Normalization and the Yoneda embedding

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What is a normalization proof?

Traditional approach: prove about *reduction*

- **normalization - existence**
- **Church-Rosser - uniqueness**

Reduction-free approach: prove about *conversion* ($\sim$):

- **there is an algorithm nf such that $t \sim nf\ t$**
- $t \sim t' \supset nf\ t = nf\ t'$

Corollary - solution of the word problem:

- $t \sim t' \leftrightarrow nf\ t = nf\ t'$

Normalization by intuitionistic representation theorem

Syntax is free model $(T, \sim)$ (classically, $T/\sim$).

Find "strict" model M with (left) inverse of unique interpretation map:

$$(T, \sim) \underset{[\![-]\!]^{-1}}{\overset{[\![-]\!]}{\rightleftarrows}} (M, =)$$

that is

$$nf\ t = [\![[\![t]\!]]\!]^{-1} \sim t$$

$$t \sim t' \supset [\![t]\!] = [\![t']\!] \supset nf\ t = nf\ t'$$

Intuitionistic framework (Martin-Löf Type Theory, etc): function = algorithm!

Normalization by Yoneda embedding?

The functor $\mathcal{Y} : \mathcal{C} \to \mathcal{S}et^{\mathcal{C}^{op}}$

$$\begin{aligned} \mathcal{Y}B &= \mathcal{C}(-, B) \\ \mathcal{Y}g &= g \circ - \end{aligned}$$

induces a bijection of hom-sets:

$$\mathcal{C}(A, B) \underset{-_A \; id_A}{\overset{\mathcal{Y}}{\rightleftarrows}} \mathcal{S}et^{\mathcal{C}^{op}}(\mathcal{Y}A, \mathcal{Y}B)$$

Monoids - one object categories:

$$M \underset{- \; id}{\overset{\mathcal{Y}}{\rightleftarrows}} \{\phi \in M^M | \phi \text{ natural}\}$$

Groups - Cayley's representation theorem:

$$G \underset{- \; id}{\overset{\mathcal{Y}}{\rightleftarrows}} \{\phi \in G^G | \phi \text{ natural iso}\}$$

View Yoneda functor as algorithm? But

$$\mathcal{C}(A,B) \underset{-_A\ id_A}{\overset{\mathcal{Y}}{\rightleftarrows}} \mathcal{S}et^{\mathcal{C}^{op}}(\mathcal{Y}A, \mathcal{Y}B)$$

only maps $g \in \mathcal{C}(A,B)$ to $g \circ id_A$!

Syntax = free category (monoid) $\mathcal{T}/\sim$. Unique interpretation functor:

$$\mathcal{C} = \mathcal{T}/\sim \xrightarrow{[\![-]\!]} \mathcal{D}$$

For presheaves, if $[\![-]\!] = \mathcal{Y}$ for atoms, then

$$(\mathcal{T}/\sim)(A,B) \underset{-_A\ id_A}{\overset{[\![-]\!] = \mathcal{Y}}{\rightleftarrows}} \mathcal{S}et^{(\mathcal{T}/\sim)^{op}}(\mathcal{Y}A, \mathcal{Y}B)$$

and now $nf\ g = [\![g]\!]\ id_A$!!

(For cccs, unique means up to isomorphism!)

Plan

1. **Earlier work**
2. **Constructive algebra: P-sets and P-monoids**
3. **The word problem for monoids**
4. **Constructive category theory: P-categories, P-Yoneda for P-cccs**
5. **The word problem for cccs**
6. **Conclusion**

Earlier work

- **Martin-Löf (1973, 1974): normalization by intuitionistic model construction (combinators and weak type theory).**

- **Berger and Schwichtenberg (1991): normalization for $\lambda\beta\eta$ by inverting set-theoretic interpretation; Friedman's theorem.**

- **T. Coquand and Dybjer (1993): footnote to Martin-Löf (1973, 1974) (algebraic aspects).**

- **C. Coquand (1993): normalization for $\lambda\beta\eta$ by inverting Kripke interpretation.**

- **Altenkirch, Hofmann, and Streicher (1995): normalization for $\lambda\beta\eta$ by Yoneda and glueing.**

Sets with equality

Bishop's distinction between *preset* and *set*.

An *E-set* (*setoid*) is a set A with an equivalence relation $\sim$.

An *E-map* from $(A, \sim)$ to $(A', \sim')$ is a function from A to A' which preserves equivalence.

Want better separation of "algorithmic" and "logical" properties - P-sets encode both "subsets" and "quotients":

A *P-set* is a set A with a per $\sim$.

A *P-map* from $(A, \sim)$ to $(A', \sim')$ is a function from A to A' which preserves pers.

P-monoid

- a P-set $(M, \sim)$;
- a binary P-map $\circ$ on $(M, \sim)$;
- an element id in M;
- such that

$$\begin{aligned} (\theta \circ \delta) \circ \gamma &\sim \theta \circ (\delta \circ \gamma) \\ id \circ \gamma &\sim \gamma \\ \gamma \circ id &\sim \gamma \end{aligned}$$

for θ in the domain of $\sim$, that is, $\theta \sim \theta$, etc.

The word problem for monoids

T is the set of binary trees generated by a set X of atoms x.

$$t ::= t \circ t \mid id \mid x$$

$\sim$ is the congruence relation between elements of T generated by identity and associativity laws.

$(T, \sim)$ is a P-free P-monoid generated by X (and an E-free E-monoid!).

Decide $t \sim t'$!

Use the constructive P-iso

$$(T, \sim) \underset{-\ id}{\overset{[\![-]\!]}{\rightleftarrows}} (T^T, \sim)$$

analogue of

$$T/\sim \underset{-\ id}{\overset{[\![-]\!]}{\rightleftarrows}} \{\phi \in (T/\sim)^{T/\sim} | \phi \text{ natural}\}$$

Hence $nf\ t = [\![t]\!]\ id \sim t$

But we also have

$$(T, \sim) \xrightarrow{[\![-]\!]} (T^T, =)$$

Hence, if $t \sim t'$**, then** $[\![t]\!] = [\![t']\!]$ **and** $nf\ t = nf\ t'$**.**

Strict notions

$(T^T, =)$ is a *strict* P-monoid.

$(T, \sim)$ and $(T^T, \sim)$ are *non-strict*.

Suggestive terminology?

$\sim$	$=$
non-strict	**strict**
abstract	**concrete**
syntactic	**semantic**
formal	**real**
static	**dynamic**

Compare category theory: $\cong$ vs $=$!

Coherence in category theory

Analogy with the coherence problem formulated as:

"every X is X-equivalent to a strict X".

X = monoidal category, bicategory (Gordon, Power, Street)

X = fibration (Bénabou)

proved by appropriate Yoneda lemmas.

Difference: for normalization we focus on syntax = free category and work intuitionistically.

Beylin and Dybjer (1995) show how coherence for monoidal categories can be derived by analyzing formal proof objects for normalization algorithm for monoids.

The word problem for groups?

G is the set of "group-expressions":

$$t ::= t \circ t \mid id \mid t^{-1} \mid x$$

$\sim$ is the congruence relation so that $(G, \sim)$ is a P-free P-group.

Try $nf\ t = [\![t]\!]\ id \sim t$! Still get $nf\ t \sim t$.

But we do not have $t \sim t'$ implies $nf\ t = nf\ t'$. Because

$$[\![x]\!] = \mathcal{Y}\ x = x \circ -$$

is only a "formal" $\sim$-iso but not a "real" $=$-iso!

P-categories

- **A *set* of objects;**
- **hom-*P-sets*;**
- **composition is a *P-map*;**
- **the category axioms refer to $\sim$.**

***Object equality*? Not part of the definition of P-category, but objects form a P-set (and E-set) under P-isomorphism.**

The P-category analogue $\mathcal{PS}et$ of the ordinary category of sets

- **P-sets as objects**
- $\mathcal{PS}et((A, \sim_A), (B, \sim_B)) = (B^A, \sim_{B^A})$, **where** $\phi \sim_{B^A} \phi'$ **iff** $a \sim_A a'$ **implies** $\phi\ a \sim_B \phi'\ a'$.

P-functor, P-natural transformations, P-ccc, P-free, ...

P-functor P-categories $\mathcal{D}^{\mathcal{C}}$ (in particular P-presheaf P-categories):

- **Objects are P-functors from $\mathcal{C}$ to $\mathcal{D}$.**
- **$\mathcal{D}^{\mathcal{C}}(F,G) = (\Pi_A \mathcal{D}(FA, GA), \sim)$ where $\phi \sim \phi'$ iff for all A, $\phi_A \sim \phi'_A$ (in $\mathcal{D}$) and ϕ and ϕ' are P-natural transformations.**

Note, we have arrows which are not P-natural transformations! But the domain (=reflexive part) of $\sim$ coincides with the P-natural transformation.

Yoneda and cccs

- **The Yoneda functor $\mathcal{Y} : \mathcal{C} \to \mathcal{S}et^{\mathcal{C}^{op}}$ preserves ccc-structure.**

  If $\mathcal{T}$ is a free ccc then there is a natural isomorphism

$$\mathcal{T} \;\; \underset{\mathcal{Y}}{\overset{[\![-]\!]}{\underset{q \Downarrow\Uparrow q^{-1}}{\longrightarrow}}} \;\; \mathcal{S}et^{\mathcal{T}^{op}}$$

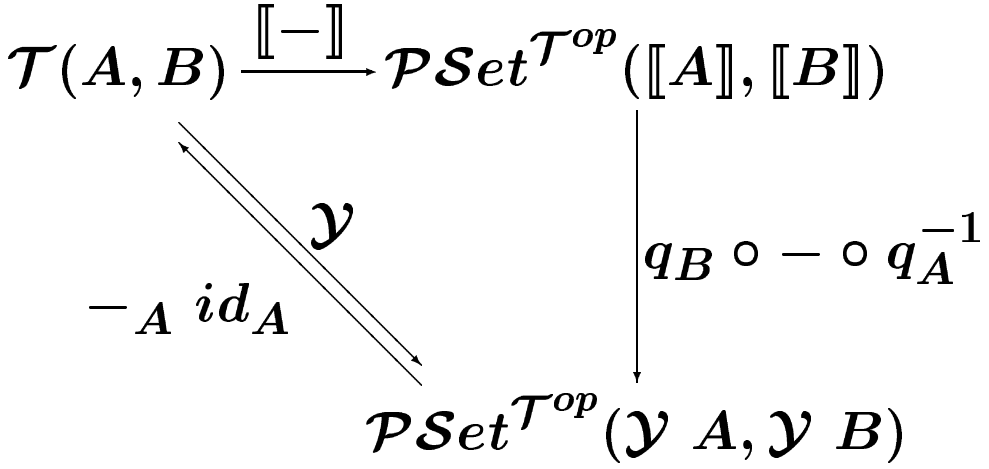
  where $[\![-]\!] = \mathcal{Y}$ on atoms.

  P-version gives normal form algorithm!

- **$\mathcal{S}et^{\mathcal{C}^{op}}$ is a ccc for any category $\mathcal{C}$.**

  P-version helps proving uniqueness of normal forms!

Normal form algorithm for cccs



is a commuting diagram of P-sets and P-maps. Hence

$$nf\ t = q_{BA}\ ([\![t]\!]\ (q_{AA}^{-1}\ id_A)) \sim t$$

$$
\begin{aligned}
[\![g \circ f]\!]_C\ a &= [\![g]\!]_C([\![f]\!]_C\ a) \\
[\![\mathtt{id}]\!]_C\ a &= a \\
[\![!]\!]_C\ a &= () \\
[\![<f, g>]\!]_C\ a &= ([\![f]\!]_C\ a, [\![g]\!]_C\ a) \\
[\![\pi]\!]_C\ (a, a') &= a \\
[\![\pi']\!]_C\ (a, a') &= a' \\
([\![f^*]\!]_C\ a)_D\ (g, a') &= [\![f]\!]_D\ ([\![A]\!]\ g\ a, a') \\
[\![\varepsilon]\!]_C\ (\theta, a) &= \theta_C\ (\mathtt{id}, a)
\end{aligned}
$$

$$
\begin{aligned}
q_{X,C}\ f &= f \\
q_{1,C}\ () &= ! \\
q_{A\times B,C}\ (a,b) &= \langle q_{A,C}\ a, q_{B,C}\ b\rangle \\
q_{A\Rightarrow B,C}\ \theta &= (q_{B,C\times A}\ (\theta_{C\times A}\ (\pi_{C,A}, q^{-1}_{A,C\times}
\end{aligned}
$$

$$
\begin{aligned}
q^{-1}_{X,C}\ f &= f \\
q^{-1}_{1,C}\ f &= () \\
q^{-1}_{A\times B,C}\ f &= (q^{-1}_{A,C}\ (\pi_{A,B}\circ f), q^{-1}_{B,C}\ (\pi'_{A,B}\circ \\
(q^{-1}_{A\Rightarrow B,C}\ f)_D\ g\ x &= q^{-1}_{B,D}\ (\varepsilon_{A,B}\circ\langle f\circ g, q_{A,D}\ x\rangle)
\end{aligned}
$$

Uniqueness of normal forms?

What about

$$t \sim t' \supset nf\ t = nf\ t'?$$

It depends on *what* P-free P-ccc we choose!

If $\mathcal{T}$ is built up by ccc-expressions (categorical combinators) under the congruence generated by the ccc-laws, then No!

If $\mathcal{T}$ is built up from the typed $\lambda\beta\eta$-calculus, then Yes - nf will return the η-long normal form of a term.

$\mathcal{T}_{\beta\eta}$ - a P-free P-ccc from the typed $\lambda\beta\eta$-calculus

objects: sequences of types $\Gamma = (A_1, \ldots, A_m)$

arrows: sequences of terms

$$(A_1, \ldots, A_m) \xrightarrow{(t_1, \ldots, t_n)} (B_1, \ldots, B_n)$$

equivalence of arrows: pointwise $\beta\eta$-convertibility

P-ccc structure:

$$\begin{aligned} 1 &= () \\ \Gamma \times \Delta &= \Gamma, \Delta \\ (B_1, \ldots, B_m)^{\Gamma} &= (B_1^{\Gamma}, \ldots, B_m^{\Gamma}) \end{aligned}$$

where

$$B^{(A_1,\ldots,A_m)} = A_1 \to \cdots \to A_m \to B$$

$\mathcal{T}_\alpha$ - the α-congruence P-category

$\mathcal{T}_\alpha$ is the P-category of sequences of λ-terms under α-congruence $\equiv$.

$\mathcal{T}_{\beta\eta}$ and $\mathcal{T}_\alpha$ have the same data part!

$\mathcal{PS}et^{\mathcal{T}_\alpha^{op}}$ is a P-ccc. Hence $t \sim t'$ implies $[\![t]\!] \equiv [\![t']\!]$.

To prove that this entails $nf\ t \equiv nf\ t'$ it remains to prove that $q_{B,A}$ and $q_{B,A}^{-1}$ preserve $\equiv$. This uses that $\mathcal{T}_\alpha$ is a P-cartesian P-category which also satisfies the ccc-law which corresponds to substitution under λ:

$$t^* \circ u \equiv (t \circ \langle u \circ \pi, \pi' \rangle)^*$$

Frequently asked questions

This is not a new result. Why do you bother?

How is this different from Altenkirch, Hofmann, and Streicher?

Your approach seems a bit mysterious. But after all there is no such thing as a free lunch, where do you hide the work?

Is the use of an intuitionistic metalanguage essential? I don't trust this airy-fairy type theory. I want a real recursive function!

Is this just a special case, where you had to work hard to coerce it into a categorical framework, or is your method generally applicable?

How does your method relate to ordinary Tait-style normalization proofs?

How does your method relate to the work by Hyland and Ong?

Why do you use P-category theory and not E-category theory? After all Aczel suggested to use E-category theory for his Galois theory in LEGO project, and Huet and Saibi have implemented a significant fragment of elementary E-category theory in COQ.

Can your work be described as "Computational Category Theory" in the sense of Burstall and Rydeheard?

What strategy does your normalization algorithm use?