A Finite Axiomatization of Inductive-Recursive Definitions

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Summary

- *Inductive-recursive* definition - a generalization of inductive definition;
- Examples include datatypes which are constructive analogues of *large cardinals* (inaccessible, hyperinaccessible, Mahlo) and other interesting datatypes in dependent type theory;
- A finite axiomatization based on categorical ideas: general *reflection principle* generalizing initial algebra semantics.
- Syntax of a very general version of *Martin-Löf type theory*.
- *Generic programming*: each construction is parameterized with respect to code for inductive-recursive definition.
- Model in *classical set theory with Mahlo cardinal*.

What is the syntax and semantics of constructive mathematics?

(a) Intuitionistic logic and BHK-semantics.

(b) Curry-Howard, Scott's Constructive Validity, and Martin-Löf type theory.

(c) Generalized inductive definition as basic constructive notion not needing justification in terms of other notions; intuitionistic type theory as extended subset of ML/Haskell:

- Typed lambda calculus + dependent types.
- Allow only well-founded inductive datatypes and structural recursive functions. Set = inductive datatype.

(d) Inductive-recursive definitions, intuitionistic model theory, etc.

Inductive and recursive definition of a set

```
Vect :: (a :: Set) -> (n :: Nat) -> Set
```

Inductive definition with constructors:

```
Nil  :: (a :: Set) ->
        Vect a 0
Cons :: (a :: Set) -> (n :: Nat) ->
        a -> Vect a n ->
        Vect a (Succ n)
```

Recursive definition:

```
Vect a 0        = ()
Vect a (Succ n) = (a,Vect a n)
```

Inductive-recursive definition

First example (Martin-Löf, 1984): universe a la Tarski

```
U :: Set
T :: U -> Set
```

Simultaneous inductive-recursive definition

```
Natcode :: U
Picode  :: (a :: U) ->
           (b :: T a -> U) ->
           U

T Natcode        = Nat
T (Picode a b) = Pi (T a) (T o b)
```

Constructive analogues of large cardinals

universe	inaccessible cardinal
superuniverse	hyperinaccessible cardinal
Mahlo universe	Mahlo cardinal

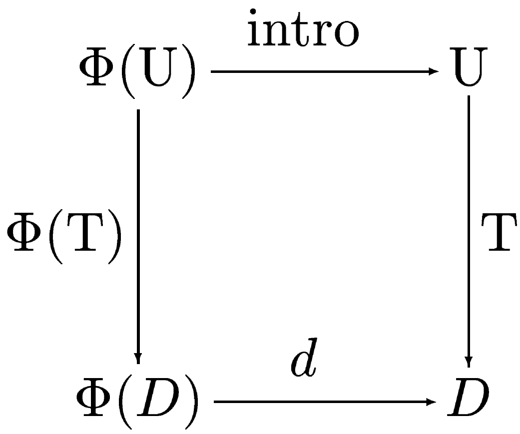
all example of inductive-recursive definitions and instances of our general formulation.

Plan

- Inductive types via initial algebras in simply typed lambda calculus
- From inductive to inductive-recursive definitions: from initial algebras to general reflection principle
- Set-theoretic semantics

Inductive types in simply typed lambda calculus

Initial algebra diagram



We only want well-founded types. Φ should be an SP-functor, consider sums of:

$$
\begin{aligned}
\Phi(D) &= 1 \\
\Phi(D) &= A \times \Phi'(D) \\
\Phi(D) &= (A \to D) \times \Phi'(D)
\end{aligned}
$$

Generic programming

Codes for SP-functors

```
data SP = Nil | Nonind Set SP | Ind Set SP
```

Decoding the object part

```
Arg :: SP -> Set -> Set

Arg Nil              d = ()
Arg (Nonind a phi) d = (a, Arg phi d)
Arg (Ind a phi)    d = (a -> d, Arg phi d)
```

Decoding the arrow part

```
map :: (phi :: SP) -> (a,b :: Set) ->
       (a -> b) ->
       Arg phi a -> Arg phi b
```

A diagram for U and T

$$\begin{array}{rcl}
(\mathtt{a} :: \mathtt{U}, \mathtt{b} :: \mathtt{Ta} \to \mathtt{U}) & \xrightarrow{\quad \mathtt{Picode} \quad} & \mathtt{U} \\
(\mathtt{a},\mathtt{b}) \mapsto (\mathtt{Ta},\mathtt{Tob}) \Big\downarrow & & \Big\downarrow \mathtt{T} \\
(\mathtt{A} :: \mathtt{Set}, \mathtt{B} :: \mathtt{A} \to \mathtt{Set}) & \xrightarrow{\quad \mathtt{Pi} \quad} & \mathtt{Set}
\end{array}$$

is not an initial algebra diagram, due to the simultaneous inductive-recursive nature of U and T.

From inductive to inductive-recursive

$$
\begin{CD}
\mathrm{Arg}_\phi(\mathrm{U}_\phi) @>{\mathrm{intro}_\phi}>> \mathrm{U}_\phi \\
@V{\mathrm{map}_\phi(\mathrm{U}_\phi, \mathrm{T}_{\phi,d})}VV @VV{\mathrm{T}_{\phi,d}}V \\
\mathrm{Arg}_\phi(D) @>{d}>> D
\end{CD}
$$

$$
\begin{CD}
\mathrm{arg}_\phi(\mathrm{U}_{\phi,d}, \mathrm{T}_{\phi,d}) @>{\mathrm{intro}_{\phi,d}}>> \mathrm{U}_{\phi,d} \\
@V{\mathrm{map}_\phi(\mathrm{U}_{\phi,d}, \mathrm{T}_{\phi,d})}VV @VV{\mathrm{T}_{\phi,d}}V \\
\mathrm{Arg}_{D,\phi} @>{d}>> D
\end{CD}
$$

More generic programming

$$\text{nil} : \text{SP}_D$$

$$\frac{A \text{ stype} \qquad \phi : A \to \text{SP}_D}{\text{nonind}(A, \phi) : \text{SP}_D}$$

$$\frac{A \text{ stype} \qquad \phi : (A \to D) \to \text{SP}_D}{\text{ind}(A, \phi) : \text{SP}_D}$$

Then define for $\phi : \text{SP}_D$, the following three (!) components

$$\begin{aligned} \text{Arg}_{D,\phi} &= \ldots \\ \text{arg}_{D,\phi}(U, T) &= \ldots \\ \text{map}_{D,\phi}(U, T) &= \ldots \end{aligned}$$

A usual functor has only two components: the object and the arrow part.

Formal rules

Formation rules:

$$\mathrm{U}_{\phi,d} : \mathrm{set}$$

$$\mathrm{T}_{\phi,d} : \mathrm{U}_{\phi,d} \to D$$

Introduction rule:

$$\frac{a : \mathrm{arg}_{\phi}(\mathrm{U}_{\phi,d}, \mathrm{T}_{\phi,d})}{\mathrm{intro}_{\phi,d}(a) : \mathrm{U}_{\phi,d}}$$

Equality rule:

$$\frac{a : \mathrm{arg}_{\phi}(\mathrm{U}_{\phi,d}, \mathrm{T}_{\phi,d})}{\mathrm{T}_{\phi,d}(\mathrm{intro}_{\phi,d}(a)) = d(\mathrm{map}_{\phi}(\mathrm{U}_{\phi,d}, \mathrm{T}_{\phi,d}, a))}$$

(Universe elimination and structural recursion ...)

Set-theoretic semantics

The rules of Martin-Löf type theory (including inductive-recursive definitions) are valid under a "naive" interpretation of a constructive concept as the corresponding classical concept with the same name.

Set is interpreted as (inductively defined) set; element as element; equal elements as equal elements; function as function (graph); Π as Π; etc.

Semantics of induction-recursion using Mahlo cardinals

We get the semantics of a $\mathrm{U}_{\phi,d}$ and $\mathrm{T}_{\phi,d}$ by iterating a monotone operator Φ to a fixed point. The only difficulty is to prove that such a fixed point exists. This can be done by using the axiom that Mahlo cardinals exist.

An inaccessible cardinal M is Mahlo if every normal function $f : \mathrm{M} \to \mathrm{M}$ has an inaccessible fixed point (assume for simplicity GCH).

Define a normal function $\theta : \mathrm{M} \to \mathrm{M}$ such that the rank of the α-th iteration of Φ is bounded by θ_α. Since M is Mahlo, θ has an inaccessible fixed point $\kappa < \mathrm{M}$. One can then show that Φ is κ-continuous and hence the κth iteration of Φ is a fixed point.