(What I Know about) the History of the Identity Type

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When was the identity type born?

- Martin-Löf 1973?
- Howard 1969? A consequence of the Curry-Howard isomorphism. (De Bruijn?)
- Lawvere 1970? Equality in hyperdoctrines?
- Kleene 1950? Realizability interpretation? (BHK?)

Howard - The formulae-as-types notion of construction

Letter to Kreisel, March 1969. Howard writes for example about the interpretation of Heyting arithmetic.

"For each of the axioms $E(x)$ or $E(x,y)$, an atomic term $H^{E(x)}$ or $H^{E}(x,y)$ is assumed to be given."

For example

$$x = x \quad \text{has a construction} \quad H^{x=x}$$

and

$$x = y \supset y = x \quad \text{has a construction} \quad H^{x=y \supset y=x}$$

Howard - functional interpretation

Two alternatives

1. "Associate to each closed equation $s = t$
   - the singleton set $\{1\}$ if $s = t$ is true
   - the null set if $s = t$ is false.
2. Associate to $s = t$ the set of all proofs of $s = t$ in, say, the quantifier-free part of Heyting arithmetic with the help of [the substitution rule]."

Howard - "cut elimination"

Howard discusses the normalization of his constructions, but has an unfinished theory of "judgemental" equality. He says that it is well-known that there is no cut-elimination theorem for Heyting arithmetic, unless the induction schema is replaced by the ω-rule.

On a related topic, Martin-Löf 1971 "Hauptsatz for the intuitionistic theory of iterated inductive definitions", shows that you have a "cut-elimination" theorem ("Hauptsatz") for the theory of inductive definitions provided you formulate it in natural deduction style. This paper is the forerunner of all later work on schemata for inductive definitions in type theory. For example, it defines the identity relation as inductively generated by the reflexivity rule.

Lawvere 1970 - Equality in hyperdoctrines and comprehension schema as an adjoint functor

"an initial study of systems of categories connected by specific kinds of adjoints of a kind that arise in formal logic, proof theory, sheaf theory, and group-representation theory."

In this second paper on hyperdoctrines Lawvere extends his work on hyperdoctrines by introducing

- "a more or less satisfactory theory of the attribute "equality"
- a "comprehension schema", "that which assigns to every formula q its "extension" $\{x : \phi(x)\}$".

Scott 1970 - Constructive Validity

"One of BROUWER'S main theses was that mathematics is not based on logic, but that logic is based on mathematics. ... If mathematics consists of mental constructions, then every mathematical theorem is the expression of a result of a successful construction. The proof of the theorem consists in the construction itself, and the steps of the proof are the same as the steps of the mathematical construction. ... Our purpose here will be to reexamine the idea of the calculus of constructions. A formalization of this calculus will be presented, and it will be applied to the problem of interpreting logical formulas in a way that, to the author at least, seems to carry out the program outlined by HEYTING above word for word."

Scott gives credit to DE BRUIJN , LÄUCHLI, and LAWVERE for example.

Scott 1970 - Constructive Validity

Scott defines boolean equality between integers by a primitive recursive definition. Then the identity type for integers is defined by

$$[n =_N m] = [\top \wedge \bot](E(n)(m))$$

This is really if then else. He allows "large elimination".

There are no universes, although at the end of the article the possibility of adding "reflection principles" is mentioned.

In a postscript Scott declares his attempt to formulate a theory of constructions unsuccessful (after discussions with GÖDEL and KREISEL): "the decidability problems definitely show that the desired reduction in logical complexity has not been obtained".

Martin-Löf 1971 - A Theory of Types, p 25

Following Russell 1903

$$I = (\lambda x \in A)(\lambda y \in A)(\Pi X \in A \rightarrow V)(X\ x \rightarrow X\ y) \in A \rightarrow A \rightarrow V$$

Called Leibnitz equality in the Calculus of Constructions.
This yields the type of identities on an arbitary type, including the identity types.

Martin-Löf 1972 - no general identity type

Equality (of numbers) is defined like in Scott 1970, but explicitly using a universe V (a la Russell).
Define

$$E \in N \longrightarrow N \longrightarrow V$$

by primitive recursion (of higher type). Then use

$$\frac{A \in V}{A\ type}$$

to obtain the identity type for natural numbers. Extensional equality of numerical functions can then be defined. Etc.

Normalization theorem!

Martin-Löf 1973 - inductive definition of the identity type on an arbitrary type

Type theory is presented in two steps - an "informal" and a "formal" account. Let's look at the informal.

$$I(x,y)$$

is a proposition if $x,y \in A$.

$$r(x) \in I(x,x)$$

The J-elimination rule. If we have a function

$$g(x) \in C(x,x,r(x))(x \in A)$$

then define

$$f(x,y,z) \in C(x,y,z)(x,y \in A, z \in I(x,y))$$

by

$$f(x,x,r(x)) = g(x) \in C(x,x,r(x))$$

Martin-Löf 1979, 1984 - Meaning explanations

A quotation from the book "Intuitionistic Type Theory" (Bibliopolis 1984, p 60).

> *"We now have to explain how to form canonical elements of $I(A,a,b)$. The standard way to know that $I(A,a,b)$ is true is to have $a = b \in A$. Thus the introduction is simply: if $a = b \in A$, then there is a canonical proof r of $I(A,a,b)$. Here r does not depend on a, b or A; it does not matter what canonical element $I(A,a,b)$ has when $a = b \in A$, as long as there is one."*

This explanation justifies identity reflection, and extensional equality of functions, and uniqueness of identity proofs.

My slogan: "meaning = extension".

Martin-Löf 1979, 1984 - Extensional polymorphic type theory

Introduction, elimination, and equality rules:

$$r : I(A,a,a)$$

$$J(c,d) : C(c)$$

$$J(r,d) = d : C(r)$$

Equality reflection

$$\frac{c : I(A,a,b)}{a = b : A}$$

It follows that equality of functions is extensional.

Uniqueness of equality proofs

$$\frac{c : I(A,a,b)}{c = r : I(A,a,b)}$$

Martin-Löf 1986 - intensional monomorphic type theory

Introduced "the logical framework". Removed identity reflection. Intensional, decidable theory (judgements involving normal terms and types are decidable).

$$I : (A : \mathrm{Set}) \to A \to A \to \mathrm{Set}$$

$$r : (A : \mathrm{Set}) \to (a : A) \to I\ A\ a\ a$$

$$\begin{aligned} J \quad : \quad & (A : \mathrm{Set}) \to (C : (x, y : A) \to I\ A\ x\ y \to \mathrm{Set}) \\ & \to ((x : A) \to (C\ x\ x\ (r\ A\ x))) \\ & \to (a, b : A) \to (c : I\ A\ a\ b) \to C\ a\ b\ c \end{aligned}$$

Hyperdoctrine - definition in Lawvere 1970

- a category **T** of "types", whose morphisms are called "terms", and which is assumed to be cartesian closed.
- for each type X there is a cartesian closed category $P(X)$ of "attributes of type X, whose morphisms are called "deductions over X"
- for each term $f : X \longrightarrow Y$ there is a functor $f \cdot (\ ) : P(Y) \longrightarrow P(X)$ called "substitution of f in $(\ )$ ...
- for each term $f : X \longrightarrow Y$, two functors $(\ )\Sigma f$ and $(\ )\Pi f$ respectively left and right adjoint to substitution, called "exístential, respectively universal, quantification along f".

The hyperdoctrine of typed predicate logic.

- **T** is the category of contexts, whose morphisms are sequences of terms.
- $P(X)$ is the preorder of formulas with free variables ranging over the context X, and we have an arrow $\phi \to \psi$ in $P(X)$ iff $\phi \vdash \psi$. It is cartesian closed since we have intuitionistic logic of conjunction, disjunction, and implication.
- The substitution functor maps a sequence of terms $f : X \to Y$ and a formula $\phi[y]$ in $P(Y)$ into the formula $\phi[f[x]]$ in $P(X)$.
- For each term $f : X \to Y$, Σf maps a formula $\phi[x]$ into the formula

$$\psi[y] \equiv \exists x . y = f[x] \wedge \phi[x]$$

and Πf maps a formula $\phi[x]$ into the formula

$$\psi[y] \equiv \forall x . y = f[x] \supset \phi[x]$$

Ordinary quantifiers are special cases of Lawvere's quantifiers

Let f be the first projection

$$fst : Y \times X \longrightarrow Y$$

(weakening). Then $\Sigma\, fst$ maps a formula $\phi[y,x]$ into the formula

$$\psi[y] \equiv \exists x . y = fst[y,x] \wedge \phi[y,x] \equiv \exists x . \phi[y,x]$$

.

Similarly for universal quantifier.

Equality as a special case of Lawvere's Σ

The equality relation is defined by

$$\Theta_X = (\Sigma\, \delta_X)\, \top$$

where

$$\delta_X : X \to X \times X$$

is the diagonal map.

Syntactically, we get

$$\Theta_X[x', x''] \equiv \exists x.(x', x'') = (x, x) \wedge \top \equiv \exists x.(x', x'') = (x, x)$$

In categorical jargon "equality is obtained from the left adjoint of substitution along the diagonal map". This is similar to saying that "equality is inductively generated by the reflexivity rule".

Lawvere p8: on quantification along a term

"Our notion of quantification along an arbitrary term seems to be a considerable generalization of the usual quantification ... The greater generality was used in defining equality, since there we quantified along a diagonal term, which is not reducible to quantification along a projection. But perhaps that is the only essential case gained by the generalization: that is, perhaps the general case ... can be expressed in terms of [equality and ordinary quantification]".

Cf the discussion of data vs idata (general vs restricted inductive families).

Examples of hyperdoctrines

- T = the category of contexts, $P(X)$ is the category of formulas (see above). "Given any theory (several sorted, intuitionistic or classical) ..."
- T = the category of small sets, $P(X) = 2^X$ = the partially ordered set of all propositional functions "or one may take suitable 'homotopy classes' of deductions".
- T = the category of small sets, $P(X) = \mathcal{S}^X$ ... "This hyperdoctrine may be viewed as a kind of set-theoretical surrogate of proof theory"
- "honest proof theory would presumably yield a hyperdoctrine with nontrivial $P(X)$, but a syntactically presented one".
- T = the category of small categories, $P(B) = \mathbf{2}^B$
- T = the category of small categories, $P(B) = \mathcal{S}^{\mathcal{B}}$
- T = the category of small groupoids, $P(B) = \mathcal{S}^{\mathcal{B}}$

Comprehension schema

In a hyperdoctrine, we can define a functor from $\mathbf{T}/B$ to $P(B)$ mapping

$$p : E \to B \quad \mapsto \quad (\Sigma p)\, 1_E \in P(B)$$

When this functor has a right adjoint, we say that the hyperdoctrine satisfies the comprehension schema, and denote this adjoint by

$$\psi \in P(B) \quad \mapsto \quad p_\psi : \{B : \psi\} \to B$$

Extensional type theory a la Lawvere

Theorem. Suppose that in a given hyperdoctrine (eed is enough) in which the comprehension schema holds, we have further the following conditions for any two terms $h_i : E \longrightarrow Y$:

1. There is at most one proof $1_E \longrightarrow h_1 \Theta h_2$
2. If there is such a proof, then $h_1 = h_2$.

Then $\{X : f_1 \Theta f_2\}$ is the equalizer of $f_i : X \longrightarrow Y$.

Lawvere p11 on equality in the hyperdoctrine where types are categories and attributes are set-valued functors (presheaves)

"This [the failure of certain meager theorems] should not be taken as indicative of a lack of vitality of $\mathcal{S}^B, B \in \mathit{Cat}$ as a hyperdoctrine, or even of a lack of a satisfactory theory of equality for it. Rather it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception. Equality should be the 'graph' of the identity term."

Schema for indexed inductive definitions – inductive families

I is an example of an inductively defined family of types (indexed inductive definition, inductive family). What is the general form of a constructively valid such inductive definition? (Pattern follows Martin-Löf 1971 for predicate logic.)

Paulin equality. We inductively define the unary predicate $I_{A,a}$ "to be equal to $a : A$".

ALF’s pattern matching, implying uniqueness of identity proofs .

Is $I\ A\ a\ b$ for arbitrary A constructively valid? Agda’s "data" vs "idata".

Streicher's K

$$
\begin{aligned}
J \quad : \quad & (A : \mathrm{Set}) \to (C : (x, y : A) \to I\ A\ x\ y \to \mathrm{Set}) \\
& \to ((x : A) \to (C\ x\ x\ (r\ A\ x))) \\
& \to (a, b : A) \to (c : I\ A\ a\ b) \to C\ a\ b\ c
\end{aligned}
$$

$$
\begin{aligned}
K \quad : \quad & (A : \mathrm{Set}) \to (C : (x : A) \to I\ A\ x\ x \to \mathrm{Set}) \\
& \to ((x : A) \to (C\ x\ (r\ A\ x))) \\
& \to (a : A) \to (c : I\ A\ a\ a) \to C\ a\ c
\end{aligned}
$$

Hofmann and Streicher

It is not generally provable in intensional type theory that there is a unique proof of an identity.

$$\nvdash I\ (I\ A\ a\ b)\ p\ q$$

p and q may be different in the groupoid model.

Hedberg 1998

If an identity is decidable, then it has a unique proof:

$$((\forall a, b : A)\ (I\ A\ a\ b \vee \neg I\ A\ a\ b)) \supset I\ (I\ A\ a\ b)\ p\ q$$

Coherence and "metacoherence"

Formalizing category theory inside constructive type theory. Two projects

- Coherence for monoidal categories, Beylin and Dybjer 1996
- Coherence for categories with families ("Internal type theory"), Dybjer 1996 (inconclusive)

Both required extensive reasoning about identities between identity proofs.

Interpretation of extensional type theory in intensional type theory - Hofmann's setoid model

Not yet satisfactory solution for universes, as far as I know.

Recent work in Nottingham by Altenkirch and McBride on reconciling intensional and extensional type theory.

Contested axioms

- identity reflection
- uniqueness of identity proofs
- universal identity

All these axioms are constructively valid in the sense of Martin-Löf 1979.