# Automated Deduction

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# SMT Related Problems – Where are we now?

- ✓ Deciding theory: Check satisfiability of a set of *literals* in $\mathcal{T}_E$

  What about in $\mathcal{T}_A$ and $\mathcal{T}_Q$?

- ? Put together theory reasoning and SAT solving

- ? What about combination of theories: Given decision procedures for theories, how can we build a decision procedure for formulas using several theories?

# Outline

# Deciding $\mathcal{T}_E$: Problems Using Non-Unit Clauses

Example: Let's try to prove the validity of the formula:

$$F: \ (a = b \vee a = c) \wedge f(a) = a \rightarrow f(f(b)) = a \vee f(c) = a.$$

This is equivalent to establishing unsatisfiability of

$$a = b \vee a = c$$
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We have a non-unit clause, so we cannot use congruence closure.

Inputs of the congruence closure algorithm are conjunctions of $\mathcal{T}_E$-literals.

# Idea: Use a SAT Solver

Add a propositional symbol to name every theory atom:

$$a = b \vee a = c$$
$$f(a) = a$$
$$f(f(b)) \neq a$$
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$$p_1 \vee p_2$$
$$p_3$$
$$\neg p_4$$
$$\neg p_5$$

$$p_1 : a = b$$
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2. If the SAT solver returns `unsatisfiable`, $\neg F$ is `unsatisfiable`.
3. If the SAT solver returns `satisfiable`, we obtain a set of literals $L_1, \ldots, L_n$ representing a model $I$ of the propositional clauses.

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4. Using the theory solver (congruence closure), check satisfiability of the theory literals corresponding to $I$.
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**DPLL($\mathcal{T}$) Algorithm for SMT**

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# Outline

# Deciding $\mathcal{T}_A$

The theory of arrays $\mathcal{T}_A$ is defined by

- a signature $\Sigma_A = \{read, write\}$ where

  $read$ is a binary function: $read(A, x)$ is the value of array $A$ at position $x$

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- the following axioms:

  *equality axioms* from $\mathcal{T}_E$

  $$x = y \rightarrow read(write(A, x, v), y) = v \qquad \text{(read-over-write 1)}$$

  $$x \neq y \rightarrow read(write(A, x, v), y) = read(A, y) \qquad \text{(read-over-write 2)}$$

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Idea:

1. $F$ contains no $write$-terms. Then, treat $read$-terms as uninterpreted function terms
2. $F$ contains $write$-terms. Then, $write$-terms occur in the context of a $read$-term, so use (read-over-write) axioms to "eliminate" $write$-terms

# Deciding $\mathcal{T}_A$: Congruence Closure Algorithm

## Algorithm for deciding $\mathcal{T}_A$

1. $F$ contains no *write*-terms:
   - use fresh function symbol $f_A$ for array variables $A$
   - replace $read(A, x)$ with $f_A(x)$ in $F$
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2. $F$ contains *write*-terms, say $read(write(A, x, v), y)$

   - Using (read-over-write 1), replace $F$ by the following formula $F_1$:

     $$F_1 : x = y \wedge F[v]$$

     where $F[v]$ denotes the formula obtained by replacing $read(write(A, x, v), y)$ with $v$ in $F$.

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   - Using (read-over-write 2), replace $F$ by the following formula $F_2$:

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   If $F_1$ and $F_2$ are $\mathcal{T}_A$-unsatisfiable, return `unsatisfiable`

# Deciding $\mathcal{T}_A$: Congruence Closure by Example

Question: Is formula $F$ given below $\mathcal{T}_A$-satisfiable?

$$x_1 = y \ \wedge \ x_1 \neq x_2 \ \wedge \ read(A, y) = v_1 \ \wedge$$
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Use (read-over-write-1), and get:

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Use (read-over-write-1), and get: `unsatisfiable`

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Use (read-over-write-2), and get:

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$F$ is thus $\mathcal{T}_A$-unsatisfiable.

# Deciding $\mathcal{T}_A$: Congruence Closure Algorithm

Summary: $\mathcal{T}_A$-satisfiability of a formula $F$ is reduced to $\mathcal{T}_E$-satisfiability

Idea:

1. $F$ contains no *write*-terms. Then, treat *read*-terms as uninterpreted function terms
2. $F$ contains *write*-terms. Then, *write*-terms occur in the context of a *read*-term, so use (read-over-write) axioms to "eliminate" *write*-terms

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Computing $\mathcal{T}_A$-satisfiability is NP-complete.

# SMT Related Problems – Where are we now?

- ✓ Deciding theory: Check satisfiability of a set of *literals* in $\mathcal{T}_E$, $\mathcal{T}_A$

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- ? What about combination of theories: Given decision procedures for theories, how can we build a decision procedure for formulas using several theories?

**We next study satisfiability of formulas in combination of theories!**

# Outline

# Reasoning with a union/combination of theories

Given:

- Theories $T_1, \ldots, T_n$ in disjoint signatures $\Sigma_1, \ldots, \Sigma_n$;
- Quantifier-free formula $F$ in $\Sigma_1 \cup \ldots \cup \Sigma_n$.

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Question:

- Is $F$ satisfiable in $T_1 \cup \ldots \cup T_n$?

# Example

Is this set of formulas satisfiable?

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- $f : int \rightarrow s$

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- Use DPLL($T$). This allows one to reduce the satisfiability problem for quantifier-free formulas to the satisfiability problem for conjunctions of literals.

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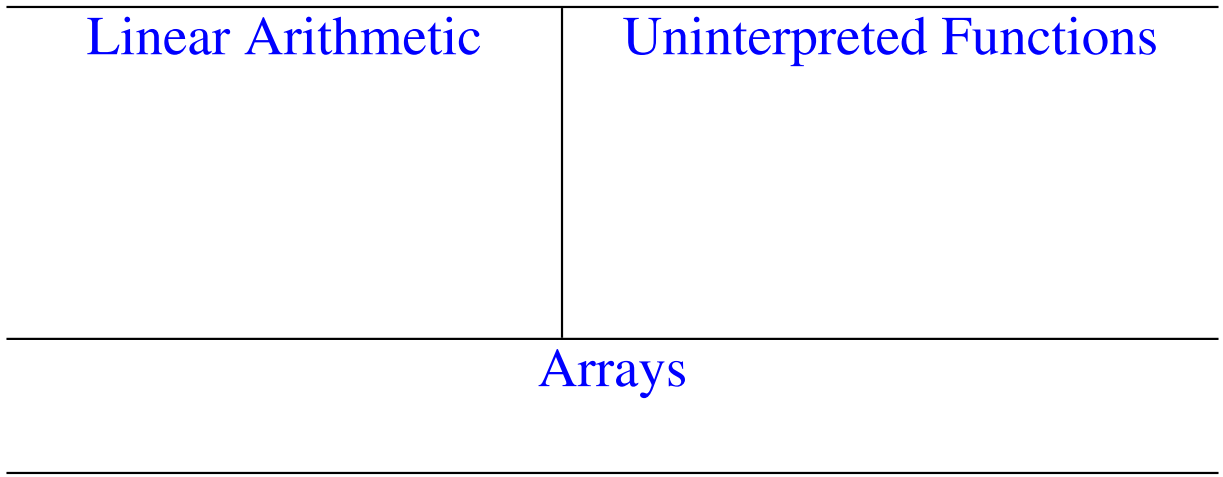
Checking satisfiability of a conjunction of literals in a combination of theories?

Idea:

- Separate reasoning in various theories
- Make reasoners exchange equalities

# Separating Reasoning

$$x + 2 = y$$
$$f(read(write(a, x, 3), y - 2)) \neq f(y - x + 1)$$



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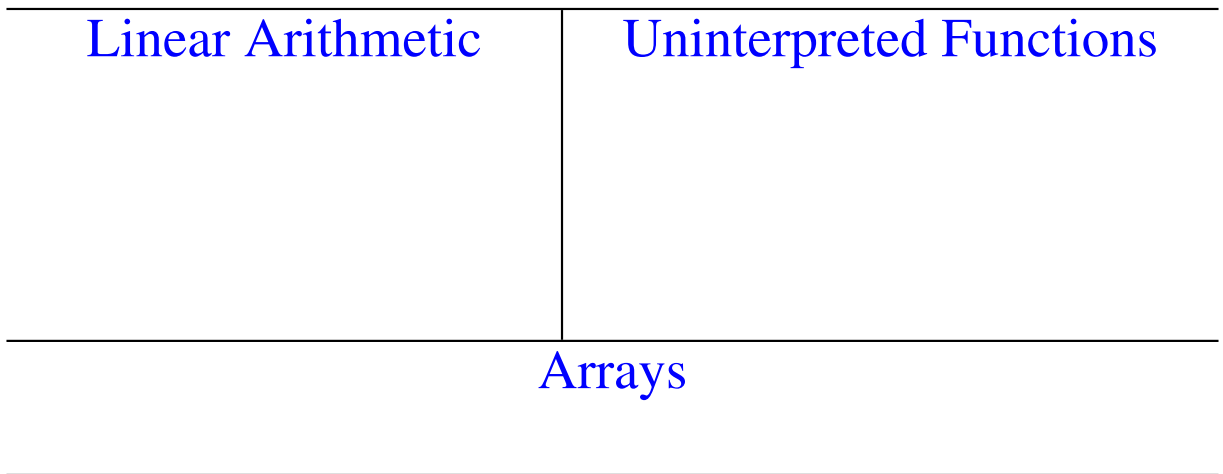

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| Linear Arithmetic | Uninterpreted Functions |
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| $y = x + 2$ | |
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- Each step obviously preserves satisfiability; moreover every model of the modified set of literals is also a model of the original set.

# Separating Reasoning

**procedure** *SeparatingReasoning*($F$)
**input**: formula $F \in \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$
**output**: formulas $F_1 \in \mathcal{T}_1, \ldots, F_n \in \mathcal{T}_n$ s.t.
$F$ and $F_1 \wedge \cdots \wedge F_n$ are equisatisfiable
**assumptions**: theory signatures $\Sigma_1, \ldots, \Sigma_n$ are disjoint
**parameters**: function $head(t)$ returning the root symbol of a term $t$
**begin**
**repeat as long as possible**
**if** $f \in \Sigma_i$ and $head(t) \in \Sigma_j$ with $i \neq j$:
rewrite $F[f(t_1, \ldots, t, \ldots, t_m)]$ into $F[f(t_1, \ldots, c, \ldots, t_m)] \wedge c = t$,
where $c$ is a fresh new variable
**if** $p \in \Sigma_i$ and $head(t) \in \Sigma_j$ with $i \neq j$:
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where $c$ is a fresh new variable
**if** $head(s) \in \Sigma_i$ and $head(t) \in \Sigma_j$ with $i \neq j$:
rewrite $F[s = t]$ into $F[s = c] \wedge c = t$,
where $c$ is a fresh new variable
**end repeat**
**return** modified $F$ as $F_1 \wedge \cdots \wedge F_n$, with each $F_i \in \mathcal{T}_i$

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- Each step obviously preserves satisfiability; moreover every model of the modified set of literals is also a model of the original set.
- In this example every "separated" subset of literals is satisfiable.
- But this does not mean that the set of all literals is satisfiable.

# System to Solve

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Not all theories can be combined (easily) in this way!