork}_{\mathsf{deg}}$. Note that a degeneracy-uniform (trivial) fibration is a (trivial) fibration that satisfies additional properties on the given lifts. Also note that the top map in the above two diagrams is epi.

2.2. **Semisimplicial sets.** We write Δ_+ for the semisimplex category, the wide subcategory of Δ of monomorphisms. The category of semisimplicial sets is $\widehat{\Delta_+}$. We write $i\colon \Delta_+ \to \Delta$ for the evident inclusion.

A semisimplicial set A is called *finite* if its set of elements is finite. In that case, its *dimension* is the maximal n such that A_n is inhabited (or -1 if none is).

Let I be the set of boundary inclusions $\partial\Delta^n \to \Delta^n$. A *trivial fibration* is an element of $I^{\pitchfork}$. Let J be the set of horn inclusions $\Lambda^n_k \to \Delta^n$. A *fibration* is an element of $J^{\pitchfork}$. Note that $I = i_! I$ and $J = i_! J$, so i^* preserves (trivial) fibrations.

2.3. **Symmetric simplicial sets.** We write Δ_{sym} for the symmetric simplex category, a skeleton of the category of non-empty finite sets. The objects of Δ_{sym} are written $[n] = \{0, ..., n\}$ for $n \geq 0$. The category of symmetric simplicial sets is $\widehat{\Delta_{\mathsf{sym}}}$. The representable on $[n]$ is written Δ^n.

We have an evident inclusion $j\colon \Delta \to \Delta_{\mathsf{sym}}$ that forgets the ordering of a non-empty finite total order. Note that $j^*\Delta^n$ is 0-coskeletal on points $0, ..., n$. Following standard convention, we also denote it $E[n]$.

Uniform (trivial) fibrations in symmetric simplicial sets are defined just as in simplicial sets. In detail, let I_{uniform} be the subcategory of arrows whose objects are levelwise decidable inclusions into a representable and whose morphisms are pullback squares. A *uniform trivial fibration* is an element of $I^{\pitchfork}_{\mathsf{uniform}}$. Let

$$J_{\mathsf{uniform}} = \{\{k\} \hookrightarrow \Delta^1 \mid k = 0, 1\} \mathbin{\widehat{\times}} I_{\mathsf{uniform}}.$$

A *uniform fibration* is an element of $J^{\pitchfork}_{\mathsf{uniform}}$.

Let $\mathcal{M}_{\mathsf{dec}}$ be the subcategory of levelwise decidable inclusions. As for simplicial sets, uniform trivial fibration structures on a map p are logically equivalent to $\mathcal{M}_{\mathsf{dec}} \pitchfork p$, and uniform fibration structures on p are logically equivalent to $\{\{k\} \hookrightarrow \Delta^1 \mid k = 0, 1\} \mathbin{\widehat{\times}} \mathcal{M}_{\mathsf{dec}} \pitchfork p$.

3. Statements

Lemma 3.1. *For $n \geq 0$, the map $i^*\partial\Delta^n \to i^*\Delta^n$ is a relative cell complex of height ω of maps, between finite semisimplicial sets, that lift against trivial fibrations.*

Proof. The map in question is levelwise a decidable inclusion and its target $i^*\Delta^n$ has finitely many elements at every level. Thus, the map presents as a relative cell complex of countably many boundary inclusions. □

Lemma 3.2. *For $n \geq 1$ and $0 \leq k \leq n$, the map $i^*\Lambda^n_k \to i^*\Delta^n$ is a relative cell complex of height ω of maps, between finite semisimplicial sets, that lift against fibrations between fibrant objects.*

Proof. Given $f\colon [m] \to [n]$, let A_f denote the subobject of $i^*\Delta^n$ given by the union of $i^*\Lambda^n_k$ with the image of $f\colon \Delta^m \to i^*\Delta^n$. This defines a diagram A of subobjects of $i^*\Delta^n$ over $i^*\Lambda^n_k$ indexed over $\Delta_+ \downarrow [n]$. Let $\mathcal{C}$ be the full subcategory of $\Delta_+ \downarrow [n]$ on those objects $f\colon [m] \to [n]$ for which $f^{-1}(k)$ is of odd cardinality. Note that $\mathcal{C}$ inherits from Δ_+ the structure of a direct category of height ω and has finite width (finitely many objects of each degree). Using standard Reedy technology, our goal will thus be proven once we verify that the latching object inclusions

$$L_f(A|_{\mathcal{C}}) \to A_f \tag{3.1}$$

for $f \in \mathcal{C}$ are pushouts of maps, between finite semisimplicial sets, that lift against fibrations between fibrant objects.

Let $f \in \mathcal{C}$ with $f\colon [m] \to [n]$. In the augmented simplex category, we have a unique decomposition $f = g_1 \star ! \star g_2$ with $g_1\colon [n_1] \to [k-1]$, $!\colon [v] \to [0]$, $g_2\colon [n_2] \to [n-k-1]$ where v is even. We may suppose g_1 and g_2 are epi for otherwise the inclusion $i^*\Lambda^n_k \to A_f$ is invertible and hence so is (3.1). By inspecting which elements of $i^*\Delta^n$ are present in A_f but not in $L_f(A|\mathcal{C})$, we observe that (3.1) is a pushout of the map h given by the pushout join, computed in augmented semisimplicial sets, of $\partial\Delta^{n_1} \to \Delta^{n_1}$, $\mathrm{Sk}^v(1) \to \mathrm{Sk}^{v-2}(1)$, and $\partial\Delta^{n_2} \to \Delta^{n_2}$. Observe that h has finite source and target.

- If $v = 0$, then h is the horn inclusion $\Lambda^m_{n_1} \to \Delta^m$, thus lifts against fibrations by definition.
- If $v \neq 0$, we may describe h as the pushout join in semisimplicial sets of $\partial\Delta^{n_1} \to \Delta^{n_1}$, $\mathrm{Sk}^v(1) \to \mathrm{Sk}^{v-2}(1)$, and $\partial\Delta^{n_2} \to \Delta^{n_2}$ where the first and last argument is omitted if $n_1 = -1$ or $n_2 = -1$, respectively. We now work with cofibrations and weak equivalences in semisimplicial sets as in [Sat18]. Clearly $\mathrm{Sk}^v(1) \to \mathrm{Sk}^{v-2}(1)$ is a cofibration. Since v is even, it is a weak equivalence by [Sat18, Lemma 3.62]. By the second part of [Sat18, Corollary 3.51], the pushout join h is then also a cofibration and weak equivalence, By [Sat18, Corollary 3.26], it then lifts against fibrations between fibrant objects. □

Proposition 3.3. *Let $Y \to X$ be a trivial fibration in semisimplicial sets between objects with levelwise decidable equality. Then $i_*Y \to i_*X$ is a degeneracy-uniform trivial fibration in simplicial sets.*

Proof. Under the adjunction $i^* \dashv i_*$, we have to construct a diagonal filler in any commuting square

$$\begin{array}{ccc} i^*\partial\Delta^n & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ i^*\Delta^n & \longrightarrow & X \end{array} \tag{3.2}$$

as indicated such that whenever the horizontal maps factor as in

$$\begin{array}{ccccc} i^*\partial\Delta^n & \longrightarrow & i^*\Delta^k & \longrightarrow & Y \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ i^*\Delta^n & \longrightarrow & i^*\Delta^k & \longrightarrow & X \end{array} \tag{3.3}$$

for a non-identity degeneracy maps $s\colon [n] \to [k]$, the constructed filler coheres with the trivial filler in the right square.

Let $L_n\Delta$ denote the latching category of Δ at level n, the full subcategory of $\Delta\backslash[n]$ restricted to non-invertible degeneracy maps $[n] \to [k]$. Note that $L_n\Delta$ forms a finite poset (of quotient maps) and has binary coproducts, which are computed as pushouts of spans of degeneracy maps in Δ and preserved by Yoneda.

Given $m \in \{0, 1, ..., \infty\}$ and $s\colon [n] \to [k]$ in $L_n\Delta$, we say that s *witnesses degeneracy at level* m if the given lifting problem (3.2) with left map restricted to m-skeleta factors as follows:

$$\begin{array}{ccccc} \mathrm{Sk}^m(i^*\partial\Delta^n) & \xrightarrow[\text{epi}]{} & \mathrm{Sk}^m(i^*\Delta^k) & \dashrightarrow & Y \\ \downarrow & & \downarrow{\simeq} & & \downarrow \\ \mathrm{Sk}^m(i^*\Delta^n) & \xrightarrow{\text{epi}} & \mathrm{Sk}^m(i^*\Delta^k) & \dashrightarrow & X. \end{array} \tag{3.4}$$

Those $s\colon [n] \to [k]$ witnessing degeneracy at level m form a finite subposet of $L_n\Delta$. Observe that it is downwards closed. We say there is *degeneracy at level* m if it is inhabited. Note that if there is degeneracy at level m, then also at all lower levels.

Note that for a pushout

$$\begin{array}{ccc} [n] & \longrightarrow & [k_1] \\ \downarrow & & \downarrow \\ [k_2] & \longrightarrow & [k_3] \end{array} \tag{3.5}$$

of a span of non-invertible degeneracy maps in Δ, the induced square

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^{k_1} \\ \downarrow & & \downarrow \\ \Delta^{k_2} & \longrightarrow & \Delta^{k_3} \end{array} \tag{3.6}$$

of epimorphisms in simplicial sets is also a pushout (just observe that the skeleton functor Sk^{n-1} in simplicial sets preserves pushouts). Since Sk^m and i^* preserve pushouts, the subposet of objects of $L_n\Delta$ witnessing degeneracy at level m is thus closed under binary coproducts. If it is inhabited, it hence is connected. If there is degeneracy at level m, we then obtain from invertibility of the middle vertical map in (3.4) a diagonal filler

$$\begin{array}{ccc} \mathrm{Sk}^m(i^*\partial\Delta^n) & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Sk}^m(i^*\Delta^n) & \longrightarrow & X. \end{array} \tag{3.7}$$

that does not depend on the choice of witnessing degeneracy s. Furthermore, this diagonal filler coheres with the corresponding one at any level lower than m (in particular also with the solid composite diagonal filler in the situation of (3.3)).

For $m \geq 0$ not ∞, note that $\mathrm{Sk}^m(i^*\Delta^n)$ and $\mathrm{Sk}^m(i^*\partial\Delta^n)$ are finite. Since X and Y have levelwise decidable equality by assumption, the factorization problem (3.4) is decidable for any s in $L_n\Delta$. Since $L_n\Delta$ is finite, degeneracy at level m is decidable.

As per Lemma 3.1, let

$$A_0 \longrightarrow A_1 \longrightarrow ...$$

be a presentation of the left map in (3.2) an ω-cell complex of maps $A_i \to A_{i+1}$ that are pushouts of maps $S_i \to T_i$, between finite semisimplicial sets, that lift against trivial fibrations. Let $v(i)$ denote the maximal dimension of $T_0, ..., T_{i-1}$ (and -1 if $i = 0$). By induction on $i \geq 0$, we construct a family of maps $A_i \to Y$ fitting into diagrams

$$\begin{array}{ccc} A_i & \dashrightarrow & Y \\ \downarrow & & \downarrow \\ i^*\Delta^n & \longrightarrow & X, \end{array} \qquad \begin{array}{ccc} A_j & \dashrightarrow & Y \\ \downarrow & \nearrow & \\ A_i & & \end{array}$$

for $j < i$ such that if there is degeneracy at level $m \geq v(i)$, then the diagram

$$\begin{array}{ccc} \mathrm{Sk}^m(A_i) & \longrightarrow & \mathrm{Sk}^m(i^*\Delta^n) \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & Y \end{array} \tag{3.8}$$

(with right map given by the diagonal filler of (3.7)) commutes; we call the latter condition *degeneracy-uniformity at stage i*.

The base case $i = 0$ is given by the top map of (3.2) with degeneracy-uniformity given by the upper left triangle of (3.7). In the induction step, we have to produce a diagonal filler

$$\begin{array}{ccccc} S_i & \longrightarrow & A_i & \longrightarrow & Y \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ T_i & \longrightarrow & A_{i+1} & \longrightarrow & X \end{array} \tag{3.9}$$

such that the induced map $A_{i+1} \to Y$ satisfies degeneracy-uniformity at stage $i+1$. We test for degeneracy at level $v(i+1)$. If there is none, we construct the diagonal filler in (3.9) using the provided lift of $S_i \to T_i$ against the trivial fibration $Y \to X$, and degeneracy-uniformity holds vacuously. If there is, we define the diagonal filler in (3.9) as the composite

$$T_i \longrightarrow \mathrm{Sk}^{v(i+1)}(i^*\Delta^n) \longrightarrow Y.$$

The upper triangle in (3.9) commutes by degeneracy-uniformity (3.8) at stage i with $m = v(i+1)$. The lower triangle in (3.9) commutes by the lower triangle of (3.7). Verifying degeneracy-uniformity at stage $i+1$ and level $m \geq v(i+1)$, we use the presentation of $\mathrm{Sk}^m(A_{i+1})$ as a pushout of $\mathrm{Sk}^m(A_i)$ and T_i to reduce the claim for the first coprojection to degeneracy-uniformity at stage i and level $m \geq v(i)$ and for the second coprojection to the remark after (3.7) on coherence of that diagonal filler between different levels.

The family of maps $A_i \to Y$ finally assembles to a diagonal filler in (3.2). For degeneracy-uniformity, we observe that coherence of diagonal fillers in the situation of (3.3) (where we have degeneracy at level $m = \infty$) restricted to A_i of is precisely the degeneracy-uniformity condition at stage i and level $m = \infty$. □

Proposition 3.4. *Let $Y \to X$ be a fibration in semisimplicial sets between fibrant objects with levelwise decidable equality. Then $i_*Y \to i_*X$ is a degeneracy-uniform fibration in simplicial sets.*

Proof. This is a verbatim copy of the proof of Proposition 3.3 with the following modifications.

- We lift against a horn $\Lambda^n_t \to \Delta^n$ instead of a boundary inclusion $\partial\Delta^n \to \Delta^n$.
- The use of Lemma 3.1 is replaced by a use of Lemma 3.2, making use of the extra assumption that X and Y are fibrant.

- For a non-invertible degeneracy map $s\colon [n] \to [k]$, the induced map $\Lambda^n_t \to \Delta^k$ is epi since s has at least two face map sections and there is only one face missing in Λ^n_t.
- We argue manually that the square

$$\begin{array}{ccc} \Lambda^n_t & \longrightarrow & \Delta^{k_1} \\ \downarrow & & \downarrow \\ \Delta^{k_2} & \longrightarrow & \ulcorner\Delta^{k_3} \end{array}$$

replacing (3.6) is a pushout in the situation of the pushout (3.5) of a span of non-invertible degeneracy maps. [Do more abstractly or cite.] By pushout pasting, it suffices to show this when $[n] \to [k_1]$ and $[n] \to [k_2]$ are generating degeneracies s_a and s_b, respectively. The case $a = b$ is trivial, so let us without loss of generality suppose $a < b$. Reducing the claim to the pushout square given by Yoneda applied to (3.5). we need to show that the simplicial equivalence relation $\sim$ on $\Delta^{k_1} + \Delta^{k_2}$ generated by

$$\tau_1(s_a d_i) \sim \tau_2(s_b d_i) \tag{3.10}$$

for $d_i\colon [n-1] \to [n]$ with $i \neq t$ already identifies $\tau_1(s_a)$ and $\tau_2(s_b)$. Pick $b' \in \{b, b+1\}$ different from t. We have

$$\begin{aligned} \tau_1(s_a) &= \tau_1(s_a d_a s_a) \sim \tau_2(s_b d_a s_a) \\ &= \tau_2(s_b d_a s_a d_{b'} s_b) \sim \tau_1(s_a d_a s_a d_{b'} s_b) \\ &= \tau_1(s_a d_{b'} s_b) \sim \tau_2(s_b d_{b'} s_b) = \tau_2(s_b) \end{aligned}$$

using (3.10) for $i = a, b'$. This derives $\tau_1(s_a) \sim \tau_2(s_b)$ in case $t \neq a$. Under the symmetry $(-)^{\mathrm{op}}\colon \Delta \to \Delta$, we obtain an analogous derivation of $\tau_1(s_a) \sim \tau_2(s_b)$ in case $t \neq b+1$. Together, this covers all cases. ☐

Proposition 3.5. *In simplicial sets, given a degeneracy-uniform trivial fibration $Y \to X$, we have a uniform trivial fibration structure on $Y \to X$.*

Proof. Present Y as a presheaf over $\int X$. The given degeneracy-uniform trivial fibration structure consists of an operation $q(x, y) \in Y(x)$ for $x \in X_n$ and a coherent family $y_f \in Y(xf)$ for $f\colon [k] \to [n]$ non-surjective such that $q(x, y)f = y_f$ for f as before that additionally for $s\colon [n'] \to [n]$ a non-invertible degeneracy map satisfies $q(xs, y') = ys$ where $x \in X_n$, $y \in Y(x)$, and $y'_f = ysf$ for $f\colon [k'] \to [n']$ non-surjective.

Fixing n, note that the category of non-surjective $f\colon [k] \to [n]$ has a final full subcategory consisting of non-identity face maps $d\colon [k] \to [n]$. We may thus equivalently take the family y in $q(x, y)$ to just be indexed over the latter category.

It will suffice to lift $Y \to X$ to an object of $\mathcal{M}^{\Uparrow}_{\mathsf{dec}}$. For this, we need to solve a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\;y\;} & Y \\ \downarrow & {\scriptstyle \bar{q}} \nearrow & \downarrow \\ B & \xrightarrow[\;x\;]{} & X \end{array} \tag{3.11}$$

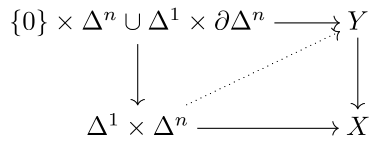
where the left map is levelwise a decidable inclusion. We define $\bar{q}(b) \in X_n$ for $b \in B_n$ by induction on $n \geq 0$. If b lies in A_n, we let $\bar{q}(b) = y(b_n)$. Otherwise, let $\bar{q}(b) = q(x(b), y)$ where $y_d = \bar{q}(bd)$ for $d\colon [k] \to [n]$ non-identity face map.

This defines levelwise maps forming a diagonal filler in (3.11). Note that $\bar{q}$ is natural with respect to face maps by construction. It remains to check naturality with respect to a degeneracy map $s\colon [n'] \to [n]$. This is proven by induction on n'. Given $b \in B_n$, the goal $\bar{q}(bs) = \bar{q}(b)s$ unfolds to $q(x(bs), y) = \bar{q}(b)s$ where

$y_{f'} = \bar{q}(bsd')$ for $d' \colon [k'] \to [n']$ non-surjective face map. It remains to show $\bar{q}(bsd') = \bar{q}(b)sd'$ for then the goal follows from degeneracy-uniformity of q. For this, we consider the Reedy factorization $sd' = ds'$ and use naturality of q with respect to the face map d and the degeneracy map s', the latter given by induction hypothesis. □

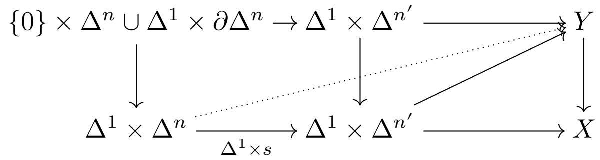
Lemma 3.6. *In simplicial sets, given a degeneracy-uniform fibration $Y \to X$ and $k \in \{0, 1\}$, then the pullback exponential with $\{k\} \hookrightarrow \Delta^1$ of $Y \to X$ is a degeneracy-uniform trivial fibration.*

Proof. We only do the case $k = 1$, the other case being analogous. We need to produce a diagonal filler in any commuting square



(3.12)

as indicated such that whenever the horizontal maps factor as in



(3.13)

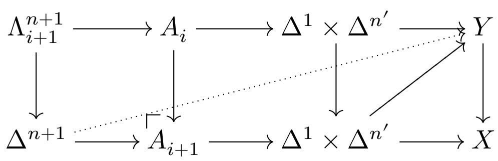
for a non-identity degeneracy maps $s \colon [n] \to [n']$, the constructed filler coheres with the trivial filler in the right square.

To construct the diagonal filler in (3.12), we use the standard presentation [GZ67, IV.2.1.1] of the open prism inclusion on the left as a relative cell complex of horn inclusions. For $0 \leq i \leq n+1$, let A_i be the union of $\{0\} \times \Delta^n \cup \Delta^1 \times \partial\Delta^n$ with the simplicial subset of $\Delta^1 \times \Delta^n$ consisting of all elements $[k] \to [1] \times [n]$ whose image consists of pairs (a, b) with $a = 1$ or $b < i$. Then

$$A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_{n+1} \tag{3.14}$$

provides a cellular presentation of the left map in (3.12) where the step $A_i \to A_{i+1}$ is a pushout of $\Lambda^{n+1}_{i+1} \to \Delta^{n+1}$ where the relevant map $\Delta^{n+1} \to \Delta^1 \times \Delta^n$ sends j to $(0, j)$ for $j \leq i$ and to $(1, j-1)$ for $j > i$. This induces the lift in (3.12) via the given degeneracy-uniform fibration structure.

Consider now the situation of (3.13) for a non-identity degeneracy map $s \colon [n] \to [n']$. Our goal will be to show that the diagonal filler constructed in the previous step makes the diagram commute. With respect to the cellular presentation (3.14), we show by induction on i that the partial diagonal filler $A_i \to Y$ factors via $\Delta^1 \times \Delta^{n'} \to Y$. In the induction step, we have to show that the dotted filler



constructed using the degeneracy-uniform fibration structure makes the diagram commute. This follows from degeneracy-uniformity as the pasting of the left and middle square decomposes as

$$\begin{array}{ccccc}
\Lambda^{n+1}_{i+1} & \longrightarrow & \Delta^{n'+1} & \longrightarrow & \Delta^1 \times \Delta^{n'} \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^{n+1} & \xrightarrow{\bar{s}} & \Delta^{n'+1} & \xrightarrow{t} & \Delta^1 \times \Delta^{n'}
\end{array}$$

where $\bar{s}\colon [n+1] \to [n'+1]$ is the non-identity degeneracy map sending j with $j \leq i$ to $s(j)$ and to $s(j-1)+1$ otherwise, and $t\colon [n'+1] \to [1] \times [n']$ sends j' with $j' \leq s(i)$ to $(0, j')$ and to $(1, j'-1)$ otherwise. □

Proposition 3.7. *In simplicial sets, given a degeneracy-uniform fibration $Y \to X$, we have a uniform fibration structure on $Y \to X$.*

Proof. Note that a uniform fibration structure on $Y \to X$ coincides with uniform trivial fibration structures on the pullback exponentials of $Y \to X$ with $\{k\} \hookrightarrow \Delta^1$ for $k = 0, 1$. With Lemma 3.6, the claim thus reduces to Proposition 3.5. □

Proposition 3.8. *Given a uniform trivial fibration $Y \to X$ in simplicial sets, then $j_* Y \to j_* X$ is a uniform trivial fibration in symmetric simplicial sets.*

Proof. This follows under the adjunction $j^* \dashv j_*$ from the trivial fact that j^* preserves levelwise decidable inclusions. □

Proposition 3.9. *Given a uniform fibration $Y \to X$ in simplicial sets, then $j_* Y \to j_* X$ is a uniform fibration in symmetric simplicial sets.*

Proof. Given a levelwise decidable inclusion $A \to B$ in symmetric simplicial sets, we have to lift the pushout product of $\{k\} \hookrightarrow \Delta^1$ and $A \to B$ against $j_* Y \to j_* X$ for $k = 0, 1$. We only do the case $k = 1$, the other case being analogous. Under the adjunction $j^* \dashv j_*$ and using bicontinuity of j^*, the claim transposes to lifting the pushout product of $j^*\{0\} \to j^*\Delta^1$ and $j^*A \to j^*B$ against $Y \to X$. Note that $j^*A \to j^*B$ is a levelwise decidable inclusion in simplicial sets.

The map $j^*\{0\} \to j^*\Delta^1$ presents as a relative cell complex of height ω consisting at stage $n \geq 1$ of the horn inclusion $\Lambda^n_0 \to \Delta^n$ where $\Delta^n \to j^*\Delta^1$ is the map $j[n] \to [1]$ sending i to its remainder after dividing by 2. The horn $\Lambda^n_0 \to \Delta^n$ is a retract of the pushout product of itself with $\{0\} \hookrightarrow \Delta^1$ [GZ67, IV.2.1.3] and also a levelwise decidable inclusion. Using cocontinuity of the pushout product in its two arguments, associativity of the pushout product, and closure of $\mathcal{M}_{\mathsf{dec}}$ under pushout product, the original left map in simplicial sets thus writes as an ω-composite of maps in $\{\delta_0, \delta_1\} \mathbin{\widehat{\times}} \mathcal{M}_{\mathsf{dec}}$, each lifting against $Y \to X$. □

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