Intuitionistic Type Theory
Lecture 1

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Intuitionistic logic and Intuitionistic Type Theory

Intuitionistic logic:

1908 **BHK.** Brouwer. Kolmogorov, a calculus of problems. Heyting, a calculus of intended constructions.

1945 Kleene, realizability model.

1968 Howard, formulas as types. De Bruijn, Automath. Lawvere, hyperdoctrines. Scott, Constructive Validity.

Intuitionistic Type Theory:

1972 Martin-Löf, intensional Intuitionistic Type Theory, universes, proof theoretic properties.

1974 Aczel, realizability model.

1979 Martin-Löf, **meaning explanations**, extensional Intuitionistic Type Theory.

1986 Martin-Löf, intensional Intuitionistic Type Theory based on a logical framework (set-type distinction)

Curry

Hilbert-style axioms of implication

$$A \supset A$$

$$A \supset B \supset A$$

$$(A \supset B \supset C) \supset (A \supset C) \supset B \supset C$$

Typed combinatory logic

$$\begin{aligned} \mathbf{I} &: A \to A \\ \mathbf{K} &: A \to B \to A \\ \mathbf{S} &: (A \to B \to C) \to (A \to C) \to B \to C \end{aligned}$$

Curry

Modus ponens

$$\frac{A \supset B \qquad A}{B}$$

Typing rule for application

$$\frac{f : A \to B \qquad a : A}{f\,a : B}$$

Natural deduction and simply typed lambda calculus

Natural deduction

$$\frac{}{\Gamma \vdash A} A \in \Gamma \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \qquad \frac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

Simply typed lambda calculus

$$\frac{}{\Gamma \vdash x : A} x : A \in \Gamma \qquad \frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x.b : A \to B} \qquad \frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash f\, a : B}$$

- formulas/propositions as types
- proofs as terms/programs
- proof normalization as term normalization

Propositions as types

Intuitionistic Type Theory is based on the Curry-Howard *identification*

$$A \supset B \quad = \quad A \to B$$

and

$$\begin{aligned} \bot &= \emptyset \\ \top &= 1 \\ A \vee B &= A + B \\ A \wedge B &= A \times B \end{aligned}$$

Gödel System T of primitive recursive functionals

Add a type N.

N-introduction

$$\Gamma \vdash 0 : N \qquad \frac{\Gamma \vdash a : N}{\Gamma \vdash s(a) : N}$$

N-elimination

$$\frac{\Gamma \vdash n : N \qquad \Gamma \vdash d : C \qquad \Gamma, y : N, z : C \vdash e : C}{\Gamma \vdash R(n, d, yz.e) : C}$$

N-equality

$$\begin{aligned} R(0, d, yz.e) &= d \\ R(s(n), d, yz.e) &= e[y := a, z := R(n, d, yz.n)] \end{aligned}$$

Gödel system T: propositional part of Intuitionistic Type Theory 1972.

Properties of Gödel System T

- (Strongly) normalizing (Tait 1967). Model of normal forms.
- Model in **Set** where $A \to B$ means the set of all set-theoretic functions from A to B

Dependent types

- Predicate = family of types = dependent type

$$x : A \vdash B \ type$$

- $\Sigma x : A.B$ - the disjoint sum of the A-indexed family of types B. Canonical elements are pairs (a, b) such that $a : A$ and $b : B[x := a]$
- $\Pi x : A.B$ the cartesian product of the A-indexed family of types B. Canonical elements of $\Pi x : A.B$ are (computable) functions $\lambda x.b$ such that $b[x := a] : B[x := a]$, whenever $a : A$.

The division theorem

As a formula in Heyting arithmetic:

$$\forall m, n. m > 0 \supset \exists q, r. mq + r = n$$

As a type in Intuitionistic Type Theory:

$$\Pi m, n : \mathbf{N}. \mathbf{GT}(m, 0) \to \Sigma q, r : \mathbf{N}. \mathbf{I}(\mathbf{N}, mq + r, n)$$

A proof of division is a program of this type:

$$\mathrm{div} : \Pi m, n : \mathbf{N}. \mathbf{GT}(m, 0) \to \Sigma q, r : \mathbf{N}. \mathbf{I}(\mathbf{N}, mq + r, n)$$

$$\mathrm{div} : (m, n, p) \mapsto (q, (r, s))$$

It's a functional program (lambda term). Program extraction.

Universal quantification and dependent function types

Natural deduction for (untyped) predicate logic

$$\frac{\Gamma \vdash_{X,x} B}{\Gamma \vdash_X \forall x.B} \qquad \frac{\Gamma \vdash_X \forall x.B}{\Gamma \vdash_X B[x := a]}$$

The lambda calculus with dependent types

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x.b : \Pi x : A.B} \qquad \frac{\Gamma \vdash f : \Pi x : A.B \qquad \Gamma \vdash a : A}{\Gamma \vdash f\, a : B[x := a]}$$

$$(\lambda x.b)\, a = b[x := a]$$

Existential quantification and dependent pair types

Natural deduction for (untyped) predicate logic

$$\frac{\Gamma \vdash_X B[x := a]}{\Gamma \vdash_X \exists x.B}$$

$$\frac{\Gamma \vdash_X \exists x.B \qquad \Gamma, B \vdash_{X,x} C}{\Gamma \vdash_X C}$$

The lambda calculus with dependent types

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[x := a]}{\Gamma \vdash \langle a, b\rangle : \Sigma x : A.B}$$

$$\frac{\Gamma \vdash c : \Sigma x : A.B \quad \Gamma, x : A, y : B \vdash d : C[z := \langle x, y\rangle]}{\Gamma \vdash E(c, xy.d) : C[z := c]}$$

Propositions as types "explain" the laws of intuitionistic logic. "On the meaning of the logical constants and the justification of the logical laws" (Siena lectures, Martin-Löf 1983)

Propositions as types

$$
\begin{aligned}
\bot &= \emptyset \\
\top &= 1 \\
A \vee B &= A + B \\
A \wedge B &= A \times B \\
A \supset B &= A \to B \\
\exists x : A.B &= \Sigma x : A.B \\
\forall x : A.B &= \Pi x : A.B
\end{aligned}
$$

Martin-Löf 1972 "An Intuitionistic Theory of Types" results by adding

N	*the type of natural numbers*
U	*the type of small types - the universe*

Natural numbers in Martin-Löf 1972

N-introduction

$$\Gamma \vdash 0 : N \qquad \frac{\Gamma \vdash a : N}{\Gamma \vdash s(a) : N}$$

N-elimination

$$\frac{\Gamma \vdash n : N \qquad \Gamma \vdash d : C[x := 0] \qquad \Gamma, y : N, z : C[x := y] \vdash e : C[x := s(y)]}{\Gamma \vdash R(n, d, yz.e) : C[x := n]}$$

Conversion rules (untyped)

$$\begin{aligned} R(0, d, yz.e) &= d \\ R(s(n), d, yz.e) &= e[y := a, z := R(n, d, yz.e)] \end{aligned}$$

Like the rules in Gödel System T, but now C depends on $x : N$.
N-elimination subsumes mathematical induction.

Rules for the type of small types U (a la Russell)

U-introduction

$$\Gamma \vdash N : U \qquad \Gamma \vdash \emptyset : U \qquad \Gamma \vdash 1 : U$$

$$\frac{\Gamma \vdash A : U \qquad \Gamma \vdash B : U}{\Gamma \vdash A + B : U}$$

$$\frac{\Gamma \vdash A : U \qquad \Gamma, x : A \vdash B : U}{\Gamma \vdash \Sigma x : A.B : U} \qquad \frac{\Gamma \vdash A : U \qquad \Gamma, x : A \vdash B : U}{\Gamma \vdash \Pi x : A.B : U}$$

U-elimination

$$\frac{\Gamma \vdash A : U}{\Gamma \vdash A}$$

Abbreviations

$$\begin{array}{rcl} A \times B & = & \Sigma x : A.B \\ A \to B & = & \Pi x : A.B \end{array}$$

The predicative universe of small types

- Martin-Löf 1971 had tried the strongly *impredicative* rule

$$type : type$$

  leading to Girard's paradox.
- Martin-Löf 1972 developed a *predicative* theory by introducing the *large* type U closed under all *small* type formers. We do *not* have $U : U$! Analogue of *Grothendieck universe* in set theory.
- U is the only source of type dependency. If it is removed the system collapses to System T.
- Identity type on N is defined in terms of U. Identity types are not primitive as in later (and earlier) versions of Intuitionistic Type Theory.

Defining a family of types by primitive recursion

Finite types N_n with n elements

$$\begin{aligned} N_0 &= \emptyset \\ N_{s(n)} &= 1 + N_n \end{aligned}$$

$$N_n = R(n, \emptyset, xy.1 + y) : U$$

Types A^n of n-tuples (vectors) of elements in A

$$\begin{aligned} A^0 &= 1 \\ A^{s(n)} &= A \times A^n \end{aligned}$$

$$A^n = R(n, 1, xy.A \times y) : U$$

Theories which are smoothly subsumed

- Gödel System T of Primitive Recursive Functions of Higher Type
- Heyting Arithmetic HA
- Heyting Arithmetic of Higher Type HA^{ω}

Defining identity of natural numbers by primitive recursion

Exercise. Define

$$I_N : N \to N \to U$$

by primitive recursion (of higher type), such that

$$\begin{aligned} I_N\,0\,0 &= 1 \\ I_N\,0\,(s\,n) &= \emptyset \\ I_N\,(s\,m)\,0 &= \emptyset \\ I_N\,(s\,m)\,(s\,n) &= I_N\,m\,n \end{aligned}$$

Hence, by U-elimination

$$I_N\,m\,n\ type$$

The Peano axioms follow. (Exercise.)

The axiom of choice is a theorem

A consequence of the BHK-interpretation of the intuitionistic quantifiers:

$$
\begin{array}{rcl}
(\Pi x : A.\Sigma y : B.C) & \to & \Sigma f : (\Pi x : A.B).\Pi x : A.C[y := f(x)] \\
g & \mapsto & (\lambda x.fst(g(x)), \lambda x.snd(g(x)))
\end{array}
$$

Remark: the "extensional axiom of choice" is not valid! Let A, B be setoids (types with equivalence relations).

Models of Intuitionistic Type Theory 1972

- Normal form model (Martin-Löf 1972, by modification of Tait's method). Decidability of the judgments (type-checking algorithm which is the basis for proof assistants).
- Model in **Set** where $\Pi x : A.B$ is the set-theoretic cartesian product of a family of types and U a Grothendieck universe.
- Realizability model (per-model). More later.
- Categorical "models". Perhaps more later.
- Etc.