Automated Deduction

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Satisfiability Modulo Theory (SMT)

- An interpretation I which which makes all axioms of $\mathcal{T}$ valid, that is $I \models A_{\mathcal{T}}$, is called a $\mathcal{T}$-interpretation.
- A formula F is valid in $\mathcal{T}$ (or $\mathcal{T}$-valid) if F is valid in every $\mathcal{T}$-interpretation, written $\mathcal{T} \models F$.
- A formula F is satisfiable in $\mathcal{T}$, or $\mathcal{T}$-satisfiable, or

  satisfiable modulo theory $\mathcal{T}$

  if there exists a $\mathcal{T}$-interpretation which satisfies F.

Cesare Tinelli:

A DPLL-Based Calculus for Ground Satisfiability Modulo Theories. JELIA 2002

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Silvio Ranise and Cesare Tinelli.
The SMT-LIB Format: An Initial Proposal. PDPAR 2003.

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Zelkova translates the policy to a first order logic formula,
then uses tools including SMT solver to find requests
that can satisfy the formula “

https://www.cs.rice.edu/rima

Related problems to SMT

- Deciding a conjunction of literals: How can we check whether a set of *literals* is satisfiable?

  In particular, in $\mathcal{T}_E$, $\mathcal{T}_A$, and $\mathcal{T}_Q$.

- How can we put theory reasoning and SAT solving together?

- Combination of theories: Given decision procedures for theories, how can we build a decision procedure for formulas using several theories?

Deciding Theory: An Example

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- From $a = c$ and (congruence), conclude $f(a) = f(c)$.
- $f(a) = f(c)$ contradicts $f(a) \neq f(c)$.

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We will now discuss specialised decision procedures for theories.

Deciding $\mathcal{T}_E$: Congruence Closure

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- a decision procedure for $\mathcal{T}_E$;

- can be extended to a decision procedure for $\mathcal{T}_A$;
- is the basis for combining theories;

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- decides formulas in the theory of equality and uninterpreted functions.

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Example: Instead of $p(x) \wedge q(y,z) \wedge a = b \rightarrow \neg q(x,z)$

we use

$$f_p(x) = t \wedge f_q(y,z) = t \wedge a = b \rightarrow f_q(x,z) \neq t,$$

where f_p, f_q are fresh functions and t is a fresh constant.

Deciding $\mathcal{T}_E$: Congruence Closure – abstract definitions

Consider a set S and a binary relation R over S.
R is a congruence relation if:

- xRx
- $xRy \rightarrow yRx$
- $xRy \wedge yRz \rightarrow xRz$
- $x_1Ry_1 \wedge \ldots \wedge x_nRy_n \rightarrow f(x_1, \ldots, x_n)Rf(y_1, \ldots, y_n)$ for all function symbols f.

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The congruence class of $t \in S$ under the congruence relation R is

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The congruence relation R defines a partition on S:

$$\left(\bigcup_{[t]_R}[t]_R\right) = S \qquad \text{and} \qquad [t_1]_R \neq [t_2]_R \rightarrow [t_1]_R \cap [t_2]_R = \emptyset$$

Deciding $\mathcal{T}_E$: Congruence Closure – abstract definitions

The congruence closure R^c of R is the congruence relation such that:

- $R \subseteq R^c$
- for all other congruence relations R' with $R \subseteq R'$, either $R^c = R'$ or $R^c \subseteq R'$

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Example: Let $S = \{a, b, c, d\}$ and $R = \{aRb, bRc, dRd\}$. Then

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Deciding $\mathcal{T}_E$: Congruence Closure Algorithm

Consider the formula F:

$$s_1 = t_1 \wedge \ldots \wedge s_n = t_n \wedge s_{n+1} \neq t_{n+1} \wedge \ldots \wedge s_m \neq t_m$$

where terms s_i, t_i are terms. Is F $\mathcal{T}_E$-satisfiable?

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Idea

1. set S is the set of subterms of F
2. construct the congruence class of each subterm of F under the binary relation $\{s_1 = t_1, \ldots, s_n = t_n\}$.

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   Congruence closure R of $\{s_1 = t_1, \ldots, s_n = t_n\}$!
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Deciding $\mathcal{T}_E$: Congruence Closure Algorithm

procedure *CongruenceClosure*(F)
input: F is $s_1 = t_1 \wedge \ldots s_n = t_n \wedge s_{n+1} \neq t_{n+1} \wedge \ldots s_m \neq t_m$
output: *satisfiable* or *unsatisfiable*
parameters: function *subterm_set*
begin
  S_F := *subterm_set*(F)
  R := $\{sRs \mid s \in S_F\}$ and $[s]_R$:= $\{s\}$, defining the partition $\{[s]_R \mid s \in S_F\}$
  for every $s_i = t_i$ in F, merge $[s_i]_R$ and $[t_i]_R$ by
    forming the union $[s_i]_R \cup [t_i]_R$
    propagate the new congruences that arise in this union
  if $s_j \; R \; t_j$ for any $j \in \{n+1, \ldots, m\}$ **then return** *unsatisfiable*
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    propagate the new congruences that arise in this union (function congruence)
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Deciding $\mathcal{T}_E$: Congruence Closure by Example

Consider the formula $F: \quad f(a,b) = a \wedge f(f(a,b),b) \neq a$.

Question: Is F $\mathcal{T}_E$-satisfiable?

Deciding $\mathcal{T}_E$: Congruence Closure by Example

Consider the formula $F: \quad f(a,b) = a \wedge f(f(a,b),b) \neq a$.

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Subterm set $S_F = \{a, b, f(a,b), f(f(a,b),b)\}$

Initial partition: $\{\{a\},\{b\},\{f(a,b)\},\{f(f(a,b),b)\}\}$

Using $f(a,b) = a$, merge $\{f(a,b)\}$ and $\{a\}$ and form partition:

$$\{\{a, f(a,b)\},\{b\},\{f(f(a,b),b)\}\}$$

From $f(a,b)\ R\ a$ and $b\ R\ b$, by congruence we have: $f(f(a,b),b)\ R\ f(a,b)$

Thus, merge $\{a, f(a,b)\}$ and $\{f(f(a,b),b)\}$ and form partition:

$$\{\{a, f(a,b), f(f(a,b),b)\},\{b\}\}$$

This is the partition of the congruence closure of $\{f(a,b) = a\}$.

F contains $f(f(a,b),b) \neq a$, but we have $f(f(a,b),b)\ R\ a$ in the congruence closure. Hence, F is $\mathcal{T}_E$-unsatisfiable.

Deciding $\mathcal{T}_E$: Congruence Closure by Example

Consider the formula $F: \quad f(a) = f(b) \wedge a \neq b$.

Question: Is F $\mathcal{T}_E$-satisfiable?

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Consider the formula
$F: \quad f(f(f(a))) = a \wedge f(f(f(f(f(a))))) = a \wedge f(a) \neq a$.

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Congruence Closure and . . .

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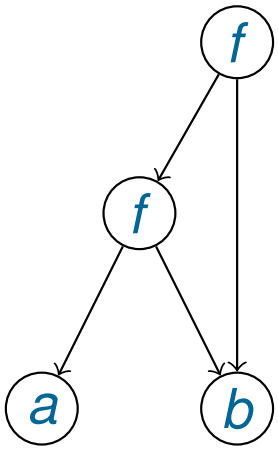

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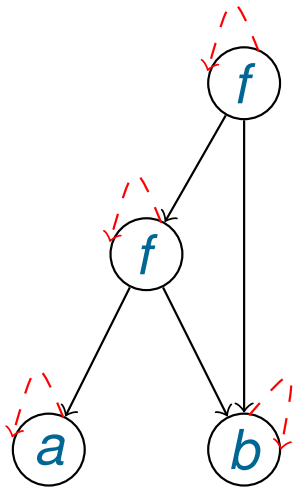

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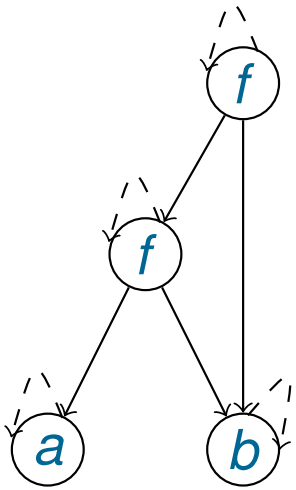

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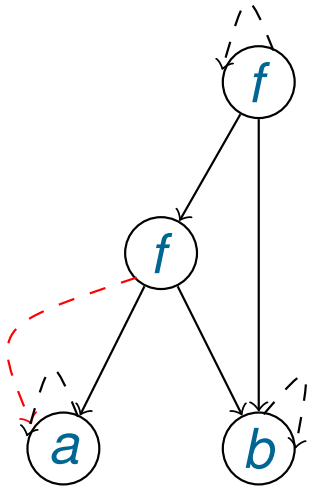

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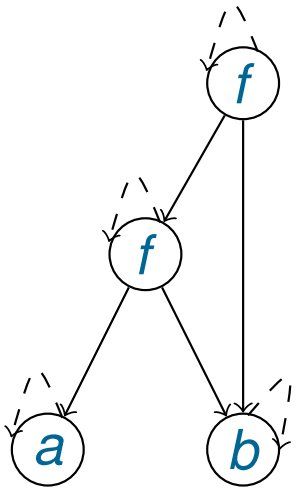

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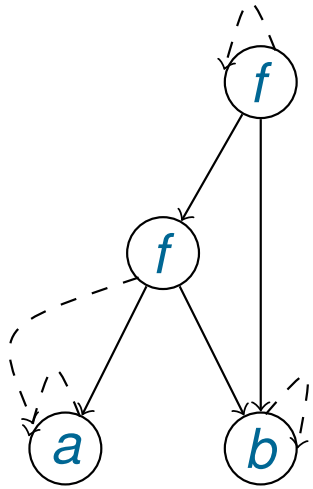

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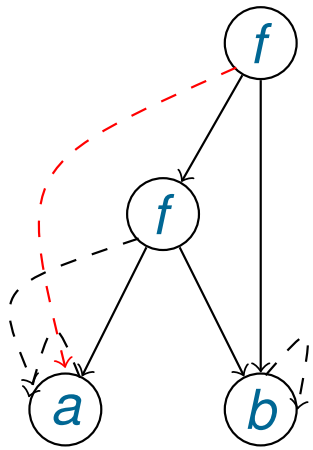

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Propagation of congruence classes.

Congruence Closure and DAGs

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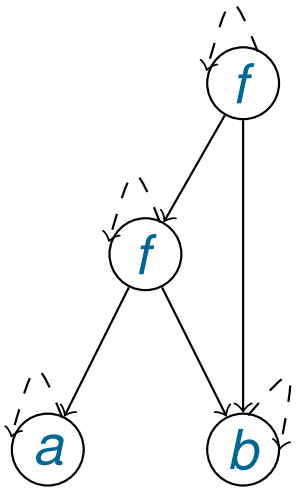

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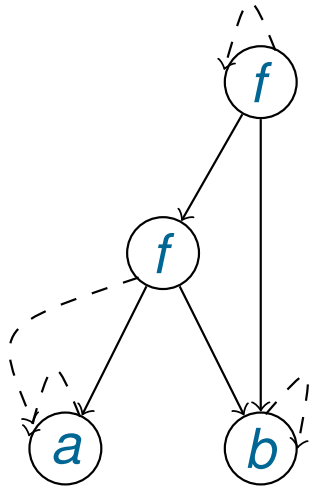

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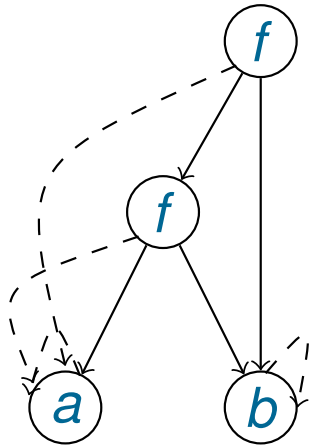

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Union-Find algorithm on the DAG for congruence closure.

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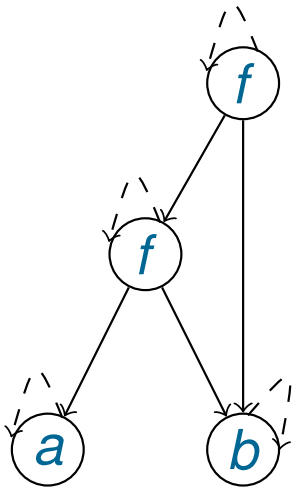

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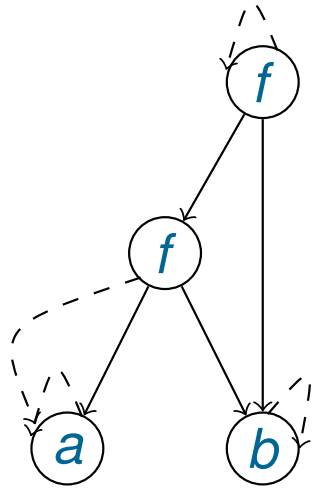
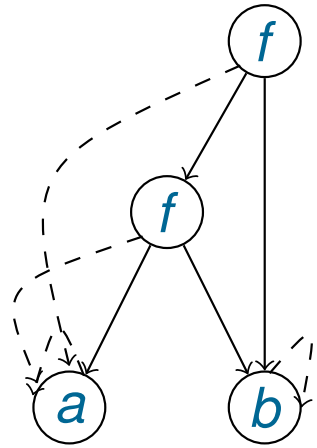


A node is a subterm of F.
Union of congruence classes.



A node is a subterm of F.
Propagation of congruence classes.
Union-Find algorithm on the DAG for congruence closure.

Congruence closure is inexpensive – it is decidable in polynomial time.

Outline

Deciding $\mathcal{T}_E$: Problems Using Non-Unit Clauses

Example: Let's try to prove the validity of the formula:

$$F: \ (a = b \vee a = c) \wedge f(a) = a \rightarrow f(f(b)) = a \vee f(c) = a.$$

This is equivalent to establishing unsatisfiability of

$$a = b \vee a = c$$
$$f(a) = a$$
$$f(f(b)) \neq a$$
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We have a non-unit clause, so we can't use congruence closure.

Inputs of the congruence closure algorithm are conjunctions of $\mathcal{T}_E$-literals.