Bag Equivalence via a Proof-Relevant Membership Relation

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ITP 2012, Princeton

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement n° 247219. This presentation does not necessarily reflect the views of the ERC or the EU. The EU is not liable for any use of the presented information. I do not hold the copyright to the EU emblem or the ERC logo.

Bag equivalence

Equality up to reordering of elements,
or equality when seen as bags:

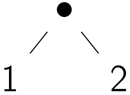
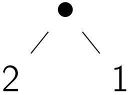
$$
\begin{aligned}
[1,2,1] &\approx_{bag} [2,1,1] \\
[1,2,1] &\not\approx_{bag} [2,1] \\
[1,2,1] &\approx_{set} [2,1]
\end{aligned}
$$

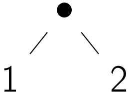
Why?

Partial specification of sorting algorithm:

$$\forall\, xs. \quad sort\ xs \ \approx_{bag}\ xs$$

Not restricted to lists

 $\approx_{bag}$ 

 $\approx_{bag} [2, 1]$

Why?

Tree sort:

$$\begin{aligned} \textit{to-search-tree} &: \textit{List}\ \mathbb{N} \to \textit{Tree}\ \mathbb{N} \\ \textit{flatten} &: \textit{Tree}\ \mathbb{N} \to \textit{List}\ \mathbb{N} \end{aligned}$$

$$\begin{aligned} \textit{tree-sort} &: \textit{List}\ \mathbb{N} \to \textit{List}\ \mathbb{N} \\ \textit{tree-sort} &= \textit{flatten} \circ \textit{to-search-tree} \end{aligned}$$

We can prove

$$\forall\ xs.\ \textit{tree-sort}\ xs \approx_{bag} xs$$

by first proving

$$\begin{aligned} &\forall\ xs.\ \textit{to-search-tree}\ xs \approx_{bag} xs \\ &\forall\ t.\ \ \textit{flatten}\ t \approx_{bag} t \end{aligned}$$

Not restricted to finite things

$$[1, 2, 1, 2, \ldots] \approx_{bag} [2, 1, 2, 1, \ldots]$$

Why?

Assume semantics of grammar given by

$$\mathcal{L} \ : \ \mathit{Grammar} \ \rightarrow \ \mathit{Colist\ String}$$

Language equivalence:

$$\mathcal{L}\ G_1 \ \approx_{set} \mathcal{L}\ G_2$$

If we want to distinguish between ambiguous and unambiguous grammars:

$$\mathcal{L}\ G_1 \ \approx_{bag} \mathcal{L}\ G_2$$

Definitions

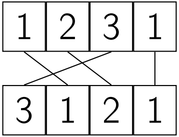
How is bag equivalence defined?

- Finite sequence of swaps of adjacent elements.
- Counting.
- Bijections.
- ...

Bag equivalence via bijections

Bijection on positions which relates equal elements:

$xs \ \approx_{bag} \ ys \ \Leftrightarrow$
$\quad \exists\, f : \text{positions of } xs \ \leftrightarrow \ \text{positions of } ys.$
$\quad \forall\, p.\ lookup\ xs\ p \ = \ lookup\ ys\ (f\ p)$



Generalises to anything with positions and *lookup*.

This talk

New definition of bag equivalence,
with the following properties:

- Many equivalences provable using "bijectional reasoning".
- Works for arbitrary unary containers (lists, streams, trees, ...).
- Generalises to set equivalence and subset and subbag preorders.
- Formalised in Agda, but the K rule is not used.

Definition

Any (Morris)

Any P xs means that *P x* holds for some *x* in *xs*.

$$
\begin{aligned}
&\mathit{Any} : (A \to \mathit{Set}) \to \mathit{List}\ A \to \mathit{Set} \\
&\mathit{Any}\ P\ [\,] = \bot \\
&\mathit{Any}\ P\ (x :: xs) = P\ x + \mathit{Any}\ P\ xs
\end{aligned}
$$

$$\mathit{Any}\ P\ [1, 2, 3] = P\ 1 + P\ 2 + P\ 3 + \bot$$

Membership

$$
\begin{aligned}
&\mathit{Any} \ : \ (A \ \to \ \mathit{Set}) \ \to \ \mathit{List}\ A \ \to \ \mathit{Set} \\
&\mathit{Any}\ P\ [\,] \qquad\qquad = \ \bot \\
&\mathit{Any}\ P\ (x :: xs) \ = \ P\ x + \mathit{Any}\ P\ xs
\end{aligned}
$$

$$
\begin{aligned}
&\_\!\in\!\_ \ : \ A \ \to \ \mathit{List}\ A \ \to \ \mathit{Set} \\
&x \in xs \ = \ \mathit{Any}\ (\lambda\ y.\ x \equiv y)\ xs
\end{aligned}
$$

$$
\begin{aligned}
&x \in [1, 2, 3] \ = \ (x \equiv 1) + (x \equiv 2) + (x \equiv 3) + \bot \\
&2 \in [2, 2] \qquad = \ (2 \equiv 2) + (2 \equiv 2) + \bot
\end{aligned}
$$

Bag equivalence

$$\begin{aligned}
&\mathit{Any} : (A \to \mathit{Set}) \to \mathit{List}\ A \to \mathit{Set} \\
&\mathit{Any}\ P\ [\,] \qquad\quad = \bot \\
&\mathit{Any}\ P\ (x :: \mathit{xs}) = P\ x + \mathit{Any}\ P\ \mathit{xs}
\end{aligned}$$

$$\begin{aligned}
&_\in_ : A \to \mathit{List}\ A \to \mathit{Set} \\
&x \in \mathit{xs} = \mathit{Any}\ (\lambda\ y.\ x \equiv y)\ \mathit{xs}
\end{aligned}$$

$$\begin{aligned}
&_\approx_{\mathit{bag}}_ : \mathit{List}\ A \to \mathit{List}\ A \to \mathit{Set} \\
&\mathit{xs} \approx_{\mathit{bag}} \mathit{ys} = \forall\ z.\quad z \in \mathit{xs} \leftrightarrow z \in \mathit{ys}
\end{aligned}$$

Caveat

What if there are several distinct proofs of $2 \equiv 2$?

$$2 \in [2, 2] \;=\; (2 \equiv 2) + (2 \equiv 2) + \bot$$

Correct

The two definitions are equivalent (without K):

$$
\begin{aligned}
&\mathit{xs} \;\approx_{\mathit{bag}}\; \mathit{ys} \;\Leftrightarrow \\
&\quad \exists\, f : \text{positions of } \mathit{xs} \;\leftrightarrow\; \text{positions of } \mathit{ys}. \\
&\quad \forall\, p.\; \mathit{lookup}\ \mathit{xs}\ p \;=\; \mathit{lookup}\ \mathit{ys}\ (f\ p)
\end{aligned}
$$

$$
\begin{aligned}
&\_\approx_{\mathit{bag}}\_ \;:\; \mathit{List}\ A \;\rightarrow\; \mathit{List}\ A \;\rightarrow\; \mathit{Set} \\
&\mathit{xs} \;\approx_{\mathit{bag}}\; \mathit{ys} \;=\; \forall\, z.\;\; z \in \mathit{xs} \;\leftrightarrow\; z \in \mathit{ys}
\end{aligned}
$$

If $\leftrightarrow$ is replaced by weak equivalence: *isomorphic*.

Bijectional reasoning

Example

Bind distributes from the left over append:

$$xs \gg\!= (\lambda\, y.\ f\ y\ +\!\!+\ g\ y)\ \approx_{bag}$$
$$(xs \gg\!= f)\ +\!\!+\ (xs \gg\!= g)$$

$$\_\!\gg\!=\!\_\ :\ List\ A\ \rightarrow\ (A\ \rightarrow\ List\ B)\ \rightarrow\ List\ B$$
$$xs \gg\!= f\ =\ concat\ (map\ f\ xs)$$

Example

Bind distributes from the left over append:

$$xs \gg\!= (\lambda\, y.\ f\ y\ +\!\!+\ g\ y)\ \approx_{bag}$$
$$(xs \gg\!= f)\ +\!\!+\ (xs \gg\!= g)$$

$$[1,2] \gg\!= (\lambda\, y.\ [y]\ +\!\!+\ [y])\ \approx_{bag}$$
$$([1,2] \gg\!= \lambda\, y.\ [y])\ +\!\!+\ ([1,2] \gg\!= \lambda\, y.\ [y])$$

Example

Bind distributes from the left over append:

$$xs \ggg= (\lambda\, y.\ f\ y\ +\!\!+\ g\ y)\ \approx_{bag}$$
$$(xs \ggg= f)\ +\!\!+\ (xs \ggg= g)$$

$$[1,1,2,2]\ \approx_{bag}$$
$$([1,2] \ggg= \lambda\, y.\, [y])\ +\!\!+\ ([1,2] \ggg= \lambda\, y.\, [y])$$

Example

Bind distributes from the left over append:

$$xs \gg\!= (\lambda\, y.\ f\ y\ +\!\!+\ g\ y)\ \approx_{bag}$$
$$(xs \gg\!= f)\ +\!\!+\ (xs \gg\!= g)$$

$$[1, 1, 2, 2]\ \approx_{bag}$$
$$[1, 2, 1, 2]$$

Outline of proof

Bijectional reasoning combinators
Any lemmas
Left distributivity

Bijectional reasoning combinators

$$
\begin{aligned}
\_\Box \quad & : (A : Set) \rightarrow A \leftrightarrow A \\
\_\leftrightarrow\langle\_\rangle\_ \quad & : (A : Set)\ \{B\ C : Set\} \rightarrow \\
& \quad A \leftrightarrow B \rightarrow B \leftrightarrow C \rightarrow A \leftrightarrow C
\end{aligned}
$$

Assume $p : A \leftrightarrow B$, $q : B \leftrightarrow C$.

$$
\begin{aligned}
& A \quad \leftrightarrow\langle\ p\ \rangle \\
& B \quad \leftrightarrow\langle\ q\ \rangle \\
& C \quad \Box
\end{aligned}
$$

Bijectional reasoning combinators

$$
\begin{aligned}
\_\,\square \quad & : (A : \mathit{Set}) \rightarrow A \leftrightarrow A \\
\_\leftrightarrow\langle\_\rangle\_ \quad & : (A : \mathit{Set})\ \{B\ C : \mathit{Set}\} \rightarrow \\
& \quad A \leftrightarrow B \rightarrow B \leftrightarrow C \rightarrow A \leftrightarrow C
\end{aligned}
$$

Assume $p : A \leftrightarrow B$, $q : B \leftrightarrow C$.

$$
C\,\square \quad : \quad C \leftrightarrow C
$$

Bijectional reasoning combinators

$$
\begin{array}{ll}
\_\Box & : (A : \mathit{Set}) \rightarrow A \leftrightarrow A \\
\_\leftrightarrow\langle\_\rangle\_ & : (A : \mathit{Set}) \{B\ C : \mathit{Set}\} \rightarrow \\
& \quad A \leftrightarrow B \rightarrow B \leftrightarrow C \rightarrow A \leftrightarrow C
\end{array}
$$

Assume $p : A \leftrightarrow B$, $q : B \leftrightarrow C$.

$$B \leftrightarrow\langle\ q\ \rangle\ (C\ \Box) \quad : \quad B \leftrightarrow C$$

Bijectional reasoning combinators

$$
\begin{aligned}
\_\,\square \quad &: (A : Set) \to A \leftrightarrow A \\
\_\leftrightarrow\langle\_\rangle\_ \quad &: (A : Set)\ \{B\ C : Set\} \to \\
&\quad A \leftrightarrow B \to B \leftrightarrow C \to A \leftrightarrow C
\end{aligned}
$$

Assume $p : A \leftrightarrow B$, $q : B \leftrightarrow C$.

$$
A \leftrightarrow\langle\ p\ \rangle\ (B \leftrightarrow\langle\ q\ \rangle\ (C\ \square)) \quad : \quad A \leftrightarrow C
$$

Bijectional reasoning combinators

$$\begin{array}{ll} \_\,\Box & : (A : Set) \to A \leftrightarrow A \\ \_\leftrightarrow\langle\_\rangle\_ & : (A : Set)\ \{B\ C : Set\} \to \\ & \quad A \leftrightarrow B \to B \leftrightarrow C \to A \leftrightarrow C \end{array}$$

Assume $p : A \leftrightarrow B$, $q : B \leftrightarrow C$.

$$\begin{array}{ll} A & \leftrightarrow\langle\ p\ \rangle \\ B & \leftrightarrow\langle\ q\ \rangle \\ C & \Box \end{array}$$

Outline of proof

Bijectional reasoning combinators
Any lemmas
Left distributivity

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\,(\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} \mathbin{+\!\!+} \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\mathit{Any}\text{-}{+\!\!+}\ P\ \mathit{xs}\ \mathit{ys} = ?$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\mathit{Any}\text{-}{+\!\!+}\ P\ [\,]\ \mathit{ys} = ?$$
$$\mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs})\ \mathit{ys} = ?$$

First lemma

$$Any\text{-}{+\!\!+} : (P : A \to Set)\,(xs\ ys : List\ A) \to$$
$$\quad Any\ P\,(xs +\!\!+ ys) \leftrightarrow Any\ P\ xs + Any\ P\ ys$$

$$\begin{aligned}
&Any\text{-}{+\!\!+}\ P\ [\,]\ \quad ys \ = \\
&\quad Any\ P\,([\,] +\!\!+ ys) \quad \leftrightarrow\langle\ ?\ \rangle \\
&\quad Any\ P\ [\,] + Any\ P\ ys \quad \square \\
&Any\text{-}{+\!\!+}\ P\ (x :: xs)\ ys \ = \ ?
\end{aligned}$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} \mathbin{+\!\!+} \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\mathit{Any}\text{-}{+\!\!+}\ P\ [\,]\ \mathit{ys} =$$
$$\quad \mathit{Any}\ P\ \mathit{ys} \quad \leftrightarrow\langle\ ?\ \rangle$$
$$\quad \mathit{Any}\ P\ [\,] + \mathit{Any}\ P\ \mathit{ys} \quad \Box$$
$$\mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs})\ \mathit{ys} = ?$$

First lemma

$$
\begin{array}{l}
\mathit{Any}\text{-}{+\!\!+} : (P : A \rightarrow \mathit{Set})\,(\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \rightarrow \\
\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}
\end{array}
$$

$$
\begin{array}{ll}
\mathit{Any}\text{-}{+\!\!+}\ P\ [\,] & \mathit{ys} \ = \\
\quad \mathit{Any}\ P\ \mathit{ys} & \leftrightarrow\langle\ ?\ \rangle \\
\quad \bot + \mathit{Any}\ P\ \mathit{ys} & \square \\
\mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs})\ \mathit{ys} & = \ ?
\end{array}
$$

First lemma

$$Any\text{-}{+\!\!+} : (P : A \to Set)\ (xs\ ys : List\ A) \to$$
$$\quad Any\ P\ (xs \mathbin{+\!\!+} ys) \leftrightarrow Any\ P\ xs + Any\ P\ ys$$

$$\begin{aligned}
&Any\text{-}{+\!\!+}\ P\ [\,]\ \ ys = \\
&\quad Any\ P\ ys \quad \leftrightarrow\langle\ \bot\ \text{identity of}\ +\ \rangle \\
&\quad \bot + Any\ P\ ys \quad \Box \\
&Any\text{-}{+\!\!+}\ P\ (x :: xs)\ ys = \ ?
\end{aligned}$$

First lemma

$$\begin{array}{l} \mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to \\ \quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys} \end{array}$$

$$\begin{array}{ll} \mathit{Any}\text{-}{+\!\!+}\ P\ [\,] & \mathit{ys} = \\ \quad \mathit{Any}\ P\ \mathit{ys} & \leftrightarrow\langle\ \bot\ \text{identity of}\ +\ \rangle \\ \quad \bot + \mathit{Any}\ P\ \mathit{ys} & \square \end{array}$$

$$\begin{array}{ll} \mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs})\ \mathit{ys} = & \\ \quad P\ x + \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) & \leftrightarrow\langle\ ?\ \rangle \\ \quad (P\ x + \mathit{Any}\ P\ \mathit{xs}) + \mathit{Any}\ P\ \mathit{ys} & \square \end{array}$$

First lemma

$$
\begin{aligned}
&\mathit{Any}\text{-}\mathbin{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to \\
&\quad \mathit{Any}\ P\ (\mathit{xs} \mathbin{+\!\!+} \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{Any}\text{-}\mathbin{+\!\!+}\ P\ []\qquad\qquad \mathit{ys} = \\
&\quad \mathit{Any}\ P\ \mathit{ys} \qquad\quad \leftrightarrow\langle\ \bot\ \text{identity of}\ +\ \rangle \\
&\quad \bot + \mathit{Any}\ P\ \mathit{ys} \quad \Box \\
&\mathit{Any}\text{-}\mathbin{+\!\!+}\ P\ (x :: \mathit{xs})\ \mathit{ys} = \\
&\quad P\ x + \mathit{Any}\ P\ (\mathit{xs} \mathbin{+\!\!+} \mathit{ys}) \qquad\qquad \leftrightarrow\langle\ \text{ind. hyp.}\ \rangle \\
&\quad P\ x + (\mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}) \quad \leftrightarrow\langle\ ?\ \rangle \\
&\quad (P\ x + \mathit{Any}\ P\ \mathit{xs}) + \mathit{Any}\ P\ \mathit{ys} \quad \Box
\end{aligned}
$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\begin{aligned}
&\mathit{Any}\text{-}{+\!\!+}\ P\ []\ \mathit{ys} =\\
&\quad \mathit{Any}\ P\ \mathit{ys} \quad \leftrightarrow\langle\ \bot\ \text{identity of}\ +\ \rangle\\
&\quad \bot + \mathit{Any}\ P\ \mathit{ys} \quad \Box
\end{aligned}$$

$$\begin{aligned}
&\mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs})\ \mathit{ys} =\\
&\quad P\ x + \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \quad \leftrightarrow\langle\ \text{ind. hyp.}\ \rangle\\
&\quad P\ x + (\mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}) \quad \leftrightarrow\langle\ +\ \text{associative}\ \rangle\\
&\quad (P\ x + \mathit{Any}\ P\ \mathit{xs}) + \mathit{Any}\ P\ \mathit{ys} \quad \Box
\end{aligned}$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$+\!\!+\text{-}\mathit{comm} : (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\qquad \mathit{xs} +\!\!+ \mathit{ys} \approx_{\mathit{bag}} \mathit{ys} +\!\!+ \mathit{xs}$$
$$+\!\!+\text{-}\mathit{comm}\ \mathrm{xs}\ \mathrm{ys} = ?$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$+\!\!+\text{-}\mathit{comm} : (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\qquad \mathit{xs} +\!\!+ \mathit{ys} \approx_{\mathit{bag}} \mathit{ys} +\!\!+ \mathit{xs}$$
$$+\!\!+\text{-}\mathit{comm}\ \mathrm{xs}\ \mathrm{ys} = \lambda\, z.$$
$$\quad z \in \mathrm{xs} +\!\!+ \mathrm{ys} \quad \leftrightarrow\langle\ ?\ \rangle$$
$$\quad z \in \mathrm{ys} +\!\!+ \mathrm{xs} \quad \square$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$+\!\!+\text{-}\mathit{comm} : (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\qquad \mathit{xs} +\!\!+ \mathit{ys} \approx_{\mathit{bag}} \mathit{ys} +\!\!+ \mathit{xs}$$
$$+\!\!+\text{-}\mathit{comm}\ \mathrm{xs}\ \mathrm{ys} = \lambda\, z.$$
$$\quad z \in \mathrm{xs} +\!\!+ \mathrm{ys} \quad \leftrightarrow\langle\ \mathit{Any}\text{-}{+\!\!+}\ \rangle$$
$$\quad z \in \mathrm{xs} + z \in \mathrm{ys} \quad \leftrightarrow\langle\ ?\ \rangle$$
$$\quad z \in \mathrm{ys} +\!\!+ \mathrm{xs} \quad \square$$

(With $P = \lambda\, y.\ z \equiv y.$)

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (xs\ ys : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (xs \mathbin{+\!\!+} ys) \leftrightarrow \mathit{Any}\ P\ xs + \mathit{Any}\ P\ ys$$

$${+\!\!+}\text{-}\mathit{comm} : (xs\ ys : \mathit{List}\ A) \to$$
$$\qquad xs \mathbin{+\!\!+} ys \approx_{bag} ys \mathbin{+\!\!+} xs$$
$${+\!\!+}\text{-}\mathit{comm}\ \mathrm{xs}\ \mathrm{ys} = \lambda\, z.$$
$$\begin{array}{ll} z \in \mathrm{xs} \mathbin{+\!\!+} \mathrm{ys} & \leftrightarrow\langle\ \mathit{Any}\text{-}{+\!\!+}\ \rangle \\ z \in \mathrm{xs} + z \in \mathrm{ys} & \leftrightarrow\langle\ ?\ \rangle \\ z \in \mathrm{ys} + z \in \mathrm{xs} & \leftrightarrow\langle\ \mathit{Any}\text{-}{+\!\!+}\ \rangle \\ z \in \mathrm{ys} \mathbin{+\!\!+} \mathrm{xs} & \Box \end{array}$$

(With $P = \lambda\, y.\ z \equiv y$.)

First lemma

$$\mathit{Any}\text{-}\mathbin{+\!\!+} : (P : A \to \mathit{Set})\,(\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} \mathbin{+\!\!+} \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\mathbin{+\!\!+}\text{-}\mathit{comm} : (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\qquad \mathit{xs} \mathbin{+\!\!+} \mathit{ys} \approx_{\mathit{bag}} \mathit{ys} \mathbin{+\!\!+} \mathit{xs}$$
$$\mathbin{+\!\!+}\text{-}\mathit{comm}\ \mathrm{xs}\ \mathrm{ys} = \lambda\, z.$$
$$\begin{array}{ll}
z \in \mathrm{xs} \mathbin{+\!\!+} \mathrm{ys} & \leftrightarrow\langle\ \mathit{Any}\text{-}\mathbin{+\!\!+}\ \rangle \\
z \in \mathrm{xs} + z \in \mathrm{ys} & \leftrightarrow\langle\ +\ \text{commutative}\ \rangle \\
z \in \mathrm{ys} + z \in \mathrm{xs} & \leftrightarrow\langle\ \mathit{Any}\text{-}\mathbin{+\!\!+}\ \rangle \\
z \in \mathrm{ys} \mathbin{+\!\!+} \mathrm{xs} & \square
\end{array}$$

(With $P = \lambda\, y.\ z \equiv y$.)

Similar lemmas

$$\begin{array}{lcl} \mathit{Any}\ P\ (\mathit{concat}\ \mathit{xss}) & \leftrightarrow & \mathit{Any}\ (\mathit{Any}\ P)\ \mathit{xss} \\ \mathit{Any}\ P\ (\mathit{map}\ f\ \mathit{xs}) & \leftrightarrow & \mathit{Any}\ (P \circ f)\ \mathit{xs} \\ \mathit{Any}\ P\ (\mathit{xs} \ggg f) & \leftrightarrow & \mathit{Any}\ (\mathit{Any}\ P \circ f)\ \mathit{xs} \end{array}$$

Proof of bind lemma:

$$\begin{array}{ll} \mathit{Any}\ P\ (\mathit{xs} \ggg f) & \leftrightarrow\langle \text{ by definition } \rangle \\ \mathit{Any}\ P\ (\mathit{concat}\ (\mathit{map}\ f\ \mathit{xs})) & \leftrightarrow\langle\ \mathit{concat}\ \rangle \\ \mathit{Any}\ (\mathit{Any}\ P)\ (\mathit{map}\ f\ \mathit{xs}) & \leftrightarrow\langle\ \mathit{map}\ \rangle \\ \mathit{Any}\ (\mathit{Any}\ P \circ f)\ \mathit{xs} & \Box \end{array}$$

More lemmas

$$Any\ P\ xs \;\leftrightarrow\; \exists\, z.\;\; P\ z \;\times\; z \in xs$$

$$\begin{aligned}
&Any\text{-}cong \;:\; (\forall\, x.\ P\ x \;\leftrightarrow\; Q\ x) \;\rightarrow \\
&\qquad\qquad\quad xs \;\approx_{bag}\; ys \;\rightarrow \\
&\qquad\qquad\quad Any\ P\ xs \;\leftrightarrow\; Any\ Q\ ys \\
&Any\text{-}cong\ p\ eq \;= \\
&\quad Any\ P\ xs \qquad\qquad\qquad\qquad \leftrightarrow\langle\ Any \rightarrow \exists\ \rangle \\
&\quad (\exists\, z.\;\; P\ z \;\times\; z \in xs) \quad \leftrightarrow\langle\ \text{assumptions}\ \rangle \\
&\quad (\exists\, z.\;\; Q\ z \;\times\; z \in ys) \quad \leftrightarrow\langle\ Any \rightarrow \exists\ \rangle \\
&\quad Any\ Q\ ys \qquad\qquad\qquad\qquad \Box
\end{aligned}$$

Outline of proof

Bijectional reasoning combinators
Any lemmas
Left distributivity

Left distributivity

$$xs \gg\!= (\lambda\, y.\ f\ y +\!\!+\ g\ y)\ \approx_{bag}$$
$$(xs \gg\!= f)\ +\!\!+\ (xs \gg\!= g)$$

Left distributivity

$$
\begin{array}{ll}
z \in xs \gg\!= (\lambda\, y.\, f\ y +\!\!+ g\ y) & \leftrightarrow \langle\ ?\ \rangle \\
z \in (xs \gg\!= f) +\!\!+ (xs \gg\!= g) & \Box
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\_{\equiv}\_\ z)\ (\mathit{xs} \gg= (\lambda\ y.\ f\ y +\!\!+\ g\ y)) & \leftrightarrow\langle\ ?\ \rangle \\
z\ \in\ (\mathit{xs} \gg= f)\ +\!\!+\ (\mathit{xs} \gg= g) & \Box
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\_\!\equiv\!\_\ z)\ (\mathit{xs} \gg\!= (\lambda\, y.\ f\ y +\!\!+ g\ y)) & \leftrightarrow\langle\ \text{bind}\ \rangle \\
\mathit{Any}\ (\mathit{Any}\ (\_\!\equiv\!\_\ z) \circ (\lambda\, y.\ f\ y +\!\!+ g\ y))\ \mathit{xs} & \leftrightarrow\langle\ ?\ \rangle \\
z\ \in\ (\mathit{xs} \gg\!= f)\ +\!\!+\ (\mathit{xs} \gg\!= g) & \Box
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\;(\_\equiv\_\;z)\;(\mathit{xs} \gg\!= (\lambda\, y.\; f\; y \mathbin{+\!\!+} g\; y)) & \leftrightarrow\langle\ \text{bind}\ \rangle \\
\mathit{Any}\;(\lambda\, y.\; z\;\in\; f\; y \mathbin{+\!\!+} g\; y)\;\mathit{xs} & \leftrightarrow\langle\ ?\ \rangle \\
z\;\in\;(\mathit{xs} \gg\!= f) \mathbin{+\!\!+} (\mathit{xs} \gg\!= g) & \square
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
Any\ (\_{\equiv}\_\ z)\ (xs \ggg= (\lambda\ y.\ f\ y +\!\!+\ g\ y)) & \leftrightarrow\langle \text{ bind } \rangle \\
Any\ (\lambda\ y.\ z\ \in\ f\ y +\!\!+\ g\ y)\ xs & \leftrightarrow\langle +\!\!+ \rangle \\
Any\ (\lambda\ y.\ z \in f\ y\ +\ z \in g\ y)\ xs & \leftrightarrow\langle \text{ ? } \rangle \\
z\ \in\ (xs \ggg= f)\ +\!\!+\ (xs \ggg= g) & \square
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
Any\ (\_{\equiv}\_\ z)\ (xs \gg\!= (\lambda\ y.\ f\ y +\!\!+\ g\ y)) & \leftrightarrow\langle\ \text{bind}\ \rangle \\
Any\ (\lambda\ y.\ z\ \in\ f\ y +\!\!+\ g\ y)\ xs & \leftrightarrow\langle\ +\!\!+\ \rangle \\
Any\ (\lambda\ y.\ z \in f\ y\ +\ z \in g\ y)\ xs & \leftrightarrow\langle\ ?\ \rangle \\
z \in xs \gg\!= f\ +\ z \in xs \gg\!= g & \leftrightarrow\langle\ +\!\!+\ \rangle \\
z\ \in\ (xs \gg\!= f)\ +\!\!+\ (xs \gg\!= g) & \square
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\_\!\equiv\!\_\ z)\ (xs \ggg\!= (\lambda\ y.\ f\ y +\!\!+\ g\ y)) & \leftrightarrow\langle\ \text{bind}\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z\ \in\ f\ y +\!\!+\ g\ y)\ xs & \leftrightarrow\langle\ +\!\!+\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z \in f\ y\ \uplus\ z \in g\ y)\ xs & \leftrightarrow\langle\ ?\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z \in f\ y)\ xs\ \uplus & \\
\quad \mathit{Any}\ (\lambda\ y.\ z \in g\ y)\ xs & \leftrightarrow\langle\ \text{bind}\ \rangle \\
z \in xs \ggg\!= f\ \uplus\ z \in xs \ggg\!= g & \leftrightarrow\langle\ +\!\!+\ \rangle \\
z\ \in\ (xs \ggg\!= f)\ +\!\!+\ (xs \ggg\!= g) & \square
\end{array}
$$

Left distributivity

$$\begin{aligned}
&\mathit{Any}\ (\lambda\ y.\ z \in f\ y\ \ +\ \ z \in g\ y)\ \mathit{xs} && \leftrightarrow\langle\ ?\ \rangle \\
&\mathit{Any}\ (\lambda\ y.\ z \in f\ y)\ \mathit{xs} + \\
&\quad \mathit{Any}\ (\lambda\ y.\ z \in g\ y)\ \mathit{xs} && \Box
\end{aligned}$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\lambda\ y.\ P\ y\ \ +\ \ z \in g\ y)\ \mathit{xs} & \leftrightarrow\langle\ ?\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ P\ y)\ \mathit{xs} + & \\
\quad \mathit{Any}\ (\lambda\ y.\ z \in g\ y)\ \mathit{xs} & \square
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\lambda\ y.\ P\ y + Q\ y)\ \mathit{xs} & \leftrightarrow \langle\ ?\ \rangle \\
\mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ Q\ \mathit{xs} & \square
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\lambda\ y.\ P\ y + Q\ y)\ \mathit{xs} & \leftrightarrow\langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
(\exists\ y.\ \ (P\ y + Q\ y)\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow\langle\ ?\ \rangle \\
\mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ Q\ \mathit{xs} & \Box
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\lambda\ y.\ P\ y + Q\ y)\ \mathit{xs} & \leftrightarrow\langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
(\exists\ y.\ \ (P\ y + Q\ y)\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow\langle\ ?\ \rangle \\
(\exists\ y.\ \ P\ y\ \ \times\ \ y \in \mathit{xs})\ \ + & \\
\quad (\exists\ y.\ \ Q\ y\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow\langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
\mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ Q\ \mathit{xs} & \Box
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\lambda\ y.\ P\ y + Q\ y)\ \mathit{xs} & \leftrightarrow \langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
(\exists\ y.\ \ (P\ y + Q\ y)\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow \langle\ \times\ \text{distrib.}\ +\ \rangle \\
(\exists\ y.\ \ P\ y\ \ \times\ \ y \in \mathit{xs}\ \ + & \\
\qquad\quad Q\ y\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow \langle\ ?\ \rangle \\
(\exists\ y.\ \ P\ y\ \ \times\ \ y \in \mathit{xs})\ \ + & \\
\quad (\exists\ y.\ \ Q\ y\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow \langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
\mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ Q\ \mathit{xs} & \Box
\end{array}
$$

Left distributivity

$$
\begin{aligned}
& \mathit{Any}\ (\lambda\ y.\ P\ y + Q\ y)\ \mathit{xs} && \leftrightarrow \langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
& (\exists\ y.\ \ (P\ y + Q\ y)\ \ \times\ \ y \in \mathit{xs}) && \leftrightarrow \langle\ \times\ \text{distrib.} + \rangle \\
& (\exists\ y.\ \ P\ y\ \ \times\ \ y \in \mathit{xs}\ + \\
& \qquad\quad Q\ y\ \ \times\ \ y \in \mathit{xs}) && \leftrightarrow \langle\ \exists\ \text{distrib.} + \rangle \\
& (\exists\ y.\ \ P\ y\ \ \times\ \ y \in \mathit{xs})\ + \\
& \quad (\exists\ y.\ \ Q\ y\ \ \times\ \ y \in \mathit{xs}) && \leftrightarrow \langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
& \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ Q\ \mathit{xs} && \square
\end{aligned}
$$

Summary of proof

Membership defined in terms of *Any*,

used *Any* lemmas,

$$
\begin{aligned}
Any\ P\ (xs \mathbin{+\!\!+} ys) &\leftrightarrow Any\ P\ xs + Any\ P\ ys, \\
Any\ P\ (xs \ggg\!= f) &\leftrightarrow Any\ (Any\ P \circ f)\ xs, \\
Any\ P\ xs &\leftrightarrow \exists\, z.\ P\ z \times z \in xs,
\end{aligned}
$$

to reduce left distributivity to

$$
\begin{aligned}
(A + B) \times C &\leftrightarrow A \times C + B \times C, \\
(\exists\, y.\ P\ y + Q\ y) &\leftrightarrow (\exists\, y.\ P\ y) + (\exists\, y.\ Q\ y).
\end{aligned}
$$

Variations

Variations

- Set equivalence:

$$xs \ \approx_{set} \ ys \ = \ \forall\, z. \ \ z \in xs \Leftrightarrow z \in ys$$

- Subset preorder:

$$xs \ \lesssim_{set} \ ys \ = \ \forall\, z. \ \ z \in xs \rightarrow z \in ys$$

- Subbag preorder:

$$xs \ \lesssim_{bag} \ ys \ = \ \forall\, z. \ \ z \in xs \rightarrowtail z \in ys$$

Variations

Other types: Change the definition of *Any*.

$$\_\approx_{bag}\_ \; : \; List\ A \;\rightarrow\; Tree\ A \;\rightarrow\; Set$$
$$xs\ \approx_{bag}\ t \;=\; \forall\, z.\;\; z \in_{List} xs \;\leftrightarrow\; z \in_{Tree} t$$

Works for arbitrary unary containers
(Abbot et al.; compare Hoogendijk & de Moor).

Conclusions

- Bag equivalence.
- Bijectional reasoning.
- Arbitrary unary containers.
- Set equivalence and
  subset and subbag preorders.

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?