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# Matroids from Modules

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**Abstract**

The aim of this work is to show that (oriented) matroid methods can be applied to many discrete geometries, namely those based on modules over integral (ordered) domains. The trick is to emulate the structure of a vector space within the module, thereby allowing matroid methods to be used as if the module were a vector space. Only those submodules which are "closed under existing divisors," and hence behave like vector subspaces, are used as subspaces of the matroid.

It is also shown that Hübler's axiomatic discrete geometry can be characterised in terms of modules over the ring of integers.

## 1 Introduction

The main approaches to image analysis and manipulation, computational geometry and related fields are based on continuous geometry. This easily leads to trouble with rounding errors and algorithms that return erroneous output, or even fail to terminate gracefully [11]. In view of this we can argue that the proper framework for many algorithms is not continuous, but discrete. Furthermore it is preferable if such a framework is axiomatically defined, so that the essential properties of the system are clearly stated and many models can share the same theory.

Matroids and related structures such as antimatroids and in particular oriented matroids (see Sections 2–4) promise to be useful as a basis upon which we can build an axiomatic discrete geometry [7,13]. In Section 6 we show that modules over integral domains have a natural matroidal structure, thereby showing that many discrete geometries can be treated within a framework based on matroids. (The existence of this matroidal structure seems to be known, but not well-known, and never applied in the context of discrete geometry. See further Section 6.2.) Matroids capture the essence of independence, but lack order and convexity. In Section 7 we extend our work

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to these concepts by showing that modules over ordered domains are easily turned into oriented matroids. If the modules are torsion free they can also be viewed as antimatroids. Finally in Section 8 we show that Hübler's axiomatic discrete geometry [6] can be treated within the framework built up in the previous sections. Hübler's geometry, possibly the only axiom system for discrete geometry presented so far, is introduced in Section 5.

Due to the lack of space many details are omitted. More details can be found in the first author's master's thesis [3].

## 2 Matroids

From the viewpoint of this article matroids capture the essence of independence as found in e.g. linear algebra (linear independence) and affine geometry (affine independence). This general treatment includes concepts such as bases, dimension, etc. Note, however, that the subject of matroids is much larger than this text might indicate.

Usually matroids have a finite ground set, but for treating independence this is an unnecessary assumption. The following definition is relatively standard. It is taken from Faure and Frölicher [5], as are the other results and definitions in this section. Oxley presents an equivalent definition of infinite matroids [9].

**Definition 2.1** A *closure space* is a pair $(M, \mathrm{cl})$, where $M$ is a set (the *ground set*) and $\mathrm{cl} : \wp(M) \to \wp(M)$ is a function *(closure operator)* satisfying (for all $A, B \subseteq M$)

(i) $A \subseteq \mathrm{cl}(A)$ *(increasing)*,

(ii) if $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ *(monotone)*, and

(iii) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ *(idempotent)*.

A *matroid* is a closure space satisfying

(iv) if $y \in \mathrm{cl}(A \cup x) \setminus \mathrm{cl}(A)$ for some $x \in M$, then $x \in \mathrm{cl}(A \cup y)$ *(the exchange property)*, and

(v) if $x \in \mathrm{cl}(A)$, then $x \in \mathrm{cl}(A')$ for some finite $A' \subseteq A$ *(finitary)*.

A closure space satisfying

(vi) $\mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{cl}(x) = \{ x \}$ for all $x \in M$

is *simple*. A *geometry* is a simple matroid.

The listed axioms/properties are independent. A matroid $(M, \mathrm{cl})$ is often denoted just by $M$.

The closed subsets of a matroid are called *subspaces*. A subset $A \subseteq M$ is said to *generate* a subspace $E$ if $\mathrm{cl}(A) = E$, and it is *independent* if it satisfies $x \notin \mathrm{cl}(A \setminus x)$ for all $x \in A$. If $A$ is independent and generates $E$, then it is a basis of $E$. An important result states that given three sets $A \subseteq D \subseteq E \subseteq M$, where $E$ is a subspace, $A$ is independent and $D$ generates $E$, there exists a basis $B$ of $E$ with $A \subseteq B \subseteq D$. Furthermore all bases of a subspace are equipotent. The *rank* $r(E)$ of a subspace $E$ is the cardinality of any of its bases. The rank function can be extended to arbitrary subsets $S$ by defining $r(S) := r\big(\mathrm{cl}(S)\big)$.

Matroids can be characterised uniquely by their subspaces. As an example we get a matroid by taking the linear subspaces of any vector space as subspaces. If we take the affine subspaces instead we get a geometry (even an *affine* geometry, see below).

Now let $E$ be a subspace of $M$. Take the quotient set $M/E$ consisting of the equivalence classes of the equivalence relation $\sim \subseteq (M \setminus E)^2$, where $x \sim y$ iff $\mathrm{cl}(E \cup x) = \mathrm{cl}(E \cup y)$. Let $\pi : M \setminus E \to M/E$ be the canonical projection. Define the closure operator $\mathrm{cl}_{M/E} : \wp(M/E) \to \wp(M/E)$ by $\mathrm{cl}_{M/E}(A) := \pi\left(\mathrm{cl}(\pi^{-1}(A) \cup E) \setminus E\right)$. The pair $(M/E, \mathrm{cl}_{M/E})$ obtained in this way is a geometry, the *quotient geometry*. The quotient geometry $M/\mathrm{cl}(\emptyset)$ is called the *canonical geometry*. The lattice of subspaces (introduced below) of $M$ is isomorphic to that of $M/\mathrm{cl}(\emptyset)$. It is instructive to work out some details about the canonical geometry of the matroid associated to the linear subspaces of a vector space (we leave this to the reader).

The *corank* of a subspace $E \subseteq M$ is $\overline{r}(E) := r(M/E)$. The corank satisfies $r(E) + \overline{r}(E) = r(M)$. The matroid $M$ itself has corank 0, and a *hyperplane* is defined as a subspace with corank 1.

The subspaces of a matroid, ordered by inclusion, is a lattice. The meet of two subspaces $E$ and $F$ is simply $E \wedge F = E \cap F$, while the join is $E \vee F = \mathrm{cl}(E \cup F)$. For any subspaces $E$, $F$ we have $r(E \wedge F) + r(E \vee F) \leq r(E) + r(F)$. A matroid is of degree $n$ ($n \in \mathbb{N}$) if equality holds in the preceding equation whenever $r(E \wedge F) \geq n$. A matroid of degree 0 is called modular since it has a modular lattice of subspaces.

In a geometry of degree 1 the subspaces of rank 2 are called *lines*, and the subspaces of rank 3 *planes*. Two lines $\ell_1$, $\ell_2$ are *parallel* ($\ell_1 \,||\, \ell_2$) iff either $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$ and $r(\ell_1 \vee \ell_2) = 3$. An *affine geometry* is a geometry $M$ of degree 1 for which for every line $\ell \subseteq M$ and point $p \in M \setminus \ell$ there is a unique line $\ell'$, parallel to $\ell$, with $p \in \ell'$. We will be liberal and use all the terminology above even if the matroid is not simple (except for affine *geometry*, of course).

## 3 Antimatroids

Convexity is treated abstractly in (at least) two different ways; via antimatroids and via oriented matroids (see the next section). The definition used here is taken from Coppel [2], who uses the term *anti-exchange alignment*.

**Definition 3.1** An *antimatroid* is a finitary closure space $(M, \mathrm{cl})$ satisfying the *anti-exchange property*: If $x, y \in M$, $x \neq y$, $S \subseteq M$, and $y \in \mathrm{cl}(S \cup x) \setminus \mathrm{cl}(S)$, then $x \notin \mathrm{cl}(S \cup y)$.

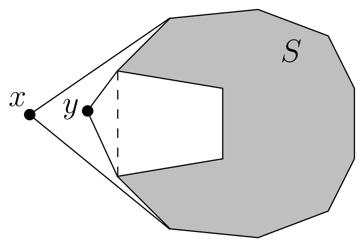


Fig. 1. Illustration of the anti-exchange property for a planar convex hull operator. The notation used is the same as in Definition 3.1. (This illustration is similar to a figure in [14].)

The standard example of an antimatroid is a vector space with the standard convex closure. Figure 1 motivates the definition of the anti-exchange property in this context.

Another, more discrete, example of an antimatroid is a (possibly infinite) tree with the closure operator that maps any subset of vertices to the smallest subtree containing the set.

## 4 Oriented Matroids

Oriented matroids add extra structure to the ordinary matroids treated in the last section. From Richter-Gebert and Ziegler we get the following description: "Roughly speaking, an oriented matroid is a matroid where in addition every basis is equipped with an orientation" [10].

We define oriented matroids as follows. This definition is a straightforward extension of one of the definitions for finite oriented matroids [1, Theorem 3.6.1].

**Definition 4.1** Let $M$ be a matroid where the complement $M \setminus H$ of each hyperplane $H$ is partitioned into two possibly empty sets $H^-$ and $H^+$, the *negative* and *positive* side of $H$. The ordered pair $(H^-, H^+)$ is a *cocircuit.* If necessary we can change the *orientation* of the cocircuit, i.e. the *opposite* $(H^+, H^-)$ is also a cocircuit. There are, by definition, no other cocircuits. The matroid $M$ together with its cocircuits is an *oriented matroid* if the following requirement is satisfied:

- Let $H$ and $K$ be two hyperplanes intersecting in a subspace of corank 2 and $x$ a point in $M \setminus (H \cup K)$. If it is possible to choose the orientations of the cocircuits associated with $H$ and $K$ such that $x \in H^+ \cap K^-$ then the hyperplane $L = x \vee (H \wedge K)$ satisfies $L^+ \subseteq H^+ \cup K^+$ and $L^- \subseteq H^- \cup K^-$, given a suitable choice of its orientation.

All the terminology used for ordinary matroids carries over to the oriented case. Some intuition behind the definition is given in Figure 2.

As with ordinary matroids there are many equivalent definitions of finite oriented matroids, and some of these do not give rise to equivalent definitions

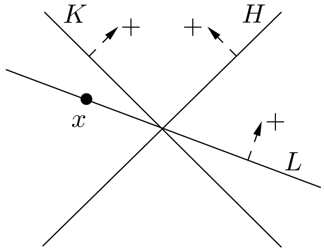


Fig. 2. The requirement which the hyperplanes and cocircuits of an oriented matroid have to satisfy. See Definition 4.1.

when relaxed to the infinite case [1,9]. The choice to use the definition above is motivated by the connection between convexity and half-spaces, which we turn to now. See Björner et al [1] for a more complete treatment of the subject.

Given a hyperplane $H$, the sets $H^-$ and $H^+$ are called *open half-spaces*. The union of an open half-space and the corresponding hyperplane is a *closed half-space*. Let $H_\mathrm{C}(M)$ denote all closed half-spaces in $M$. We can define a *convex closure operator* $[\cdot] : \wp(M) \to \wp(M)$ by

$$[S] = \begin{cases} \bigcap \{\, H \in H_\mathrm{C}(M) \mid S \subseteq H \,\}, & S \text{ finite},\\ \bigcup \{\, [S'] \mid S' \subseteq S,\ S' \text{ finite} \,\}, & \text{otherwise}. \end{cases} \tag{1}$$

Here we use the convention that $\bigcap \emptyset = M$. The reason for having different cases depending on the cardinality of $S$ is that this definition makes $[\cdot]$ finitary.

## 5 Hübler's Axiomatic Discrete Geometry

In this section Hübler's work about axiomatic discrete geometry [6] is briefly summarised. Since Hübler's thesis is unpublished, and in German, this section is more detailed than the other sections with background material. Unless otherwise noted, all the results in this section come from Hübler's thesis.

Let $\mathcal{P}$ be a set of *points*, and $\mathcal{L} \subseteq \wp(\mathcal{P})$ a nonempty set of *lines*. The first axiom is that for any pair $p$, $q$ of distinct points there is exactly one line $\ell$ for which the points *lie on* the line ($p, q \in \ell$). Let $\ell(p, q)$ denote that unique line. The second axiom says that for any line $\ell$ there exist two different points $p, q \in \ell$ and one point $r \notin \ell$.

The third axiom states that there is an equivalence relation $||$, parallelity, on $\mathcal{L}$ for which for any line $\ell$ and point $p$ there exists exactly one line $\ell'$ with $p \in \ell'$ and $\ell \,||\, \ell'$. The corresponding equivalence classes are called *directions*.

A *translation* is defined as a bijection $T$ on $\mathcal{P}$ that either equals the identity bijection $\mathsf{id}$ or has the following properties (referred to simply as the first, second, and third translation properties).

(i) For all lines $\ell$, $T(\ell) := \{\, T(p) \mid p \in \ell \,\}$ is a line parallel to $\ell$.

(ii) For all points $p$, $p \neq T(p)$.

(iii) The set $\{ \ell(p, T(p)) \mid p \in \mathcal{P} \}$ is a direction.

The fourth axiom now states that for any two points $p$, $q$ there exists a translation $T$ with $T(p) = q$. This translation can be shown to be unique. Another result is that two lines $\ell$ and $\ell'$ are parallel iff there exists a translation $T$ such that $T(\ell) = \ell'$. Furthermore the set $\mathcal{T}$ of all translations on $\mathcal{P}$ is an abelian group under the group operation composition $(\circ)$. Hence $\mathcal{T}$ can be made into a $\mathbb{Z}$-module in the standard way.

Now let, for each line, a total order $\leq$ be defined on the set of points of the line (together with the standard variations $<$, $>$ and $\geq$). A betweenness relation $B$ is then defined as follows: for three different points $p$, $q$ and $r$ on a line, $B(p, q, r)$ holds if $p < q < r$ or $r < q < p$. The fifth axiom enforces infinite sets $\mathcal{P}$: For each line $\ell$ and point $p \in \ell$ there are points $q, r \in \ell$ such that $B(q, p, r)$.

The next, sixth, axiom introduces discreteness: For any two points $p$ and $p'$ there is at most a finite number of points $q$ such that $B(p, q, p')$. This means e.g. that every line is a countably infinite set of points.

The seventh axiom is as follows: Let $\ell_1$, $\ell_2$ and $\ell_3$ be different, parallel lines, and $\ell$ and $\ell'$ lines that have points $p_i$ and $p'_i$, respectively, in common with all the lines $\ell_i$, $i \in \{1, 2, 3\}$. Then $B(p_1, p_2, p_3)$ holds iff $B(p'_1, p'_2, p'_3)$. This axiom rules out *cyclic translations*, i.e. translations $T \neq \mathsf{id}$ for which, for some $n \in \mathbb{Z}^+$, $T^n = \mathsf{id}$.

For each line $\ell$ there exists a translation $G$ (a *generator*) such that $\ell = \{ G^n(p) \mid n \in \mathbb{Z} \}$ for any point $p \in \ell$. For such a triple $(\ell, G, p)$ the relation $B(G^i(p), G^j(p), G^k(p))$ holds iff $i < j < k$ or $k < j < i$ $(i, j, k \in \mathbb{Z})$. Furthermore each line has exactly two generators ($G$ and $G^{-1}$), and two lines are parallel iff they have the same generators.

Let $\ell_1$, $\ell_2$, and $\ell_3$ be different, parallel lines. The line $\ell_2$ is said to *lie between* $\ell_1$ and $\ell_3$ ($B(\ell_1, \ell_2, \ell_3)$) if there are a translation $T$ and $i, j \in \mathbb{Z}^+$ such that $T^i(\ell_1) = \ell_2$ and $T^j(\ell_2) = \ell_3$.

A *planar set* is a point set $S$

(i) whose points do not all belong to one line, and

(ii) for which for any four, different points $p_i \in S$, $i \in \{1, 2, 3, 4\}$, one has for the lines $\ell_i$, $i \in \{1, 2, 3\}$ with $\ell_1 = \ell(p_1, p_2)$, $\ell_2 \,||\, \ell_1$ with $p_3 \in \ell_2$, and $\ell_3 \,||\, \ell_1$ with $p_4 \in \ell_3$ that if the lines are different, then one of the lines lies between the other two lines.

A *plane* is a planar set $P$ for which $P \cup x$ is not planar for any $x \in \mathcal{P} \setminus P$. For each planar set $S$ there is exactly one plane $P$ with $S \subseteq P$. Furthermore $\ell(p, q) \subseteq P$ holds for any two different points $p$ and $q$ in a plane $P$, and $P_1 \cap P_2 = \ell(p, q)$ holds for two different planes $P_1$ and $P_2$ whose intersection contains at least two different points $p$ and $q$.

In his report Hübler demonstrates a model of the first seven axioms which, in some sense, is not discrete. This "model" is not a model [3], but nevertheless motivates why Hübler introduced an eighth axiom. This axiom states that there is only a finite number of lines between two parallel lines. The eighth axiom actually makes axiom six, the other discreteness axiom, unnecessary, because it can be deduced from the other axioms. The independence of the remaining axioms does not seem to have been investigated. The axiom system is consistent; there are models satisfying all the axioms (as we will see later).

A *Hübler geometry* is a collection of a point set $\mathcal{P}$, a line set $\mathcal{L}$, a parallelity relation $||$, and a set of total orders $\leq$ for each line $\ell \in \mathcal{L}$, such that all the eight axioms are satisfied. Such a geometry is uniquely defined by $\mathcal{P}$ and the set $\mathcal{T}$ of all translations on $\mathcal{P}$.

## 6 Matroids from Modules

This section explores some closure operators defined on modules over integral domains[2] and the associated matroids and geometries.

### *6.1 Submodule Closure*

The effect of a closure operator is determined by its closed sets (since every set is mapped to the smallest closed set containing it; this set has to be unique). As mentioned above, for a vector space you get a matroid by choosing the vector subspaces as closed sets. Choosing the affine subspaces also yields a matroid, in fact an affine geometry (naturally). This approach does not in general work for modules. The submodules do not always yield a matroid, as we will now prove.

Let us first characterise the closure operator that corresponds to using submodules as subspaces (proof omitted).

**Lemma 6.1** *Let $M$ be an $R$-module, where $R$ is a ring, and let $\langle\cdot\rangle_{\mathrm{s}} : \wp(M) \to \wp(M)$ take any subset to the smallest submodule containing it. Then $\langle\cdot\rangle_{\mathrm{s}}$ is a well-defined closure operator with the explicit characterisation*

$$\langle S\rangle_{\mathrm{s}} = \left\{ \sum_{i=1}^{n} a_i s_i \;\middle|\; a_i \in R,\ s_i \in S,\ n \in \mathbb{N} \right\}.$$

(The empty sum $\sum_{i=1}^{0} a_i s_i$ is interpreted as 0.)

Let us now consider the standard $\mathbb{Z}$-module over $\mathbb{Z}$ (i.e. the ring of integers). Define $n\mathbb{Z} := \{\, nm \,|\, m \in \mathbb{Z} \,\}$. Observe that $2 \in \langle 10, 3\rangle_{\mathrm{s}} = \mathbb{Z}$ (since $1 \in \langle 10, 3\rangle_{\mathrm{s}}$), $2 \notin \langle 10\rangle_{\mathrm{s}} = 10\mathbb{Z}$, and $3 \notin \langle 10, 2\rangle_{\mathrm{s}} = \{\, 10m + 2n \,|\, m, n \in \mathbb{Z} \,\} = 2\,\{\, 5m + n \,|\, m, n \in \mathbb{Z} \,\} = 2\mathbb{Z}$. Hence the exchange property does not hold, and $\langle\cdot\rangle_{\mathrm{s}}$ is, in general, not a matroidal closure operator.

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[2] An *integral domain* is a nontrivial commutative ring $R$ with multiplicative unit satisfying $r_1 r_2 \neq 0$ for all $r_1, r_2 \in R \setminus 0$.

### 6.2 D-submodule Closure

We now know that we cannot (in general) use submodules as subspaces of a matroid. However, by restricting ourselves to d-submodules and modules over integral domains we get a matroid.

**Definition 6.2** A *d-submodule* of an $R$-module $M$ is a submodule $S$ with the property that if $rm \in S$ for any $r \in R \setminus 0$ and $m \in M$, then $m \in S$.

We say that a d-submodule is closed under existing divisors, i.e. under those divisors which happen to exist. Thus it is easy to see, intuitively, why this approach works; d-submodules emulate vector subspaces. In Section 6.5 below we formalise this statement.

**Theorem 6.3** *Let $M$ be an $R$-module, where $R$ is an integral domain, and let $\langle\cdot\rangle_{\mathrm{d}} : \wp(M) \to \wp(M)$ take any subset to the smallest d-submodule containing it. Then $\langle\cdot\rangle_{\mathrm{d}}$ is a well-defined matroidal closure operator with the explicit characterisation*

$$\langle S\rangle_{\mathrm{d}} = \left\{ m \in M \;\middle|\; bm = \sum_{i=1}^{n} a_i s_i,\ \ s_i \in S,\ \ a_i, b \in R,\ \ b \neq 0,\ \ n \in \mathbb{N} \right\}.$$

**Proof.** It is straightforward to show that $\langle\cdot\rangle_{\mathrm{d}}$ is a closure operator with the explicit characterisation given above (here we use the fact that $R$ is an integral domain).

For the exchange property we use the explicit characterisation of $\langle\cdot\rangle_{\mathrm{d}}$. Take any $y \in \langle S \cup x\rangle_{\mathrm{d}} \setminus \langle S\rangle_{\mathrm{d}}$. Then $by = \sum_{i=1}^{n} a_i s_i + ax$ for some $a, b, a_i \in R$, $b \neq 0$, $s_i \in S$, and $n \in \mathbb{N}$. Furthermore $a \neq 0$, because otherwise $y \in \langle S\rangle_{\mathrm{d}}$. Thus we have $ax = \sum_{i=1}^{n}(-a_i)s_i + by$ where $a \neq 0$, and hence $x \in \langle S \cup y\rangle_{\mathrm{d}}$. This means that the fourth axiom is satisfied.

To show that $\langle\cdot\rangle_{\mathrm{d}}$ is finitary, assume that $x \in \langle S\rangle_{\mathrm{d}}$. Then $bx = \sum_{i=1}^{n} a_i s_i$ for some $b, a_i \in R$, $b \neq 0$, $s_i \in S$, and $n \in \mathbb{N}$, and we have that $x \in \langle S'\rangle_{\mathrm{d}}$, where $S' := \{\, s_i \mid i \in \mathbb{N},\ 1 \le i \le n \,\}$ is a finite subset of $S$. $\square$

Aside from an exercise in Faure and Fröhlicher's book [5, Exercise 3.7.4.5] the authors have not found any reference to this construction, neither in the module nor in the discrete geometry literature. In fact the exercise claims that the construction works for modules over *left Ore domains*. The ideas behind a-closure (next to come) also come from [5], but that text only treats the case of vector spaces over division rings, not modules over integral domains (except for some specific examples).

### 6.3 A-submodule Closure

We note immediately that the matroid obtained from $\langle\cdot\rangle_{\mathrm{d}}$ is not simple, since $\langle\emptyset\rangle_{\mathrm{d}} = \{\, 0 \,\}$. Furthermore all subspaces contain 0, which ensures that they cannot be interpreted as affine lines, planes, etc. To get something reminiscent of an affine geometry we define a-submodules.

**Definition 6.4** An *a-submodule* $A$ of a module $M$ is a subset of the form $A = D + m$ where $D \subseteq M$ is a d-submodule and $m \in M$ is any element.

Addition of an element to a set is defined in the obvious way as $D + m := \{ d + m \mid d \in D \}$. Subtraction of an element from a set is defined analogously. Do not confuse this subtraction, written using $-$, with set difference, which we always write using $\setminus$.

It is easy to check that if $D$ is a d-submodule with $m \in D$, then $D+m = D$. Furthermore, if $A$ is an a-submodule, then for any $m \in A$ the set $A - m$ is a d-submodule, and all d-submodules obtained from $A$ in this way are equal. We get that $A$ is an a-submodule with $a \in A$ iff $A - a$ is a d-submodule. Given this relation between a- and d-submodules the following theorem is relatively straightforward to prove.

**Theorem 6.5** *Let $M$ be an $R$-module, where $R$ is an integral domain, and let $\langle\cdot\rangle_{\mathrm{a}} : \wp(M) \to \wp(M)$ take any nonempty subset to the smallest a-submodule containing it and $\emptyset$ to $\emptyset$. Then $\langle\cdot\rangle_{\mathrm{a}}$ is a well-defined matroidal closure operator with the explicit characterisation*

$$\langle S\rangle_{\mathrm{a}} = \left\{ m \in M \;\middle|\; bm = \sum_{i=1}^{n} a_i s_i,\ s_i \in S,\ a_i, b \in R,\ b = \sum_{i=1}^{n} a_i \neq 0,\ n \in \mathbb{Z}^+ \right\}.$$

*Furthermore $\langle S\rangle_{\mathrm{a}} = \langle S - s\rangle_{\mathrm{d}} + s$ for any $s \in \langle S\rangle_{\mathrm{a}}$, and $\langle S\rangle_{\mathrm{a}} = \langle S - s\rangle_{\mathrm{a}} + s$ for any $s \in M$.*

**Corollary 6.6** *The matroid defined in Theorem 6.5 is a geometry iff the underlying module is torsion free.*[3]

**Proof.** Since $\langle\emptyset\rangle_{\mathrm{a}} = \emptyset$ we have to check when we have $\langle m\rangle_{\mathrm{a}} = \{ m \}$ for arbitrary $m \in M$. Since $\langle m\rangle_{\mathrm{a}} = \{ n \in M \mid bn = bm,\ b \neq 0 \}$ the corollary follows immediately. □

Let us simplify the terminology slightly. Every concept relating to d-submodules is prefixed with d- (d-closure, d-matroids, etc.), and similarly for a-submodules (a-).

Before we go on note that the d- and a-matroids over a certain module are closely related. For instance, it is straightforward to check that a module has d-rank $n$ iff it has a-rank $n + 1$.

### *6.4 Representations*

From now on let all modules be modules over integral domains.

[3] An $R$-module $M$ is *torsion free* if $r \in R \setminus 0$ and $m \in M \setminus 0$ implies $rm \neq 0$.

The following theorem shows that the *representation* of an element with respect to a certain basis is uniquely defined up to a scalar factor.

**Theorem 6.7 (The Representation Theorem)** *Let $B$ be an a-basis for the $R$-module $M$. Assume that $p \in M$ has the representation*

$$cp = \sum_{i=1}^{n} a_i b_i,\ n \in \mathbb{Z}^+,\ c, a_i \in R \setminus 0,\ c = \sum_{i=1}^{n} a_i,\ b_i \in B,\ b_i \neq b_j \text{ if } i \neq j$$

*in this basis. Then the only other representations of $p$ in this basis are $dp = \sum_{i=1}^{n} \frac{da_i}{c} b_i$, where $d \in R \setminus 0$ and all $\frac{da_i}{c}$ are assumed to be well-defined.*

(Here $\frac{da_i}{c}$ is shorthand for any solution $x \in R$ to the equation $da_i = cx$; such a solution is unique if it exists.)

**Proof.** The theorem is straightforward to prove. Just assume that there are two different representations and deduce that the basis is not independent. □

Note that if the module is not torsion free, then any particular representation does not necessarily stand for a unique module element. Note also that since we require $c = \sum_{i=1}^{n} a_i$ the representations should really be termed *a-representations*. The theorem also holds for *d-representations*, where the requirement $c = \sum_{i=1}^{n} a_i$ is dropped and $n = 0$ is allowed.

### *6.5 Embedding*

Let $M$ be a module over an integral domain $R$. Define an equivalence relation on $M \times (R \setminus 0)$ by $(m, r) \sim (m', r')$ iff there is an $s \in R \setminus 0$ such that $s(r'm - rm') = 0$. The equivalence classes are the elements of the *module of fractions* $F(M)$. Use the notation $\frac{m}{r}$ for the equivalence class containing $(m, r)$, and identify $m$ with $\frac{m}{1}$ for all $m \in M$. By defining addition in this new structure by $\frac{m}{r} + \frac{m'}{r'} := \frac{mr'+m'r}{rr'}$ and multiplication by $\frac{m}{r}\frac{m'}{r'} := \frac{mm'}{rr'}$ we get an $F(R)$-module, where $F(R)$ is the field of quotients [8] associated with $R$. The structure of $M$ is preserved within $F(M)$ iff $M$ is torsion free. This construction is described by Taylor [12].

The canonical map $\pi : M \to F(M)$ is defined by $\pi(m) := \frac{m}{1}$. Denote the preimage of $\pi$ by $\pi^{-1}$, i.e. $\pi^{-1}\left(\frac{m}{r}\right) = \{\, m' \in M \mid \pi(m') = \frac{m}{r} \,\}$. For notational convenience let us use the convention that $\pi^{-1}(S) := \bigcup \{\, \pi^{-1}(s) \mid s \in S \,\}$ for subsets $S \subseteq F(M)$. If $M$ is torsion free then its ground set can be seen as a subset of $F(M)$, and we get that $\pi$ is essentially just the identity, while $\pi^{-1}$ for subsets $S \subseteq F(M)$ satisfies $\pi^{-1}(S) = S \cap M$.

We also need to define another function, $\mu : F(M) \to M$, which maps 0 to 0 and an element $m' \in F(M) \setminus 0$ to an arbitrary (but fixed) element in the nonempty set $\{\, m \in M \setminus 0 \mid r \in R,\ m = rm' \,\}$.

Let us now show in what way the d-matroid structure carries over to the associated vector space. Note first that in a vector space the d-submodule closure $\langle\cdot\rangle_{\mathrm{d}}$ equals the vector subspace (linear) closure. Furthermore independence, bases, etc. match the corresponding d-matroid concepts exactly.

**Theorem 6.8** *Let $M$ be an $R$-module, and let $F(M)$ be the module of fractions associated with $M$. Denote the d-submodule closure in $F(M)$ by $\langle\cdot\rangle_{\mathrm{D}}$. Then we have the following properties.*

(i) *For any subset $S \subseteq M$ the equality $\langle S\rangle_{\mathrm{d}} = \pi^{-1}\big(\langle\pi(S)\rangle_{\mathrm{D}}\big)$ holds.*

(ii) *Let $D$ be a d-submodule of $M$ with d-basis $B$. Then $F(R)\pi(D) = F(D)$ is a vector subspace with basis $\pi(B)$, and the d-rank of $D$ equals the dimension of $F(D)$.*

(iii) *Let $S$ be a vector subspace of $F(M)$ with basis $B$. Then $\pi^{-1}(S)$ is a d-submodule of $M$ with d-basis $\mu(B)$, and the dimension of $S$ equals the d-rank of $\pi^{-1}(S)$.*

**Proof.** Straightforward to prove using the explicit characterisations of $\langle\cdot\rangle_{\mathrm{d}}$, $\langle\cdot\rangle_{\mathrm{D}}$, $\pi$, and $\pi^{-1}$, together with the definition of the equivalence relation that $F(M)$ is based on. $\square$

**Corollary 6.9** *The lattice of subspaces of the d-matroid $M$ is isomorphic to that of $F(M)$, and the d-rank of $M$ coincides with the dimension of $F(M)$.*

Given some knowledge about vector space theory (see e.g. [5]) this corollary immediately implies that all d-matroids are modular.

Since a-matroids have subspaces which are just translations of d-subspaces all the results above can be transformed to an a-matroid context. For instance, for any subset $S \subseteq M$ and any $s \in \langle S\rangle_{\mathrm{a}}$ we have

$$\begin{aligned}\langle S\rangle_{\mathrm{a}} &= \langle S-s\rangle_{\mathrm{d}} + s = \pi^{-1}\big(\langle\pi(S-s)\rangle_{\mathrm{D}}\big) + s \\ &= \pi^{-1}\big(\langle\pi(S)-\pi(s)\rangle_{\mathrm{D}} + \pi(s)\big) = \pi^{-1}\big(\langle\pi(S)\rangle_{\mathrm{A}}\big),\end{aligned} \tag{2}$$

where $\langle\cdot\rangle_{\mathrm{A}}$ is a-submodule closure in $F(M)$. Furthermore the lattice of subspaces of any a-matroid $M$ is isomorphic to that of the a-matroid over $F(M)$, and by using some simple theory about matroid degrees which we cannot go into here (see [5]) we get that all a-matroids are of degree 1.

### *6.6 Affine Geometry*

Given that all a-matroids are of degree 1, and that all a-matroids over modules that are torsion free are geometries, is an a-geometry an affine geometry? Not necessarily, as we will show.

Take the standard $\mathbb{Z}$-module over $\mathbb{Z}^2$. This module is torsion free and is hence an a-geometry of degree 1. Take a look at the line configuration in Figure 3. According to the definition of parallelity (see Section 2) both $\ell_1$ and $\ell_2$ are parallel to $\ell$, and since they intersect the geometry is not affine.

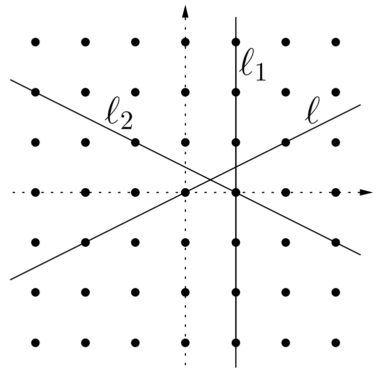


Fig. 3. An example demonstrating why the standard $\mathbb{Z}$-module over $\mathbb{Z}^2$ is not affine. The lines $\ell_1$ and $\ell_2$ are both parallel to $\ell$.

The $\mathbb{Z}$-module over $\mathbb{Z}^2$ clearly has an affine feel to it. The reason why it is not affine is that two lines which are non-parallel in the associated vector space can be disjoint, and hence parallel; the problem lies in the discreteness of the structure. The following proposition shows that it is easy to define a notion of parallelity which at least satisfies some of the usual requirements of a parallelity relation. Of course this definition is influenced by the fact that the same approach gives the proper parallelity relation in an a-geometry over a vector space.

**Proposition 6.10** *Let $M$ be an $R$-module. Define a binary relation $|||$ on the lines of $M$ by $\ell \,|||\, \ell'$ iff there is some $p \in M$ such that $\ell + p = \ell'$. Then $|||$ is an equivalence relation, and for any point $p \in M$ and line $\ell \subseteq M$ there is a unique line $\ell'$ such that $p \in \ell'$ and $\ell \,|||\, \ell'$.*

**Proof.** Straightforward using the facts that d-submodules are subgroups (since they contain 0 and are closed under inverse and addition) and two right cosets of a subgroup are either disjoint or equal. □

The question about what would make a suitable definition of a discrete affine geometry remains open. However, we can at least motivate why $|||$ seems to be a valid parallelity relation (although, in general, it is not). It is straightforward to show that $||| \subseteq ||$, and in an a-geometry over a vector space we have $||| = ||$. By using the module of fractions construction we get the following result (proof omitted).

**Proposition 6.11** *Let $M$ be an $R$-module, and let $\ell_1, \ell_2 \subseteq M$ be two lines. Then $\ell_1 \,|||\, \ell_2$ iff $\pi(\ell_1) \,|||\, \pi(\ell_2)$, i.e. iff $\pi(\ell_1) \,||\, \pi(\ell_2)$, where $\pi$ is the canonical map into the vector space $F(M)$ as defined in Section 6.5.*

### *6.7 Generators and Isomorphism*

The generator properties defined below can perhaps serve as an indication of whether a geometry is discrete or not. They are based on a generalisation of Hübler's generators, see Section 5. Let us say that a set $S$ s-generates a submodule $M$ if $M = \langle S \rangle_{\mathrm{s}}$, where $\langle \cdot \rangle_{\mathrm{s}}$ is the closure operator from Lemma 6.1. (The standard terminology in algebra is just to say that $S$ generates $M$, but the prefix s- reduces the risk of confusion.)

**Definition 6.12** Let $M$ be a d-matroid over an $R$-module with d-rank at least $n$. This matroid has the *rank $n$ generator property* if all d-submodules of d-rank $n$ are s-generated by a set of points of cardinality $n$. The elements of this set are called *generators*.

Note that the rank $n$ generator property implies that all a-submodules of rank $n+1$ are s-generated by a set of cardinality $n$ (plus the usual translation). Hence, when $n = 1$ we use the term *line generator property*, or more often just generator property. For $n = 2$ we use the term *plane generator property*.

Let us show that in some cases the properties for different $n$ are not independent. In fact, while we are indulging in the theory of finitely s-generated, torsion free modules over principal ideal domains [4] we might as well throw in a result about isomorphism as well. A module is *finitely s-generated* if there is some finite subset which s-generates the module.

**Theorem 6.13** *All finitely s-generated torsion free modules of d-rank $n$ over a principal ideal domain $R$ are isomorphic to the $R$-module over $R^n$, and the corresponding d-matroids satisfy the rank $m$ generator property for all $m \leq n$.*

**Proof.** Use the theory about modules over principal ideal domains as presented by e.g. Taylor [12]. $\square$

**Corollary 6.14** *Let $M$ be a d-matroid over a torsion free $R$-module satisfying the rank $n$ generator property, where $R$ is a principal ideal domain and $n$ is finite. Then the rank $m$ generator property holds for all $m \leq n$.*

# 7 Modules with Order

In this section we assume that all modules are modules over ordered domains. [5]

## 7.1 Ordered Lines

By using the order of an ordered domain we can easily induce an order on the set of points of a line.

**Theorem 7.1** *Let $M$ be an $R$-module and $\ell = \langle p, q \rangle_{\mathrm{a}} \subseteq M$ a line. Take any two a-rank 1 subspaces $r_1, r_2 \subseteq M$ with elements $r_1' \in r_1$, $r_2' \in r_2$ given by $b_i r_i' = a_{i1}p + a_{i2}q$, $b_i, a_{ij} \in R$, $b_i = a_{i1} + a_{i2} > 0$, $i, j \in \{1, 2\}$. These subspaces*

[4] A *principal ideal domain* is an integral domain $R$ in which every *ideal* (subset closed under addition and multiplication) $I$ is *principal* (i.e. $I = \{ r'r \mid r' \in R \}$ for some $r \in R$).

[5] An *ordered domain* is an integral domain $R$ together with a nonempty subset $R^+ \subseteq R$ of *positive elements* such that $R^+$ is closed under addition and multiplication and exactly one of $r \in R^+$, $r = 0$, and $-r \in R^+$ holds for any $r \in R$. Given this $r_1 < r_2$ iff $r_2 - r_1 \in R^+$.

*satisfy* $r_1 = r_2$ *iff* $a_{12}a_{21} = a_{11}a_{22}$*, independently of which members* $r_1'$, $r_2'$ *were chosen.*

*Define* $r_1 < r_2$ *iff* $a_{12}a_{21} < a_{11}a_{22}$*. This is well-defined, and the relation* $\leq := < \bigcup =$ *is a total order on the a-rank 1 subspaces of* $\ell$*. By swapping* $p$ *and* $q$ *we get the opposite order (*$\geq$*), and these two orders are invariant when going to another basis of* $\ell$*.*

**Proof.** Relatively straightforward using the Representation Theorem 6.7 and multiplicative isotony. To prove transitivity it might be helpful to pass to the field of quotients $F(R)$ [8] and use that $a_{12}a_{21} < a_{11}a_{22}$ is equivalent to $b_1a_{21} < b_2a_{11}$. □

Note that in the case of torsion free modules this result means that the set of points of any line can be totally ordered. For convenience we define $r_1' < r_2'$ ($r_1' \leq r_2'$) iff $r_1' \in r_1$, $r_2' \in r_2$ and $r_1 < r_2$ ($r_1 \leq r_2$). Beware that when the module is not torsion free these new relations do not have all their usual properties, though.

### *7.2 Convexity and Antimatroids*

We will now treat convexity. We depart from the procedure used when introducing the d- and a-closure. Instead of defining the closed sets first, and then deducing the closure operator, we just define the closure operator. Note that this convex hull operator is related to the standard vector space convex hull operator (as given in [5]),

$$[S]_{\mathrm{V}} := \left\{ \sum_{i=1}^{n} a_i s_i \,\middle|\, s_i \in S,\ a_i \in R,\ \sum_{i=1}^{n} a_i = 1,\ 0 \leq a_i \leq 1,\ n \in \mathbb{Z}^+ \right\}, \quad (3)$$

in the same way as the d- and a-closures are related to the corresponding vector space closures. Of course the operators are chosen to agree in the case where the module is a vector space.

**Theorem 7.2** *Let* $M$ *be an* $R$*-module. Define the* convex hull operator $[\cdot] : \wp(M) \to \wp(M)$ *by*

$$[S] := \left\{ m \in M \,\middle|\, \begin{array}{r} bm = \sum_{i=1}^{n} a_i s_i,\ s_i \in S,\ b, a_i \in R, \\ b = \sum_{i=1}^{n} a_i > 0,\ 0 \leq a_i \leq b,\ n \in \mathbb{Z}^+ \end{array} \right\}.$$

*Then* $(M, [\cdot])$ *is a finitary closure space satisfying* $[\emptyset] = \emptyset$*. The subspaces are called* convex*. Furthermore the following properties are equivalent:*

(i) *$M$ is torsion free,*

(ii) *$(M, [\cdot])$ is an antimatroid, and*

(iii) *$(M, [\cdot])$ is simple.*

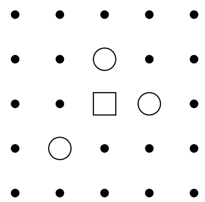

Fig. 4. The figure shows an excerpt of the standard $\mathbb{Z}$-module over $\mathbb{Z}^2$. The set consisting of the three circles is betweenness closed, but the convex hull of these points also includes the square. (This example is presented in e.g. [6].)

Maybe it is inappropriate to use the term convex if the anti-exchange property does not hold, but that is a minor issue. As an example of why non-torsion free modules do not yield antimatroids, consider an infinite discrete cylinder represented as the standard $\mathbb{Z}$-module over $\mathbb{Z}\times\mathbb{Z}_n$ for some $n \in \mathbb{Z}^+\setminus 1$. We get that $[(i,j)] = \{\, (i,j') \mid j' \in \mathbb{Z}_n \,\}$, i.e. the convex hull of a single point is a "circle" going around the cylinder. This clearly violates the anti-exchange property.

Most parts of the theorem are easy to prove by using the module of fractions construction, since we know that $(F(M), [\cdot]_{\mathrm{V}})$ is an antimatroid. We only state the following lemma, the rest of the proof is omitted.

**Lemma 7.3** *Let $M$ be an $R$-module with convex hull operator $[\cdot]$, and let $[\cdot]_{\mathrm{V}}$ be the convex hull operator of $F(M)$. Then $[S] = \pi^{-1}\big([\pi(S)]_{\mathrm{V}}\big)$, where $\pi$ and $\pi^{-1}$ are the functions defined in Section 6.5.*

Let us also give some properties of the convex sets. Let $M$ be an $R$-module with $p, q \in M$. If $\{\, p, q \,\}$ is a-independent then the *line segment* $\langle p, q\rangle_\ell$ is defined by $\langle p, q\rangle_\ell := \{\, r \in \langle p, q\rangle_{\mathrm{a}} \mid p \le r \le q$ or $p \ge r \ge q \,\}$. Here $\le$ and $\ge$ are the two point orders for $\langle p, q\rangle_{\mathrm{a}}$ as given in Theorem 7.1. If $\{\, p, q \,\}$ is not a-independent then $\langle p, q\rangle_\ell := \langle p, q\rangle_{\mathrm{a}}$. It is easy to verify that $\langle p, q\rangle_\ell = [p, q]$. It follows that a convex set is betweenness closed; i.e. if $p, q \in [S]$ then $\langle p, q\rangle_\ell \subseteq [S]$. The converse is not true, a betweenness closed set may not be convex. See Figure 4.

### *7.3 Oriented Matroids*

As mentioned before several abstract frameworks for convexity are used. The antimatroid approach was treated above, and now we will take another approach by using oriented matroid theory. First we have to show in what way we get an oriented matroid from a module.

**Definition 7.4** Let $M$ be an $R$-module, $H \subseteq M$ a d-hyperplane, and $x \in M \setminus H$. Define the *open half-spaces* $H^+$ and $H^-$ by

$$H^+ := \{\, m \in M \mid bm = h + ax,\ a, b \in R,\ h \in H,\ ab > 0 \,\} \text{ and}$$
$$H^- := \{\, m \in M \mid bm = h + ax,\ a, b \in R,\ h \in H,\ ab < 0 \,\}.$$

For ease of reference let us denote $x$ as the *positive point* of $H$.

Let us now show that this definition is consistent with oriented matroid terminology, and that it gives rise to an oriented matroid structure on $M$.

**Lemma 7.5** *The open half-spaces of Definition 7.4 are both non-empty, and they satisfy $H^+ \cup H^- = M \setminus H$ and $H^+ \cap H^- = \emptyset$. In other words $(H^-, H^+)$ is a cocircuit, as is the opposite $(H^+, H^-)$. Furthermore the half-spaces are independent of the choice of positive point $x \in M \setminus H$; given another $x' \in H^+$ we get the same cocircuit, and given $x'' \in H^-$ we get the opposite cocircuit.*

**Proof.** Straightforward using the Representation Theorem 6.7 and some properties regarding matroidal bases. $\square$

**Lemma 7.6** *Let $M$ be an $R$-module with two different d-hyperplanes $H$ and $K$ intersecting in a subspace of d-corank 2. Let $x$ be any point in $M \setminus (H \cup K)$. Choose the orientation of the cocircuits associated with $H$ and $K$ such that $x \in H^+ \cap K^-$. Then the hyperplane $L = x \vee (H \wedge K)$ satisfies $L^+ \subseteq H^+ \cup K^+$ and $L^- \subseteq H^- \cup K^-$, given a suitable choice of its orientation.*

**Proof.** Relatively straightforward by using $x$ as the positive point for $H$, $-x$ for $K$, and any point in $H^+ \cap K$ for $L$. $\square$

We can sum up the preceding results in a theorem.

**Theorem 7.7** *Every module over an ordered domain has an oriented matroid structure given by its d-matroid structure and the open half-spaces as defined in Definition 7.4.*

Note that for some modules, e.g. the $\mathbb{Z}$-module over $(\mathbb{Z}_n)^m$, the oriented matroid structure obtained is trivial. This follows since the underlying matroid is trivial and does not have any hyperplanes.

It is easy to extend the results above to a-matroids. Given a hyperplane $H$ of an a-matroid $M$ with $h \in H$ and $x \in M \setminus H$ we get that $H - h$ is a d-hyperplane, and by letting $x - h$ be the positive point we get the cocircuit $((H-h)^-, (H-h)^+)$. Hence we can define the cocircuit associated with $H$ with $x$ as positive point to be $(H^-, H^+) := ((H-h)^- + h, (H-h)^+ + h)$. It is straightforward to check that this is a proper cocircuit, that it is independent of the choice of $h \in H$, and that the resulting structure is an oriented matroid.

Denote the convex closure obtained from this oriented a-matroid (see Section 4) by $[\cdot]_{\mathrm{OM}}$. Using some simple algebraic manipulations it is easy to show that $[S] \subseteq [S]_{\mathrm{OM}}$ for any $S \subseteq M$. It is unknown whether the converse holds though, although it seems likely.

## 8 Modules and Hübler Geometries

Given the previous sections it is relatively straightforward to characterise Hübler geometries in terms of $\mathbb{Z}$-modules. Due to the lack of space we only give the main results. First note that there is a superficially significant difference between a-geometries and Hübler geometries in that the latter have both points and translations, while the a-geometries only have points. However, by choosing an origin we get that each Hübler point corresponds to a unique translation (by the first four axioms). The following theorem assumes that an origin has been chosen, thereby avoiding the differences by identifying points and translations.

**Theorem 8.1** *The models of Hübler's first seven axioms are exactly those a-geometries over $\mathbb{Z}$-modules that exhibit the generator property and have d-rank at least 2, if the lines of the a-geometries are used as Hübler lines, the "parallelity" $|||$ as Hübler parallelity, and the orders as defined in Theorem 7.1 as the orders on the set of points of a line. Furthermore the Hübler planes are exactly the a-geometry planes. Given a model of the first seven axioms the eighth axiom is equivalent to the plane generator property.*

Recall that it is unknown, at least to the authors, whether the eighth axiom is independent from the others or not.

Given this result it is possible to show that there are Hübler geometries of any d-rank greater than or equal to 2, finite or infinite. Just let Dim be any set with $|\text{Dim}| \geq 2$, and take the $\mathbb{Z}$-module $\text{Dim} \to \mathbb{Z}$ with the obvious addition and scalar multiplication. This module is easily seen to be torsion free with d-rank at least 2. It is trickier to show that the geometry satisfies the line and plane generator properties, and we omit the proof here.

Hübler also discusses convexity, although most of his discussions are limited to the d-rank 2 case. Given the results in Section 7 it is easy to see that a general discussion of convexity is possible for Hübler geometries.

## 9 Conclusions and Future Work

This exposition makes it pretty clear that many (discrete) geometries can be treated within a framework based on matroids or oriented matroids, even though more work clearly needs to be done on the oriented matroid aspect. In fact infinite oriented matroid theory is a neglected area which deserves more attention. Even in the finite case the relation between the two approaches to convexity—antimatroids and oriented matroids—is not fully explored. However, it has been shown that some finite oriented matroids satisfy the anti-exchange property [4,1], and given this result it should not be too hard to relate the two convex closures given here properly.

The class of discrete geometries that are also modules over integral domains may not be as large as one would wish, it is e.g. hard to model the geometry of a cylinder as a module, and the same applies to many finite geometries as well. We hope that a more general framework, one that is still based on matroids but not on modules, will be capable of handling also those geometries.

Finally it is an open question whether generator properties are the right way to go for characterising discreteness.

## Acknowledgement

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