Calculating Bag Equalities

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Bag equality

Equality up to reordering of elements,
or equality when seen as bags:

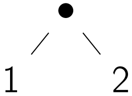
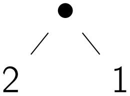
$$
\begin{aligned}
[1,2,1] &\approx_{\mathit{bag}} [2,1,1] \\
[1,2,1] &\not\approx_{\mathit{bag}} [2,1] \\
[1,2,1] &\approx_{\mathit{set}} [2,1]
\end{aligned}
$$

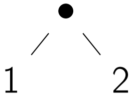
Why?

Partial specification of sorting algorithm:

$$\forall\, xs. \quad sort\ xs \ \approx_{bag}\ xs$$

Not restricted to lists

 $\approx_{bag}$ 

 $\approx_{bag} [2, 1]$

Why?

Tree sort:

$$\begin{array}{lcl} \textit{to-search-tree} & : & \textit{List}\ \mathbb{N} \ \to\ \textit{Tree}\ \mathbb{N} \\ \textit{flatten} & : & \textit{Tree}\ \mathbb{N} \ \to\ \textit{List}\ \mathbb{N} \end{array}$$

$$\begin{array}{lcl} \textit{tree-sort} & : & \textit{List}\ \mathbb{N} \ \to\ \textit{List}\ \mathbb{N} \\ \textit{tree-sort} & = & \textit{flatten} \circ \textit{to-search-tree} \end{array}$$

We can prove

$$\forall\ xs.\ \textit{tree-sort}\ xs \ \approx_{bag}\ xs$$

by first proving

$$\begin{array}{lll} \forall\ xs.\ \textit{to-search-tree}\ xs & \approx_{bag} & xs \\ \forall\ t.\ \ \textit{flatten}\ t & \approx_{bag} & t \end{array}$$

Not restricted to finite things

$$[1, 2, 1, 2, \ldots] \approx_{bag} [2, 1, 2, 1, \ldots]$$

Why?

Assume semantics of grammar given by

$$\mathcal{L} \ : \ \mathit{Grammar} \ \rightarrow \ \mathit{Colist\ String}$$

Language equivalence:

$$\mathcal{L}\ G_1 \ \approx_{set} \mathcal{L}\ G_2$$

If we want to distinguish between ambiguous and unambiguous grammars:

$$\mathcal{L}\ G_1 \ \approx_{bag} \mathcal{L}\ G_2$$

Definitions

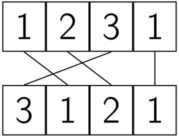
How is bag equality defined?

- Finite sequence of swaps of adjacent elements.
- Counting.
- Bags in the Boom hierarchy:
  append commutative.
- Bijections.
- ...

Bag equality via bijections

Bijection on positions which relates equal elements:

$xs \approx_{bag} ys \Leftrightarrow$
$\quad \exists\, f : \text{positions of } xs \leftrightarrow \text{positions of } ys.$
$\quad \forall\, p.\ lookup\ xs\ p = lookup\ ys\ (f\ p)$



Generalises to anything with positions and *lookup*.

This talk

New definition of bag equality,
with the following properties:

- Many equalities provable using "bijectional reasoning", calculations with bijections instead of equalities.
- Works for arbitrary unary containers (lists, streams, trees, ...).
- Generalises to set equality and subset and subbag preorders.
- Works well in mechanised proofs.

Definition

Any (Morris)

$Any\ P\ xs$ means that $P\ x$ holds for some x in xs.

$$
\begin{array}{lcl}
Any & : & (A \to Set) \to List\ A \to Set \\
Any\ P\ [\,] & = & \bot \\
Any\ P\ (x :: xs) & = & P\ x + Any\ P\ xs
\end{array}
$$

$$Any\ P\ [1, 2, 3] = P\ 1 + P\ 2 + P\ 3 + \bot$$

Membership

$$
\begin{aligned}
&\mathit{Any} : (A \to \mathit{Set}) \to \mathit{List}\ A \to \mathit{Set} \\
&\mathit{Any}\ P\ [\,] = \bot \\
&\mathit{Any}\ P\ (x :: \mathit{xs}) = P\ x + \mathit{Any}\ P\ \mathit{xs}
\end{aligned}
$$

$$
\begin{aligned}
&\_\!\in\!\_ : A \to \mathit{List}\ A \to \mathit{Set} \\
&x \in \mathit{xs} = \mathit{Any}\ (\lambda\ y.\ x \equiv y)\ \mathit{xs}
\end{aligned}
$$

$$
\begin{aligned}
&x \in [1,2,3] = (x \equiv 1) + (x \equiv 2) + (x \equiv 3) + \bot \\
&x \in [1,1] = (x \equiv 1) + (x \equiv 1) + \bot
\end{aligned}
$$

Bag equality

$$\begin{aligned}
&\mathit{Any} : (A \to \mathit{Set}) \to \mathit{List}\ A \to \mathit{Set} \\
&\mathit{Any}\ P\ [\,] = \bot \\
&\mathit{Any}\ P\ (x :: xs) = P\ x + \mathit{Any}\ P\ xs
\end{aligned}$$

$$\begin{aligned}
&_\in_ : A \to \mathit{List}\ A \to \mathit{Set} \\
&x \in xs = \mathit{Any}\ (\lambda\, y.\ x \equiv y)\ xs
\end{aligned}$$

$$\begin{aligned}
&_\approx_{bag}_ : \mathit{List}\ A \to \mathit{List}\ A \to \mathit{Set} \\
&xs \approx_{bag} ys = \forall\, z.\ \ z \in xs \leftrightarrow z \in ys
\end{aligned}$$

Bijectional reasoning

Example

Bind distributes from the left over append:

$$xs \gg\!= (\lambda\, y.\ f\ y\ +\!\!+\ g\ y)\ \approx_{bag}$$
$$(xs \gg\!= f)\ +\!\!+\ (xs \gg\!= g)$$

$$\_\gg\!=\_\ :\ List\ A\ \rightarrow\ (A\ \rightarrow\ List\ B)\ \rightarrow\ List\ B$$
$$xs \gg\!= f\ =\ concat\ (map\ f\ xs)$$

Example

Bind distributes from the left over append:

$$xs \gg\!= (\lambda\, y.\ f\ y\ +\!\!+\ g\ y)\ \approx_{bag}$$
$$(xs \gg\!= f)\ +\!\!+\ (xs \gg\!= g)$$

$$[1,2] \gg\!= (\lambda\, y.\ [y]\ +\!\!+\ [y])\ \approx_{bag}$$
$$([1,2] \gg\!= \lambda\, y.\ [y])\ +\!\!+\ ([1,2] \gg\!= \lambda\, y.\ [y])$$

Example

Bind distributes from the left over append:

$$xs \gg= (\lambda\, y.\ f\ y\ +\!\!+\ g\ y)\ \approx_{bag}$$
$$(xs \gg= f)\ +\!\!+\ (xs \gg= g)$$

$$[1, 1, 2, 2]\ \approx_{bag}$$
$$([1, 2] \gg= \lambda\, y.\ [y])\ +\!\!+\ ([1, 2] \gg= \lambda\, y.\ [y])$$

Example

Bind distributes from the left over append:

$$xs \gg\!= (\lambda\, y.\ f\ y\ +\!\!+\ g\ y)\ \approx_{bag}$$
$$(xs \gg\!= f)\ +\!\!+\ (xs \gg\!= g)$$

$$[1, 1, 2, 2]\ \approx_{bag}$$
$$[1, 2, 1, 2]$$

Outline of proof

Bijectional reasoning combinators
Removing structure from *Any*'s list argument
Left distributivity

Bijectional reasoning combinators

$$
\begin{aligned}
\_\,\square \quad &: (A : Set) \to A \leftrightarrow A \\
\_\leftrightarrow\langle\_\rangle\_ \quad &: (A : Set)\ \{B\ C : Set\} \to \\
&\quad A \leftrightarrow B \to B \leftrightarrow C \to A \leftrightarrow C
\end{aligned}
$$

Assume $p : A \leftrightarrow B$, $q : B \leftrightarrow C$.

$$
\begin{aligned}
&A \quad \leftrightarrow\langle\ p\ \rangle \\
&B \quad \leftrightarrow\langle\ q\ \rangle \\
&C \quad \square
\end{aligned}
$$

Bijectional reasoning combinators

$$\_\,\square \quad : \ (A : Set) \ \rightarrow \ A \leftrightarrow A$$

$$\_\leftrightarrow\langle\_\rangle\_ \ : \ (A : Set)\ \{B\ C : Set\} \ \rightarrow$$
$$A \leftrightarrow B \ \rightarrow \ B \leftrightarrow C \ \rightarrow \ A \leftrightarrow C$$

Assume $p : A \leftrightarrow B$, $q : B \leftrightarrow C$.

$$C\,\square \quad : \quad C \leftrightarrow C$$

Bijectional reasoning combinators

$$\_\Box \quad : (A : Set) \rightarrow A \leftrightarrow A$$

$$\_\leftrightarrow\langle\_\rangle\_ : (A : Set)\ \{B\ C : Set\} \rightarrow$$
$$A \leftrightarrow B \rightarrow B \leftrightarrow C \rightarrow A \leftrightarrow C$$

Assume $p : A \leftrightarrow B$, $q : B \leftrightarrow C$.

$$B \leftrightarrow\langle\ q\ \rangle\ (C\ \Box) \quad : \quad B \leftrightarrow C$$

Bijectional reasoning combinators

$$
\begin{aligned}
\_\,\square \quad &: (A : \mathit{Set}) \rightarrow A \leftrightarrow A \\
\_\leftrightarrow\langle\_\rangle\_ \quad &: (A : \mathit{Set})\ \{B\ C : \mathit{Set}\} \rightarrow \\
&\quad A \leftrightarrow B \rightarrow B \leftrightarrow C \rightarrow A \leftrightarrow C
\end{aligned}
$$

Assume $p : A \leftrightarrow B$, $q : B \leftrightarrow C$.

$$
A \leftrightarrow\langle\ p\ \rangle\ (B \leftrightarrow\langle\ q\ \rangle\ (C\ \square)) \quad : \quad A \leftrightarrow C
$$

Bijectional reasoning combinators

$$
\begin{aligned}
\_\,\square \quad & : (A : Set) \rightarrow A \leftrightarrow A \\
\_\leftrightarrow\langle\_\rangle\_ \quad & : (A : Set)\,\{B\ C : Set\} \rightarrow \\
& \quad A \leftrightarrow B \rightarrow B \leftrightarrow C \rightarrow A \leftrightarrow C
\end{aligned}
$$

Assume $p : A \leftrightarrow B$, $q : B \leftrightarrow C$.

$$
\begin{aligned}
& A \quad \leftrightarrow\langle\ p\ \rangle \\
& B \quad \leftrightarrow\langle\ q\ \rangle \\
& C \quad \square
\end{aligned}
$$

Outline of proof

Bijectional reasoning combinators
Removing structure from *Any*'s list argument
Left distributivity

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\,(\mathit{xs}\;\mathit{ys} : \mathit{List}\;A) \to$$
$$\quad \mathit{Any}\;P\,(\mathit{xs} \mathbin{+\!\!+} \mathit{ys}) \leftrightarrow \mathit{Any}\;P\;\mathit{xs} + \mathit{Any}\;P\;\mathit{ys}$$

$$\mathit{Any}\text{-}{+\!\!+}\;P\;\mathit{xs}\;\mathit{ys} = ?$$

First lemma

$$
\begin{aligned}
&\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\,(\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to \\
&\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{Any}\text{-}{+\!\!+}\ P\ [\,]\ \ \ \ \ \ \ \ \ \ \ \ \ \mathit{ys} = ? \\
&\mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs})\ \mathit{ys} = ?
\end{aligned}
$$

First lemma

$$Any\text{-}{+\!\!+} : (P : A \to Set)\ (xs\ ys : List\ A) \to$$
$$\quad Any\ P\ (xs +\!\!+ ys) \leftrightarrow Any\ P\ xs + Any\ P\ ys$$

$$
\begin{aligned}
& Any\text{-}{+\!\!+}\ P\ [\,]\ \ ys = \\
& \quad Any\ P\ ([\,] +\!\!+ ys) \quad \leftrightarrow\langle\ ?\ \rangle \\
& \quad Any\ P\ [\,] + Any\ P\ ys \quad \square \\
& Any\text{-}{+\!\!+}\ P\ (x :: xs)\ ys = \ ?
\end{aligned}
$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\mathit{Any}\text{-}{+\!\!+}\ P\ [\,]\ \mathit{ys} =$$
$$\quad \mathit{Any}\ P\ \mathit{ys} \quad \leftrightarrow\langle\ ?\ \rangle$$
$$\quad \mathit{Any}\ P\ [\,] + \mathit{Any}\ P\ \mathit{ys} \quad \Box$$
$$\mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs})\ \mathit{ys} = \ ?$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\begin{array}{ll}
\mathit{Any}\text{-}{+\!\!+}\ P\ [\,] & \mathit{ys} = \\
\quad \mathit{Any}\ P\ \mathit{ys} & \leftrightarrow\langle\ ?\ \rangle \\
\quad \bot + \mathit{Any}\ P\ \mathit{ys} & \square \\
\mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs}) & \mathit{ys} = ?
\end{array}$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\begin{aligned}
&\mathit{Any}\text{-}{+\!\!+}\ P\ [\,]\ && \mathit{ys} = \\
&\quad \mathit{Any}\ P\ \mathit{ys} && \leftrightarrow\langle\ \bot\ \text{identity of}\ +\ \rangle \\
&\quad \bot + \mathit{Any}\ P\ \mathit{ys} && \square \\
&\mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs})\ && \mathit{ys} = ?
\end{aligned}$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\begin{array}{ll} \mathit{Any}\text{-}{+\!\!+}\ P\ [\,]\ \ \mathit{ys} = & \\ \quad \mathit{Any}\ P\ \mathit{ys} & \leftrightarrow\langle\ \bot\ \text{identity of}\ +\ \rangle \\ \quad \bot + \mathit{Any}\ P\ \mathit{ys} & \Box \end{array}$$

$$\begin{array}{ll} \mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs})\ \mathit{ys} = & \\ \quad P\ x + \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) & \leftrightarrow\langle\ ?\ \rangle \\ \quad (P\ x + \mathit{Any}\ P\ \mathit{xs}) + \mathit{Any}\ P\ \mathit{ys} & \Box \end{array}$$

First lemma

$$Any\text{-}{+\!\!+} : (P : A \to Set)\,(xs\;ys : List\;A) \to$$
$$\quad Any\;P\,(xs \mathbin{+\!\!+} ys) \leftrightarrow Any\;P\;xs + Any\;P\;ys$$

$$
\begin{aligned}
& Any\text{-}{+\!\!+}\;P\;[\,]\qquad ys = \\
& \quad Any\;P\;ys \qquad \leftrightarrow\langle\ \bot\ \text{identity of} + \rangle \\
& \quad \bot + Any\;P\;ys \quad \Box \\
& Any\text{-}{+\!\!+}\;P\;(x :: xs)\;ys = \\
& \quad P\;x + Any\;P\;(xs \mathbin{+\!\!+} ys) \qquad \leftrightarrow\langle\ \text{ind. hyp.}\ \rangle \\
& \quad P\;x + (Any\;P\;xs + Any\;P\;ys) \quad \leftrightarrow\langle\ ?\ \rangle \\
& \quad (P\;x + Any\;P\;xs) + Any\;P\;ys \quad \Box
\end{aligned}
$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\begin{array}{ll}
\mathit{Any}\text{-}{+\!\!+}\ P\ [\,] & \mathit{ys} = \\
\quad \mathit{Any}\ P\ \mathit{ys} & \leftrightarrow\langle\ \bot\ \text{identity of}\ +\ \rangle \\
\quad \bot + \mathit{Any}\ P\ \mathit{ys} & \square
\end{array}$$

$$\begin{array}{ll}
\multicolumn{2}{l}{\mathit{Any}\text{-}{+\!\!+}\ P\ (x :: \mathit{xs})\ \mathit{ys} =} \\
\quad P\ x + \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) & \leftrightarrow\langle\ \text{ind. hyp.}\ \rangle \\
\quad P\ x + (\mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}) & \leftrightarrow\langle\ +\ \text{associative}\ \rangle \\
\quad (P\ x + \mathit{Any}\ P\ \mathit{xs}) + \mathit{Any}\ P\ \mathit{ys} & \square
\end{array}$$

First lemma

$$\mathit{Any}\text{-}\mathbin{+\!\!+} : (P : A \to \mathit{Set})\,(\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} \mathbin{+\!\!+} \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\mathbin{+\!\!+}\text{-}\mathit{comm} : (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\qquad \mathit{xs} \mathbin{+\!\!+} \mathit{ys} \approx_{\mathit{bag}} \mathit{ys} \mathbin{+\!\!+} \mathit{xs}$$
$$\mathbin{+\!\!+}\text{-}\mathit{comm}\ \mathrm{xs}\ \mathrm{ys} = ?$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} \mathbin{+\!\!+} \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$\mathbin{+\!\!+}\text{-}\mathit{comm} : (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\qquad \mathit{xs} \mathbin{+\!\!+} \mathit{ys} \approx_{\mathit{bag}} \mathit{ys} \mathbin{+\!\!+} \mathit{xs}$$
$$\mathbin{+\!\!+}\text{-}\mathit{comm}\ \mathrm{xs}\ \mathrm{ys} = \lambda\, z.$$
$$\quad z \in \mathrm{xs} \mathbin{+\!\!+} \mathrm{ys} \quad \leftrightarrow\langle\ ?\ \rangle$$
$$\quad z \in \mathrm{ys} \mathbin{+\!\!+} \mathrm{xs} \quad \square$$

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (xs\ ys : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (xs +\!\!+ ys) \leftrightarrow \mathit{Any}\ P\ xs + \mathit{Any}\ P\ ys$$

$$+\!\!+\text{-}\mathit{comm} : (xs\ ys : \mathit{List}\ A) \to$$
$$\qquad xs +\!\!+ ys \approx_{bag} ys +\!\!+ xs$$
$$+\!\!+\text{-}\mathit{comm}\ \mathrm{xs}\ \mathrm{ys} = \lambda\, z.$$
$$\begin{array}{ll} z \in \mathrm{xs} +\!\!+ \mathrm{ys} & \leftrightarrow\langle\ \mathit{Any}\text{-}{+\!\!+}\ \rangle \\ z \in \mathrm{xs}\ +\ z \in \mathrm{ys} & \leftrightarrow\langle\ ?\ \rangle \\ z \in \mathrm{ys} +\!\!+ \mathrm{xs} & \square \end{array}$$

(With $P = \lambda\, y.\ z \equiv y.$)

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\quad \mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$+\!\!+\text{-}\mathit{comm} : (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\qquad \mathit{xs} +\!\!+ \mathit{ys} \approx_{\mathit{bag}} \mathit{ys} +\!\!+ \mathit{xs}$$
$$+\!\!+\text{-}\mathit{comm}\ \mathrm{xs}\ \mathrm{ys} = \lambda\, z.$$
$$\begin{array}{ll} z \in \mathrm{xs} +\!\!+ \mathrm{ys} & \leftrightarrow\langle\ \mathit{Any}\text{-}{+\!\!+}\ \rangle \\ z \in \mathrm{xs} + z \in \mathrm{ys} & \leftrightarrow\langle\ ?\ \rangle \\ z \in \mathrm{ys} + z \in \mathrm{xs} & \leftrightarrow\langle\ \mathit{Any}\text{-}{+\!\!+}\ \rangle \\ z \in \mathrm{ys} +\!\!+ \mathrm{xs} & \Box \end{array}$$

(With $P = \lambda\, y.\ z \equiv y$.)

First lemma

$$\mathit{Any}\text{-}{+\!\!+} : (P : A \to \mathit{Set})\ (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to$$
$$\mathit{Any}\ P\ (\mathit{xs} +\!\!+ \mathit{ys}) \leftrightarrow \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ P\ \mathit{ys}$$

$$
\begin{aligned}
&{+\!\!+}\text{-}\mathit{comm} : (\mathit{xs}\ \mathit{ys} : \mathit{List}\ A) \to \\
&\qquad\qquad \mathit{xs} +\!\!+ \mathit{ys} \approx_{\mathit{bag}} \mathit{ys} +\!\!+ \mathit{xs} \\
&{+\!\!+}\text{-}\mathit{comm}\ \mathrm{xs}\ \mathrm{ys} = \lambda\, z. \\
&\quad z \in \mathrm{xs} +\!\!+ \mathrm{ys} && \leftrightarrow\langle\ \mathit{Any}\text{-}{+\!\!+}\ \rangle \\
&\quad z \in \mathrm{xs} + z \in \mathrm{ys} && \leftrightarrow\langle\ + \text{ commutative } \rangle \\
&\quad z \in \mathrm{ys} + z \in \mathrm{xs} && \leftrightarrow\langle\ \mathit{Any}\text{-}{+\!\!+}\ \rangle \\
&\quad z \in \mathrm{ys} +\!\!+ \mathrm{xs} && \square
\end{aligned}
$$

(With $P = \lambda\, y.\, z \equiv y$.)

Similar lemmas

$$\begin{array}{lcl}
\mathit{Any}\ P\ (\mathit{concat}\ \mathit{xss}) & \leftrightarrow & \mathit{Any}\ (\mathit{Any}\ P)\ \mathit{xss} \\
\mathit{Any}\ P\ (\mathit{map}\ f\ \mathit{xs}) & \leftrightarrow & \mathit{Any}\ (P \circ f)\ \mathit{xs} \\
\mathit{Any}\ P\ (\mathit{xs} \ggg f) & \leftrightarrow & \mathit{Any}\ (\mathit{Any}\ P \circ f)\ \mathit{xs}
\end{array}$$

Proof of bind lemma:

$$\begin{array}{ll}
\mathit{Any}\ P\ (\mathit{xs} \ggg f) & \leftrightarrow \langle \text{ by definition } \rangle \\
\mathit{Any}\ P\ (\mathit{concat}\ (\mathit{map}\ f\ \mathit{xs})) & \leftrightarrow \langle\ \mathit{concat}\ \rangle \\
\mathit{Any}\ (\mathit{Any}\ P)\ (\mathit{map}\ f\ \mathit{xs}) & \leftrightarrow \langle\ \mathit{map}\ \rangle \\
\mathit{Any}\ (\mathit{Any}\ P \circ f)\ \mathit{xs} & \Box
\end{array}$$

More lemmas

$$Any\ P\ xs \;\leftrightarrow\; \exists\, z.\; P\ z \;\times\; z \in xs$$

$$\begin{aligned}
&Any\text{-}cong \;:\; (\forall\, x.\, P\ x \;\leftrightarrow\; Q\ x) \;\rightarrow \\
&\qquad xs \;\approx_{bag}\; ys \;\rightarrow \\
&\qquad Any\ P\ xs \;\leftrightarrow\; Any\ Q\ ys \\
&Any\text{-}cong\ p\ eq \;= \\
&\quad Any\ P\ xs && \leftrightarrow\langle\ Any \rightarrow \exists\ \rangle \\
&\quad (\exists\, z.\; P\ z \;\times\; z \in xs) && \leftrightarrow\langle\ \text{assumptions}\ \rangle \\
&\quad (\exists\, z.\; Q\ z \;\times\; z \in ys) && \leftrightarrow\langle\ Any \rightarrow \exists\ \rangle \\
&\quad Any\ Q\ ys && \square
\end{aligned}$$

Outline of proof

Bijectional reasoning combinators
Removing structure from *Any*'s list argument
Left distributivity

Left distributivity

$$xs \gg\!= (\lambda\, y.\ f\ y \mathbin{+\!\!+} g\ y) \ \approx_{bag}$$
$$(xs \gg\!= f) \mathbin{+\!\!+} (xs \gg\!= g)$$

Left distributivity

$$z \in xs \ggg= (\lambda\, y.\, f\ y +\!\!+\ g\ y) \quad \leftrightarrow \langle\ ?\ \rangle$$
$$z \in (xs \ggg= f) +\!\!+ (xs \ggg= g) \quad \square$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\_\equiv\_\ z)\ (\mathit{xs} \ggg\!= (\lambda\ y.\ f\ y \mathbin{+\!\!+} g\ y)) & \leftrightarrow\langle\ ?\ \rangle \\
z\ \in\ (\mathit{xs} \ggg\!= f)\ \mathbin{+\!\!+}\ (\mathit{xs} \ggg\!= g) & \Box
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
Any\ (\_\equiv\_\ z)\ (xs \gg\!= (\lambda\ y.\ f\ y +\!\!+ g\ y)) & \leftrightarrow\langle\ \text{bind}\ \rangle \\
Any\ (Any\ (\_\equiv\_\ z) \circ (\lambda\ y.\ f\ y +\!\!+ g\ y))\ xs & \leftrightarrow\langle\ ?\ \rangle \\
z\ \in\ (xs \gg\!= f)\ +\!\!+\ (xs \gg\!= g) & \square
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\_\equiv\_\ z)\ (\mathit{xs} \ggg= (\lambda\ y.\ f\ y \mathbin{+\!\!+} g\ y)) & \leftrightarrow\langle\ \text{bind}\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z\ \in\ f\ y \mathbin{+\!\!+} g\ y)\ \mathit{xs} & \leftrightarrow\langle\ ?\ \rangle \\
z\ \in\ (\mathit{xs} \ggg= f)\ \mathbin{+\!\!+}\ (\mathit{xs} \ggg= g) & \Box
\end{array}
$$

Left distributivity

$$\begin{array}{ll}
\mathit{Any}\ (_\equiv_\ z)\ (xs \ggg= (\lambda\ y.\ f\ y +\!\!+\ g\ y)) & \leftrightarrow\langle\ \text{bind}\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z\ \in\ f\ y +\!\!+\ g\ y)\ xs & \leftrightarrow\langle +\!\!+ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z \in f\ y\ +\ z \in g\ y)\ xs & \leftrightarrow\langle\ ?\ \rangle \\
z\ \in\ (xs \ggg= f)\ +\!\!+\ (xs \ggg= g) & \square
\end{array}$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\_{\equiv}\_\ z)\ (\mathit{xs} \gg= (\lambda\ y.\ f\ y \mathbin{+\!\!+} g\ y)) & \leftrightarrow\langle\ \text{bind}\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z\ \in\ f\ y \mathbin{+\!\!+} g\ y)\ \mathit{xs} & \leftrightarrow\langle\ \mathbin{+\!\!+}\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z \in f\ y\ \uplus\ z \in g\ y)\ \mathit{xs} & \leftrightarrow\langle\ ?\ \rangle \\
z \in \mathit{xs} \gg= f\ \uplus\ z \in \mathit{xs} \gg= g & \leftrightarrow\langle\ \mathbin{+\!\!+}\ \rangle \\
z\ \in\ (\mathit{xs} \gg= f)\ \mathbin{+\!\!+}\ (\mathit{xs} \gg= g) & \Box
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\_\!\equiv\!\_\ z)\ (\mathit{xs} \gg\!= (\lambda\ y.\ f\ y +\!\!+\ g\ y)) & \leftrightarrow\langle\ \text{bind}\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z\ \in\ f\ y +\!\!+\ g\ y)\ \mathit{xs} & \leftrightarrow\langle\ +\!\!+\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z \in f\ y\ \uplus\ z \in g\ y)\ \mathit{xs} & \leftrightarrow\langle\ ?\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z \in f\ y)\ \mathit{xs}\ \uplus & \\
\quad \mathit{Any}\ (\lambda\ y.\ z \in g\ y)\ \mathit{xs} & \leftrightarrow\langle\ \text{bind}\ \rangle \\
z \in \mathit{xs} \gg\!= f\ \uplus\ z \in \mathit{xs} \gg\!= g & \leftrightarrow\langle\ +\!\!+\ \rangle \\
z\ \in\ (\mathit{xs} \gg\!= f)\ +\!\!+\ (\mathit{xs} \gg\!= g) & \square
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\lambda\ y.\ z \in f\ \ y\ \ +\ \ z \in g\ \ y)\ \mathit{xs} & \leftrightarrow\langle\ ?\ \rangle \\
\mathit{Any}\ (\lambda\ y.\ z \in f\ \ y)\ \mathit{xs} + & \\
\quad \mathit{Any}\ (\lambda\ y.\ z \in g\ \ y)\ \mathit{xs} & \Box
\end{array}
$$

Left distributivity

$$\begin{array}{ll} \mathit{Any}\ (\lambda\ y.\ P\ y\ \ +\ \ z \in g\ y)\ \mathit{xs} & \leftrightarrow\langle\ ?\ \rangle \\ \mathit{Any}\ (\lambda\ y.\ P\ y)\ \mathit{xs} + & \\ \quad \mathit{Any}\ (\lambda\ y.\ z \in g\ y)\ \mathit{xs} & \Box \end{array}$$

Left distributivity

$$\begin{array}{ll} \mathit{Any}\ (\lambda\ y.\ P\ y + Q\ y)\ \mathit{xs} & \leftrightarrow\langle\ ?\ \rangle \\ \mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ Q\ \mathit{xs} & \square \end{array}$$

Left distributivity

$$
\begin{array}{ll}
Any\ (\lambda\ y.\ P\ y + Q\ y)\ xs & \leftrightarrow\langle\ Any \rightarrow \exists\ \rangle \\
(\exists\ y.\ \ (P\ y + Q\ y)\ \ \times\ \ y \in xs) & \leftrightarrow\langle\ ?\ \rangle \\
Any\ P\ xs + Any\ Q\ xs & \square
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\lambda\ y.\ P\ y + Q\ y)\ \mathit{xs} & \leftrightarrow \langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
(\exists\ y.\ \ (P\ y + Q\ y)\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow \langle\ ?\ \rangle \\
(\exists\ y.\ \ P\ y\ \ \times\ \ y \in \mathit{xs})\ + & \\
\quad (\exists\ y.\ \ Q\ y\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow \langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
\mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ Q\ \mathit{xs} & \square
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\lambda\ y.\ P\ y + Q\ y)\ \mathit{xs} & \leftrightarrow\langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
(\exists\ y.\ \ (P\ y + Q\ y)\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow\langle\ \times\ \text{distrib.} + \rangle \\
(\exists\ y.\ \ P\ y\ \ \times\ \ y \in \mathit{xs}\ \ + & \\
\qquad\quad Q\ y\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow\langle\ ?\ \rangle \\
(\exists\ y.\ \ P\ y\ \ \times\ \ y \in \mathit{xs})\ \ + & \\
\quad (\exists\ y.\ \ Q\ y\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow\langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
\mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ Q\ \mathit{xs} & \square
\end{array}
$$

Left distributivity

$$
\begin{array}{ll}
\mathit{Any}\ (\lambda\ y.\ P\ y + Q\ y)\ \mathit{xs} & \leftrightarrow\langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
(\exists\ y.\ \ (P\ y + Q\ y)\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow\langle\ \times\ \text{distrib.}\ +\ \rangle \\
(\exists\ y.\ \ P\ y\ \ \times\ \ y \in \mathit{xs}\ \ + & \\
\qquad\quad Q\ y\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow\langle\ \exists\ \text{distrib.}\ +\ \rangle \\
(\exists\ y.\ \ P\ y\ \ \times\ \ y \in \mathit{xs})\ \ + & \\
\quad (\exists\ y.\ \ Q\ y\ \ \times\ \ y \in \mathit{xs}) & \leftrightarrow\langle\ \mathit{Any} \rightarrow \exists\ \rangle \\
\mathit{Any}\ P\ \mathit{xs} + \mathit{Any}\ Q\ \mathit{xs} & \square
\end{array}
$$

Summary of proof

Membership defined in terms of *Any*,

used *Any* lemmas,

$$
\begin{aligned}
Any\ P\ (xs\ +\!\!\!+\ ys) &\leftrightarrow\ Any\ P\ xs + Any\ P\ ys, \\
Any\ P\ (xs \ggg\!= f) &\leftrightarrow\ Any\ (Any\ P \circ f)\ xs, \\
Any\ P\ xs &\leftrightarrow\ \exists\, z.\ \ P\ z\ \times\ z \in xs,
\end{aligned}
$$

to reduce left distributivity to

$$
\begin{aligned}
(A + B) \times C &\leftrightarrow\ A \times C\ +\ B \times C, \\
(\exists\, y.\ P\ y + Q\ y) &\leftrightarrow\ (\exists\, y.\ P\ y) + (\exists\, y.\ Q\ y).
\end{aligned}
$$

Variations

Variations

- Set equality:

$$xs \approx_{set} ys = \forall z.\ \ z \in xs \Leftrightarrow z \in ys$$

- Subset preorder:

$$xs \lesssim_{set} ys = \forall z.\ \ z \in xs \rightarrow z \in ys$$

- Subbag preorder:

$$xs \lesssim_{bag} ys = \forall z.\ \ z \in xs \rightarrowtail z \in ys$$

Variations

Other types: Change the definition of *Any*.

$$
\begin{aligned}
\_\approx_{bag}\_ &: List\ A \ \rightarrow\ Tree\ A \ \rightarrow\ Set \\
xs\ \approx_{bag}\ t &= \forall\, z.\ \ z \in_{List} xs \ \leftrightarrow\ z \in_{Tree} t
\end{aligned}
$$

Works for arbitrary unary containers
(Abbot et al.; compare Hoogendijk & de Moor).

Conclusions

- Bag equality.
- Bijectional reasoning.
- Arbitrary unary containers.
- Set equality and subset and subbag preorders.
- Mechanised proofs.

Conclusions

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- Arbitrary unary containers.
- Set equality and subset and subbag preorders.
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Bonus slides

Swapping definition for streams

Naive definition (coinductive):

$$\frac{}{xs \approx_{bag} xs} \qquad \frac{xs +\!\!+ x :: y :: ys \approx_{bag} zs}{xs +\!\!+ y :: x :: ys \approx_{bag} zs}$$

Problem: All streams equal.
Can build infinite derivation showing $xs \approx_{bag} zs$.

Bag equality for streams

For streams *Any* can be defined inductively:

$$\frac{P\ x}{Any\ P\ (x :: xs)} \qquad \frac{Any\ P\ xs}{Any\ P\ (x :: xs)}$$

$$\begin{aligned} \_\in\_ &: A \to Stream\ A \to Set \\ x \in xs &= Any\ (\lambda\ y.\ x \equiv y)\ xs \end{aligned}$$

$$\begin{aligned} \_\approx_{bag}\_ &: Stream\ A \to Stream\ A \to Set \\ xs \approx_{bag} ys &= \forall\ z.\quad z \in xs \leftrightarrow z \in ys \end{aligned}$$

The full code of *Any*-++

```
Any-++ : {A : Set} (P : A → Set) (xs ys : List A) →
             Any P (xs ++ ys) ↔ Any P xs ⊎ Any P ys
Any-++ P [] ys =
   Any P ys          ↔⟨ sym ⊎-left-identity ⟩
   ⊥ ⊎ Any P ys   □
Any-++ P (x :: xs) ys =
   P x ⊎ Any P (xs ++ ys)          ↔⟨ ⊎-cong (P x □)
                                            (Any-++ P xs ys) ⟩
   P x ⊎ (Any P xs ⊎ Any P ys)   ↔⟨ ⊎-assoc ⟩
   (P x ⊎ Any P xs) ⊎ Any P ys   □
```

Variations

Can define parametrised notion of equality:

$$\_\rightsquigarrow[\_]\_ \;:\; \mathit{Set} \to \mathit{Kind} \to \mathit{Set} \to \mathit{Set}$$

$$A \rightsquigarrow[\ \text{implication}\ ]\ B \;=\; A \to B$$

$$A \rightsquigarrow[\ \text{equivalence}\ ]\ B \;=\; A \Leftrightarrow B$$

$$A \rightsquigarrow[\ \text{injection}\ ]\ B \;=\; A \rightarrowtail B$$

$$A \rightsquigarrow[\ \text{bijection}\ ]\ B \;=\; A \leftrightarrow B$$

$$\_\sim[\_]\_ \;:\; \mathit{List}\ A \to \mathit{Kind} \to \mathit{List}\ A \to \mathit{Set}$$

$$xs \sim[\ k\ ]\ ys \;=\; \forall\, z.\quad z \in xs \rightsquigarrow[\ k\ ]\ z \in ys$$

Variations

Can prove preservation properties uniformly:

$$
\begin{aligned}
&\gg\!=\text{-}cong \;:\; (xs\; ys \;:\; List\; A)\,(f\; g \;:\; A \to List\; B) \to \\
&\qquad xs \sim[\,k\,]\; ys \;\to\; (\forall\, x.\; f\; x \sim[\,k\,]\; g\; x) \;\to \\
&\qquad xs \gg\!= f \sim[\,k\,]\; ys \gg\!= g \\
&\gg\!=\text{-}cong\; xs\; ys\; f\; g\; eq_1\; eq_2 \;=\; \lambda\, z. \\
&\quad z \in xs \gg\!= f \qquad\qquad\qquad \leftrightarrow\langle\, \text{bind} \,\rangle \\
&\quad Any\; (\lambda\, x.\; z \in f\; x)\; xs \quad \leadsto\langle\, Any\text{-}cong \,\rangle \\
&\quad Any\; (\lambda\, x.\; z \in g\; x)\; ys \quad \leftrightarrow\langle\, \text{bind} \,\rangle \\
&\quad z \in ys \gg\!= g \qquad\qquad\qquad \square
\end{aligned}
$$

Parsing

Parser:

$$parse : Grammar\ A \to String \to List\ A$$

Semantics of grammar G:

$$Semantics\ G\ x\ s$$

A predicate stating when x is one possible result of parsing s.

Parsing

Correctness of parser:

$$\forall\, s.\quad parse\ G\ s\ \ \approx_{bag}\ \wr\, x \mid Semantics\ G\ x\ s\, \int$$

Parsing

Correctness of parser:

$$\forall\, s.\quad parse\ G\ s\ \ \approx_{bag}\ \wr\ x \mid Semantics\ G\ x\ s\ \int$$

What does $\wr \ldots \int$ mean? How is $\_\approx_{bag}\_$ defined?

$$\forall\, s\ x.\quad x \in_{List} parse\ G\ s\ \ \leftrightarrow\ \ Semantics\ G\ x\ s$$

Compulsory message

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