Matroids from Modules

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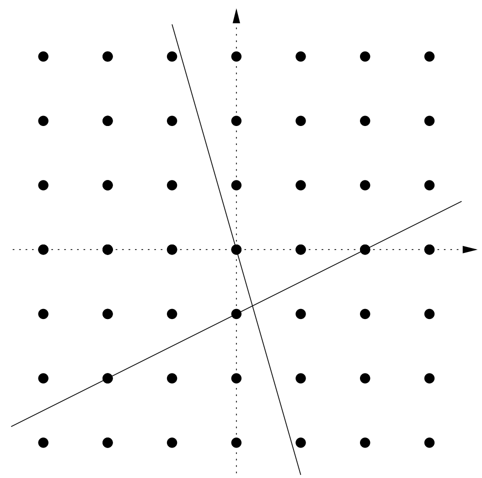
July 18, 2002

Motivation

- *Matroids* capture the essence of independence, dimension, etc.
- Standard example: Vector spaces.
- For discrete/digital geometry we do not have a vector space, but often a *module over an integral domain*.
- Standard example: $\mathbb{Z}$-module over $\mathbb{Z}^n$. (Compare with images made up of pixels.)



Modules

- Let R be a ring. An *R-module* is an abelian group M together with a scalar multiplication $R \times M \to M$ satisfying
  1. $r(m_1 + m_2) = rm_1 + rm_2$,
  2. $(r_1 + r_2)m = r_1 m + r_2 m$,
  3. $r_1(r_2 m) = (r_1 r_2)m$, and
  4. $1m = m$.
- An *integral domain* is a nontrivial commutative ring with no zero divisors ($xy \neq 0$ for all $x, y \neq 0$).

Matroids

- Ground set M, possibly infinite.
- Closure operator $\mathrm{cl} : \wp(M) \to \wp(M)$:
  - Monotone: $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
  - Increasing: $A \subseteq \mathrm{cl}(A)$.
  - Idempotent: $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
- Finitary: $x \in \mathrm{cl}(A) \Rightarrow x \in \mathrm{cl}(A')$ for some finite $A' \subseteq A$.
- Exchange property: $y \in \mathrm{cl}(A \cup x) \setminus \mathrm{cl}(A) \Rightarrow x \in \mathrm{cl}(A \cup y)$.

Infinite?

- The standard definition of matroids requires a finite ground set.
- Having an infinite set of e.g. points is often natural/useful in geometry.
- Infinite matroids retain some properties of finite matroids, but not all.
- References for infinite matroids:
  - Faure and Frölicher, *Modern Projective Geometry*, 2000.
  - Coppel, *Foundations of Convex Geometry*, 1998.

Subspaces

- A closure operator is determined by its closed sets (*subspaces*).
- Vector spaces: You get a matroid from the vector subspaces, and also from the affine subspaces.
- Modules: The submodules do not necessarily yield a matroid.
- Counterexample: The $\mathbb{Z}$-module over $\mathbb{Z}$; the exchange property fails:
  - $2 \in \langle 10, 3 \rangle_{\mathrm{s}} = \mathbb{Z}$,
  - $2 \notin \langle 10 \rangle_{\mathrm{s}} = 10\mathbb{Z}$,
  - $3 \notin \langle 10, 2 \rangle_{\mathrm{s}} = 2\mathbb{Z}$.

D-submodules

- Solution: Emulate the vector subspaces by including existing divisors.
- A *d-submodule* D of the R-module M is a submodule which is closed under existing divisors:

$$r \in R \setminus 0,\ m \in M,\ rm \in D \Rightarrow m \in D.$$

- When R is an integral domain this yields a matroid. Closure operator:

$$\langle S \rangle_{\mathrm{d}} = \left\{ m \in M \,\middle|\, bm = \sum_{i=1}^{n} a_i s_i,\ b, a_i \in R \setminus 0,\ s_i \in S,\ n \in \mathbb{N} \right\}.$$

- From now on: Let all rings be integral domains.

"Affine" Geometry

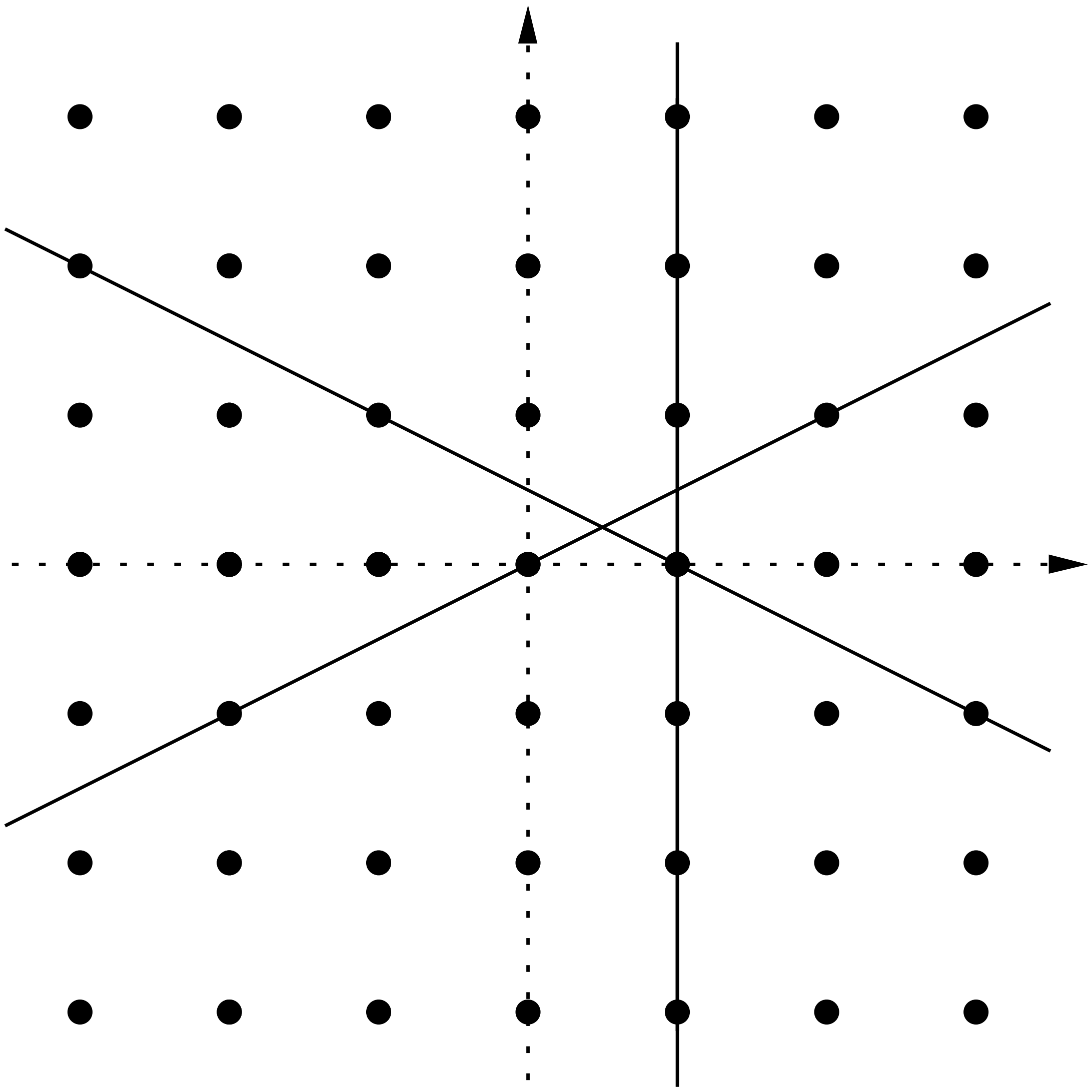
- "Affine" submodules—*a-submodules*—are translated d-submodules.
- The a-submodules plus $\emptyset$ also yield a matroid. Closure operator: $\langle\emptyset\rangle_{\mathrm{a}} = \emptyset$, $\langle S\rangle_{\mathrm{a}} = \langle S - s\rangle_{\mathrm{d}} + s$ for any $s \in \langle S\rangle_{\mathrm{a}}$.
- Extra properties: $\langle\emptyset\rangle_{\mathrm{a}} = \emptyset$ (obviously), $\langle\{\, p \,\}\rangle_{\mathrm{a}} = \{\, p \,\}$ (iff $rm \neq 0$ for any $r \in R \setminus 0$ and $m \in M \setminus 0$).
- Thus we get a *geometry* (matroid with the two extra properties).

Bases, Rank

- A subset B is *independent* if $\forall x \in B \,.\, x \notin \mathrm{cl}(B \setminus x)$.
- If B is independent and $\mathrm{cl}(B) = A$, then B is a *basis* of A.
- Every closed set A has a basis, and all bases of A are equipotent.
- The cardinality of any basis of A is the *rank* of A.

Lines, Planes, Parallelity

- *Lines* are subspaces of rank 2, *planes* subspaces of rank 3.
- An a-submodule geometry is not in general affine since lines in the same plane can cross without intersecting.
- Something reminiscent of affine parallelity can still be defined; two lines ℓ, ℓ' are *pseudo-parallel* ($\ell \,|||\, \ell'$) if there is some $p \in M$ such that $\ell = \ell' + p$.
- For any point $p \in M$ and line $\ell \subseteq M$ there is a unique line ℓ' such that $p \in \ell'$ and $\ell \,|||\, \ell'$.



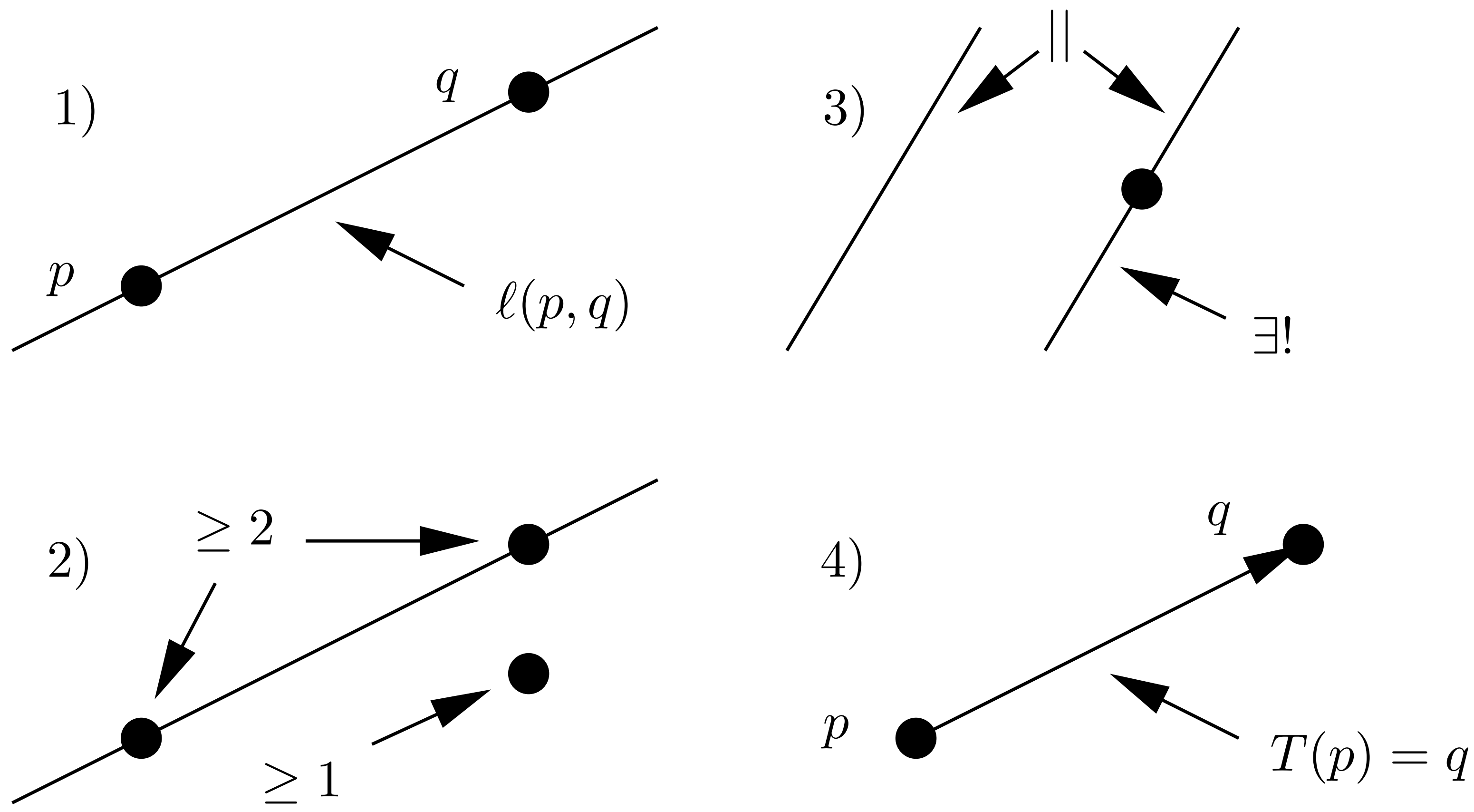
Degrees and Affine Geometry

- A geometry is of degree n if it satisfies, for any subspaces E, F: If $r(E \wedge F) \geq n$ then $r(E \wedge F) + r(E \vee F) = r(E) + r(F)$.
- $E \wedge F = E \cap F$, $E \vee F = \mathrm{cl}(E \cup F)$.
- A-submodule geometries are of degree 1.
- Two lines are parallel if they are equal, or if they are disjoint and span a plane.
- A geometry of degree 1 is affine if for every line $\ell \subseteq M$ and point $p \in M \setminus \ell$ there is a unique line ℓ', parallel to ℓ, with $p \in \ell'$.
- The figure on slide 11 shows that some a-submodule geometries are not affine.

Hübler's Axiomatic Discrete Geometry

- Hübler has developed an axiom system with the intention to capture the essence of discrete geometry as utilised in image processing and computer graphics.
- Albrecht Hübler, *Diskrete Geometrie für die Digitale Bildverarbeitung*, Habilitationsschrift, Friedrich-Schiller-Universität, Jena, 1989.

Axioms 1–4



Axioms 1–3

The axiom system assumes the existence of a point set $\mathcal{P}$ and a nonempty line set $\mathcal{L} \subseteq \wp(\mathcal{P})$.

1. For each pair of distinct points p, q there is a unique line $\ell(p, q)$ including the points.

2. For each line there are at least two points included in the line, and at least one point not included in the line.

3. There exists an equivalence relation on $\mathcal{L}$, parallelity $(||)$. For each line and point there is a unique line, including the point, which is parallel to the first line.

Translations

Translations are bijections T on $\mathcal{P}$ satisfying either $T = \mathsf{id}$ or

1. $T(\ell) \,||\, \ell$ for all $\ell \in \mathcal{L}$ (lines are mapped bijectively onto parallel lines),
2. $T(p) \neq p$ for all $p \in \mathcal{P}$, and
3. $\{\, \ell(p, T(p)) \,|\, p \in \mathcal{P} \,\}$ is an equivalence class of $||$.

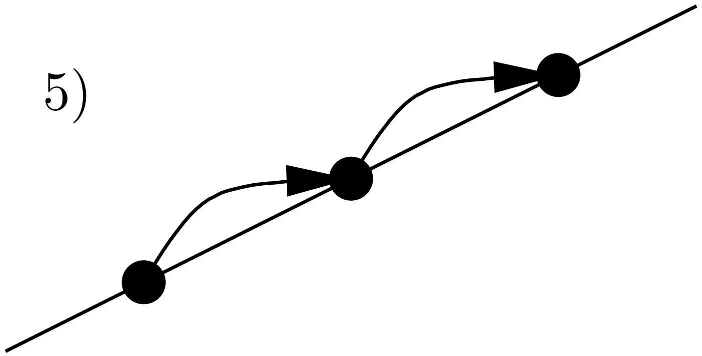
Axiom 4

4. For each pair of points p, q there is a translation mapping p to q.

- This translation turns out to be unique. Choose an origin $0 \Rightarrow$ we can identify points and translations.
- The set of all translations ($\mathcal{T}$) is an abelian group under function composition $\Rightarrow$ we have a $\mathbb{Z}$-module.
- $\ell \,||\, \ell' \Leftrightarrow \exists T \in \mathcal{T} \,.\, \ell = T(\ell')$. Compare with $|||$.

Axioms 5–8

5)



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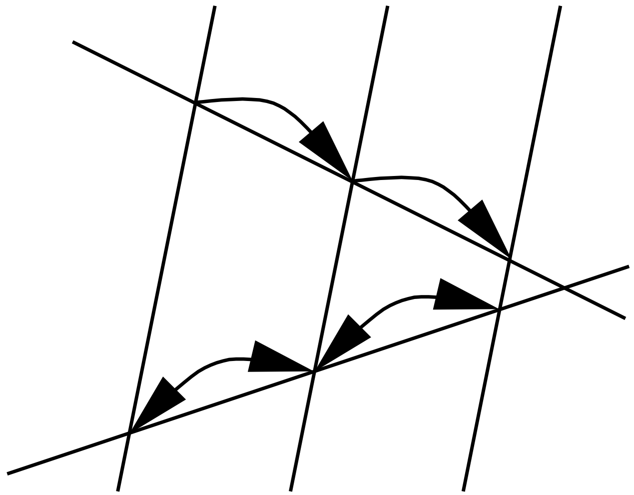

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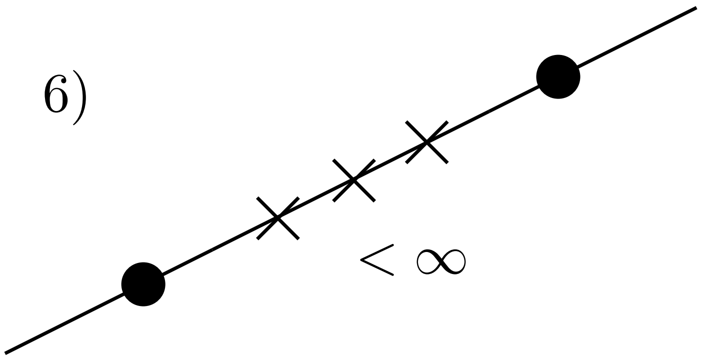
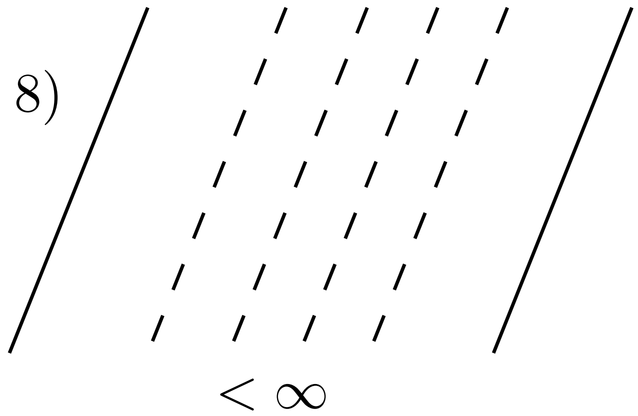


8)



Axioms 5–7

A further assumption is the existence of two opposite total orders $\leq$, $\geq$ defined on the points of each line.

5. For each point p on a line ℓ there are two other, different points $q, r \in \ell$ with $q < p < r$.
6. Given two points p, q on a line ℓ, the set of all points $r \in \ell$ satisfying $p < r < q$ is finite.
7. Let ℓ_1, ℓ_2, and ℓ_3 be different, parallel lines, and ℓ and ℓ' lines that have points p_i and p'_i, respectively, in common with all the lines $\ell_i,\ i \in \{1, 2, 3\}$. Then $p_1 < p_2 < p_3$ holds iff $p'_1 < p'_2 < p'_3$ or $p'_1 > p'_2 > p'_3$.

Generators

- All lines ℓ can be written in the form

$$\ell = \{\, G^n(p) \mid n \in \mathbb{Z} \,\}$$

for an arbitrary point $p \in \ell$ and a unique (up to inverses) translation (*generator*) G.

Axiom 8

A line ℓ is between two other parallel, different lines if a fourth line intersects all the other lines and the intersection with ℓ is between the other intersections.

8. The set of all lines between two different, parallel lines is finite.

- Axiom 6 is made redundant by Axiom 8.
- These two axioms are included to make the geometry discrete.
- Planes also have generators.

Correspondence

The Hübler geometries are exactly the a-submodule geometries M over $\mathbb{Z}$-modules with rank ≥ 3 satisfying

1. for every line ℓ,

$$\ell = \{\, p + ng \mid n \in \mathbb{Z} \,\}$$

   for some $p, g \in M$, and

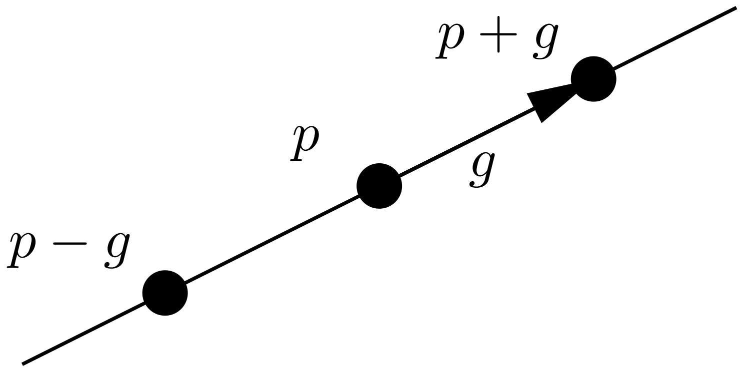
2. for every plane P,

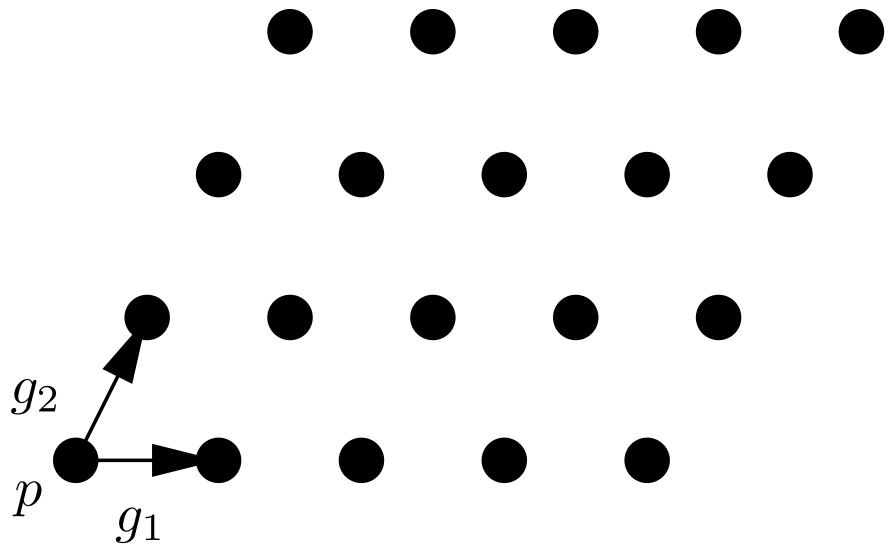
$$P = \{\, p + n_1 g_1 + n_2 g_2 \mid n_1, n_2 \in \mathbb{Z} \,\}$$

   for some $p, g_1, g_2 \in M$.

Here $\mathcal{P} = M$, the lines are the rank 2 subspaces, $|| = |||$, and $p + n_1 g \leq p + n_2 g$ iff $n_1 \leq n_2$.

Correspondence





Conclusion

- Easy to generalise vector space matroids to modules over integral domains.
- The module approach allows discrete structures to be modelled. No need to embed these structures in e.g. Euclidean space.
- We have demonstrated this by giving a characterisation of Hübler's geometries which is arguably easier to understand than the original axioms.

Possible Future Work

- It is easy to define a convex hull operator, analogously to the standard vector space convex hull.
- A natural next step is to connect modules to *oriented* matroids. The theory for infinite oriented matroids does not seem to be well-developed, though.