Fibrancy of universe from equivalence extension

Abstract

This expository note concerns the cartesian cubical model of type theory.[1] We explain how to derive fibrancy of the universe from the equivalence extension property in the style of working "on the right", that is, not explicitly looking at lifting problems against trivial cofibrations, and not using internal languages.

1 Setting and assumptions

1.1 Local cartesian closure

For a map $f\colon A \to B$, we write $f_! \dashv f^* \dashv f_*$ for the adjunctions between slices induced by f. Note that $f_! \dashv f^*$ is a cartesian adjunction: both functors preserve pullbacks and the unit and counit are cartesian. In the special case that B is the terminal object, we identify the slice over B with the underlying category and just write $A_! \dashv A^* \dashv A_*$.

For an object I, the unit of $I_! \dashv I^*$ evaluated at the terminal object is called the *generic element inclusion* $g_I\colon 1 \to I^*I$ over I. Its underlying map is the diagonal $I \to I \times I$.

Lemma 1.1. *The exponential action $(-)^{g_I}$ is isomorphic to the counit of $I^* \dashv I_*$.*

Proof. Passing from right to left adjoints, the claim becomes that $g_I \times -$ is isomorphic to the unit of $I_! \dashv I^*$. This expresses that the unit is cartesian. □

Lemma 1.2. *For an object I, the following functors on arrows are naturally isomorphic:*

(1) the composite of I^*, pullback exponential with g_I, and $I_!$,

(2) the composite of I^*, pullback application of the counit of $I^* \dashv I_*$, and $I_!$,

(3) pullback application of the counit of $I \times - \dashv (-)^I$.

Proof. This follows formally using "Leibniz calculus". For relating conditions (1) and (2), we use Lemma 1.1. For relating, Conditions (2) and (3), we observe that the adjunction $I \times - \dashv (-)^I$ is the composite of the adjunction $I_! \dashv I^*$ with the adjunction $I^* \dashv I_*$. The counit of $I_! \dashv I^*$ is cartesian, so has trivial pullback evaluation. □

1.2 Trivial fibrations

We assume trivial fibrations have been defined in the usual manner. We try to work in a manner that treats trivial fibrations as abstractly as possible, avoiding mentions of the cofibration classifier when possible.

Trivial fibrations are closed under the following operations:

- compositions,
- pullbacks,
- retracts,
- pushforward (f_* for any map $f\colon A \to B$)

Every trivial fibration admits a section.

[1] A previous shorter note applies to the connection-based cubical model.

1.3 Fibrations

We now fix a suitable interval object I. A map $p\colon Y \to X$ is a *fibration* if the pullback exponential with g_I of I^*p is a trivial fibration. We may re-express this condition more elegantly as follows.

Lemma 1.3. *Pullback evaluation of the counit of $I \times - \dashv (-)^I$ creates fibrations from trivial fibrations.*

Proof. Use the isomorphism of Lemma 1.2 between conditions (1) and (3). □

Spelled out, a map $p\colon Y \to X$ is a fibration exactly if the pullback gap map in the square

$$\begin{array}{ccc} I \times Y^I & \xrightarrow{\mathrm{ev}_Y} & Y \\ \big\downarrow{\scriptstyle I\times p^I} & & \big\downarrow{\scriptstyle p} \\ I \times X^I & \xrightarrow{\mathrm{ev}_X} & X \end{array}$$

is a trivial fibration.

Lemma 1.4. *Pullback exponential with $\langle \mathrm{id}, \mathrm{id}\rangle\colon I \to I \times I$ sends fibrations to trivial fibrations.*

Proof. Write $\widehat{\epsilon}$ for the pullback evaluation of the counit of $I^* \dashv I_*$. The operation in question decomposes as $I_* \circ \widehat{\epsilon} \circ I^*$. From Lemma 1.2, we know that $I_! \circ \widehat{\epsilon} \circ I^*$ sends fibrations to trivial fibrations. The claim follows since $I_!$ creates and I_* preserves trivial fibrations. □

The following statement requires diagonal cofibrations.

Lemma 1.5. *Every trivial fibration is a fibration.* □

1.4 Homotopy equivalences

Given an object Γ, the notion of *homotopy equivalence* $A \simeq B$ between fibrant objects A and B over Γ is defined in any of a variety of equivalent ways. This is a pullback-stable notion. We recall some of their closure properties.

Lemma 1.6. *Over any object Γ:*

(1) trivial fibrations are homotopy equivalences,

(2) homotopy equivalences admit inverses up to homotopy,

(3) homotopy equivalences are closed under composition.

1.5 Universe

We write $p\colon \tilde{\mathsf{U}} \to \mathsf{U}$ for a chosen fibration. It has to have sufficient closure properties for Lemma 1.9 to be derivable.

1.6 Equivalence classifier

We build a classifier for homotopy equivalences between fibers of p. This is a span

$$\mathsf{U} \xleftarrow{s_0} \mathsf{Equiv} \xrightarrow{s_1} \mathsf{U}$$

that is Reedy fibrant at its summit.

Lemma 1.7 (Fragment of classifying property)**.** *Consider Γ with $A_0, A_1\colon \Gamma \to U$ and a homotopy equivalence $(A_0)^*p \simeq (A_1)^*p$ over Γ. We have a filler as follows:*

$$\begin{array}{ccccc} & & \Gamma & & \\ & \swarrow{\scriptstyle A_0} & \vdots & {\scriptstyle A_1}\searrow & \\ \mathsf{U} & \xleftarrow[s_0]{} & \mathsf{Equiv} & \xrightarrow[s_1]{} & \mathsf{U} \end{array}$$

Corollary 1.8. *The relation* Equiv *on* U *is reflexive.*

Proof. Apply Lemma 1.7 with $\Gamma = U$ and $A_0 = A_1 = \mathrm{id}$ and the identity homotopy equivalence. □

The main technical work is hidden in the following statement. It depends on the definition of the homotopy equivalence classifier (which may ultimately be defined in terms of a contractibility classifier).

Lemma 1.9 (Equivalence extension property)**.** *s_1 is a trivial fibration.*

2 Equivalence extension

The goal of this section is to prove that $s_1 \colon \mathsf{Equiv} \to \mathsf{U}$ is a trivial fibration.

Here are some tools we expect to use:

- aligning for trivial fibrations and fibrations
- closure of trivial fibrations under pushforward
- switching back and forth between different representations of homotopy equivalences.

3 Fibrancy of universe

A *reflexive relation* Y on an object X consists of the following data:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\mathrm{id}} & \downarrow{\scriptstyle r} & \searrow^{\mathrm{id}} & \\ X & \xleftarrow[s_0]{} & Y & \xrightarrow[s_1]{} & X. \end{array} \tag{3.1}$$

Definition 3.1. Let Y be a reflexive relation on an object X. Consider the reflexive relation on $I \times X^I$ obtained by restricting the terminal relation on I. We say that Y has *generalized paths* if

$$\mathrm{ev}_X \colon I \times X^I \to X$$

lifts to a morphism of reflexive relations. □

Using the notation of (3.1), this definition unfolds to a morphism e fitting into the squares

$$\begin{array}{ccc} I \times I \times X^I & \xrightarrow{\pi_k \times X^I} & I \times X^I \\ \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle \mathrm{ev}_X} \\ Y & \xrightarrow{s_k} & X \end{array} \tag{3.2}$$

for $k = 0, 1$ and

$$\begin{array}{ccc} I \times X^I & \xrightarrow{\langle \mathrm{id}, \mathrm{id} \rangle \times X^I} & I \times I \times X^I \\ \Big\downarrow{\scriptstyle \mathrm{ev}_X} & & \Big\downarrow{\scriptstyle e} \\ X & \xrightarrow{r} & Y. \end{array} \tag{3.3}$$

Lemma 3.2. *Let Y be a reflexive relation on X (denoted as in* (3.1)*). If*

$$\langle s_0, s_1 \rangle \colon Y \to X \times X$$

is a fibration, then Y has generalized paths.

Proof. A map e satisfying (3.2) and (3.3) amounts to the following diagonal filler:

$$\begin{array}{ccccc} I \times X^I & \xrightarrow{\mathrm{ev}_X} & X & \xrightarrow{r} & Y \\ \Big\downarrow{\scriptstyle \langle \mathrm{id}, \mathrm{id} \rangle \times X^I} & & {\scriptstyle e} \nearrow & & \Big\downarrow{\scriptstyle \langle s_0, s_1 \rangle} \\ I \times I \times X^I & \xrightarrow[\langle \pi_0 \times X^I, \pi_1 \times X^I \rangle]{} & (I \times X^I) \times (I \times X^I) & \xrightarrow[\mathrm{ev}_X \times \mathrm{ev}_X]{} & X \times X. \end{array}$$

The left map is the pushout product of $\langle \mathrm{id}, \mathrm{id} \rangle \colon I \to I \times I$ with $0 \to X^I$. The right map is a fibration by assumption. By adjointness and Lemma 1.4, this lifting is equivalent to lifting $0 \to X^I$ against a trivial fibration. For this, we use that every trivial fibration admits a section. □

Lemma 3.3. *In the situation of Definition 3.1, read*

$$\begin{tikzcd} Y \arrow[r, "s_0"] \arrow[d, "s_1"'] & X \arrow[d] \\ X \arrow[r] & 1 \end{tikzcd}$$

as a horizontal morphism of arrows. Applying the functorial action of pullback evaluation of the counit ev *of the adjunction $I \times - \dashv (-)^I$ yields a split epimorphism.*

Proof. We introduce notation for the pullback evaluation of ev at s_1:

$$\begin{tikzcd} I \times Y^I \arrow[rrd, bend left, "I \times (s_1)^I"] \arrow[rd, dotted, "\widehat{\mathrm{ev}}_{s_1}"] \arrow[rdd, bend right, "\mathrm{ev}_Y"'] & & \\ & P \arrow[r, "\pi_0"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & I \times X^I \arrow[d, "\mathrm{ev}_X"] \\ & Y \arrow[r, "s_1"] & X \end{tikzcd}$$

The morphism of arrows in question is the solid part of the below diagram:

$$\begin{tikzcd} I \times X^I \arrow[r, dashed, "\langle \pi_0, \overline{e} \rangle"] \arrow[d, "\langle \pi_0, \mathrm{ev}_X \rangle"] & I \times Y^I \arrow[r, "I \times (s_0)^I"] \arrow[d, "\widehat{\mathrm{ev}}_{s_1}"] & I \times X^I \arrow[d, "\langle \pi_0, \mathrm{ev}_X \rangle"] \\ I \times X \arrow[r, dashed, "\langle I \times \overline{\pi_1}, r\pi_1 \rangle"] & P \arrow[r, "\langle \pi_0\pi_0, s_0\pi_1 \rangle"] & I \times X. \end{tikzcd}$$

For the indicated section, we make use of the map e after Definition 3.1 and write $\overline{(-)}$ for the transpose with respect to $I \times - \dashv (-)^I$.

- The bottom dashed map is well-defined because $\mathrm{ev}_X \circ (I \times \overline{\pi_1}) = \pi_1 = s_1 r \pi_1$.
- The bottom row composes to the identity because $s_0 r \pi_1 = \pi_1$.
- Let us check that the top row composes to the identity:

$$\begin{tikzcd} I \times X^I \arrow[r, "\overline{e}"] \arrow[rd, "\pi_1"'] & Y^I \arrow[d, "(s_1)^I"] \\ & X^I. \end{tikzcd}$$

  This transposes to

$$\begin{tikzcd} I \times I \times X^I \arrow[r, "e"] \arrow[d, "I \times \pi_1"'] & Y \arrow[d, "s_1"] \\ I \times X^I \arrow[r, "\mathrm{ev}_X"] & X, \end{tikzcd}$$

  which is (3.2) for $k = 0$.
- Let us check that the left square commutes after postcomposing with π_0:

$$\begin{tikzcd} I \times X^I \arrow[r, "\overline{e}"] \arrow[d, "\mathrm{ev}_X"] & Y^I \arrow[d, "(s_1)^I"] \\ X \arrow[r, "\overline{\pi_1}"] & X^I. \end{tikzcd}$$

This transposes to:

$$\begin{array}{ccc} I \times I \times X^I & \xrightarrow{e} & Y \\ \downarrow{\scriptstyle I \times \mathrm{ev}_X} & & \downarrow{\scriptstyle (s_1)^I} \\ I \times X & \xrightarrow{\pi_1} & X, \end{array}$$

which is (3.2) for $k = 1$.

- Let us check that the left square commutes after postcomposing with π_1:

$$\begin{array}{ccc} I \times X^I & \xrightarrow{\langle \pi_0, \overline{e} \rangle} & I \times Y^I \\ \downarrow{\scriptstyle \mathrm{ev}_X} & & \downarrow{\scriptstyle \mathrm{ev}_Y} \\ X & \xrightarrow{r} & Y. \end{array}$$

This is (3.3) after substituting $e = \mathrm{ev}_Y \circ (I \times \overline{e})$. □

Corollary 3.4. *In the situation of Definition 3.1, if s_1 is a fibration, then X is fibrant.*

Proof. Apply Lemmata 1.3 and 3.3 and note that trivial fibrations are closed under retract. □

Corollary 3.5. *Consider an object X with a reflexive relation (denoted as in (3.1)). If*

$$\langle s_0, s_1 \rangle : Y \to X \times X,$$
$$s_1 : Y \to X$$

are fibrations, then X is fibrant.

Proof. Combine Lemma 3.2 and Corollary 3.5. □

Corollary 3.6. *The object U is fibrant.*

Proof. Use Corollary 3.5 with Corollary 1.8 and Lemma 1.9. □