

Automated Deduction

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Outline

Combinations of Theories

Combination of theories: Models of Satisfiable Sets

Running an SMT Solver

First-Order Theorem Proving - An Example

Example (recap)

$$x + 2 = y$$

$$f(\text{read}(\text{write}(a, x, 3), y - 2)) \neq f(y - x + 1)$$

For deciding satisfiability:

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- ▶ Make reasoners **exchange equalities**

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- ▶ Make reasoners **exchange equalities**
 - derive and use **equalities among shared variables**

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Linear Arithmetic	Uninterpreted Functions
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Arrays	
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- ▶ Each step obviously **preserves satisfiability**; moreover every model of the modified set of literals is also a model of the original set.
- ▶ In this example every “separated” subset of literals is satisfiable.
- ▶ But this does not mean that the set of all literals is satisfiable.

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Problem with combining theories, as described, that are not stably infinite. Why?

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For efficiency, work with **minimal disjunctions**.

Combination of theories: The Nelson-Oppen method

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- ▶ for convex theories \mathcal{T}_i with decision procedures in P , the Nelson-Oppen method is in P ;
- ▶ for theories \mathcal{T}_i with decision procedures in NP , the Nelson-Oppen method is in NP .

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Combination of theories: **Models of Satisfiable Sets**

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Take models of:

- ▶ the literals in every theory plus the set of derived equalities.

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The most efficient SMT solver: Z3, <http://rise4fun.com/Z3>.

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Easy to use.

SMT Related Problems — Where are we now?

- ✓ **Deciding theory:** Check satisfiability of a set of *literals* in $\mathcal{T}_E, \mathcal{T}_A$
- ✓ **SMT and non-unit formulas:** Put together theory reasoning and SAT solving
- ✓ **Reasoning in combination of theories:** Given decision procedures for theories, we can build a decision procedure for the combination of these theories.

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Go to

<https://vprover.github.io/download.html>

and pick the route most suitable to you.

Notes:

- ▶ For Linux users, a binary is probably the easiest route
- ▶ For Mac users, you need to build from source
 - ▶ `run make vampire_rel`
- ▶ For Windows users, the easiest route for this tutorial is a virtual machine and then use Linux

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Group theory theorem: if a group satisfies the identity $x^2 = 1$, then it is commutative.

More formally: in a group “**assuming** that $x^2 = 1$ for all x **prove** that $x \cdot y = y \cdot x$ holds for all x, y .”

First-Order Theorem Proving. An Example

Group theory theorem: if a group satisfies the identity $x^2 = 1$, then it is commutative.

More formally: in a group “**assuming** that $x^2 = 1$ for all x **prove** that $x \cdot y = y \cdot x$ holds for all x, y .”

What is implicit: axioms of the group theory.

$$\forall x(1 \cdot x = x)$$

$$\forall x(x^{-1} \cdot x = 1)$$

$$\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$$

Formulation in First-Order Logic

Axioms (of group theory):	$\forall x(1 \cdot x = x)$ $\forall x(x^{-1} \cdot x = 1)$ $\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$
Assumptions:	$\forall x(x \cdot x = 1)$
Conjecture:	$\forall x \forall y(x \cdot y = y \cdot x)$

In the TPTP Syntax

The **TPTP** library (**T**housands of **P**roblems for **T**heorem **P**rovers), <http://www.tptp.org> contains a large collection of first-order problems. For representing these problems it uses the **TPTP syntax**, which is understood by all modern theorem provers, including Vampire.

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```
%---- 1 * x = x
fof(left_identity,axiom,
    ! [X] : mult(e,X) = X).
%---- i(x) * x = 1
fof(left_inverse,axiom,
    ! [X] : mult(inverse(X),X) = e).
%---- (x * y) * z = x * (y * z)
fof(associativity,axiom,
    ! [X,Y,Z] : mult(mult(X,Y),Z) = mult(X,mult(Y,Z))).
%---- x * x = 1
fof(group_of_order_2,hypothesis,
    ! [X] : mult(X,X) = e).
%---- prove x * y = y * x
fof(commutativity,conjecture,
    ! [X] : mult(X,Y) = mult(Y,X)).
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Running Vampire on a TPTP file

is easy: simply use

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One can also run Vampire with various options, some of them will be explained later. For example, save the group theory problem in a file `group.tptp` and try

```
vampire --thanks TUWien group.tptp
```

Proof by Vampire (Slightly Modified)

Refutation found.

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