Automated Deduction

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Outline

Satisfiability and Randomisation

- SAT solving of randomly generated set of clauses are very hard for DPLL-based SAT solvers;
- Some small randomly generated set of clauses cannot be satisfied by DPLL-based SAT solvers, whereas random SAT algorithms can satisfy them;
- Small randomly generated set of clauses are useful for debugging SAT solvers;
- Experiments show that randomly generated set of clauses become from satisfiable to unsatisfiable in a very narrow region.

Random Clause Generation

How can one generate a random clause?
Let's first generate a random literal.
A random clause is a collection of random literals.

- Fix a number n of boolean variables;
- Select a literal among $p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n$ with an equal probability.
- Fix the length k of the clause;

Suppose we generate random clauses one after one. How does the set of models of this set change?

SAT and k-SAT

SAT is the problem of satisfiability checking for sets of clauses.

k-SAT is the problem of satisfiability checking for sets of clauses of length k.

- SAT is NP-complete;
- 2-SAT is decidable in linear time;
- 3-SAT is NP-complete.

There is a simple reduction of SAT to 3-SAT based on the same ideas as used for generating short clausal forms (naming). Take a clause having more than 3 literals:

$$L_1 \vee L_2 \vee L_3 \vee L_4 \ldots$$

And replace it by two clauses:

$$L_1 \vee L_2 \vee n$$
$$\neg n \vee L_3 \vee L_4 \ldots$$

where n is a new variable.

Example (obtained by a program) for $n = 5$ and $k = 2$

	p_1	p_2	p_3	p_4	p_5
$\neg p_2 \vee \neg p_3$	0	0	0	0	0
$\neg p_2 \vee p_1$	0	0	0	0	1
$\neg p_2 \vee p_2$	0	0	0	1	0
$p_1 \vee p_1$	0	0	0	1	1
$\neg p_5 \vee p_5$	0	0	1	0	0
$p_4 \vee p_5$	0	0	1	0	1
$\neg p_5 \vee \neg p_3$	0	0	1	1	0
$p_2 \vee \neg p_4$	0	0	1	1	1
$p_5 \vee \neg p_2$	0	1	0	0	0
$p_5 \vee p_2$	0	1	0	0	1
$\neg p_1 \vee \neg p_4$	0	1	0	1	0
$p_5 \vee p_2$	0	1	0	1	1
$\neg p_1 \vee \neg p_5$	0	1	1	0	0
	0	1	1	0	1
	0	1	1	1	0
	0	1	1	1	1

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1
1	1	1	0	0
1	1	1	0	1
1	1	1	1	0
1	1	1	1	1

Number of models: 32242012974310

This set of 13 clauses is unsatisfiable.

Random Clause Generation

We are interested in the probability that a set of clauses of a given size is unsatisfiable.

Fix:

- Number n of boolean variables;
- Number k of literals per clause, so we will generate k-SAT instances;
- Number m of the clauses. Real number r: ratio of clauses per variable.

Generate $m[rn]$ clauses, each one has k literals randomly generated among $p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n$ with an equal probability.

Note that the probability is a monotone function: the more clauses we generate, the higher chance we have that the set is unsatisfiable.

Exercise: What are the probabilities that the resulting sets are unsatisfiable for $m = 1$ and $m = 2$? Denote by $\pi(r, n)$ the probability that a randomly generated set of $[rn]$ k-clauses is unsatisfiable.

SAT Roulette

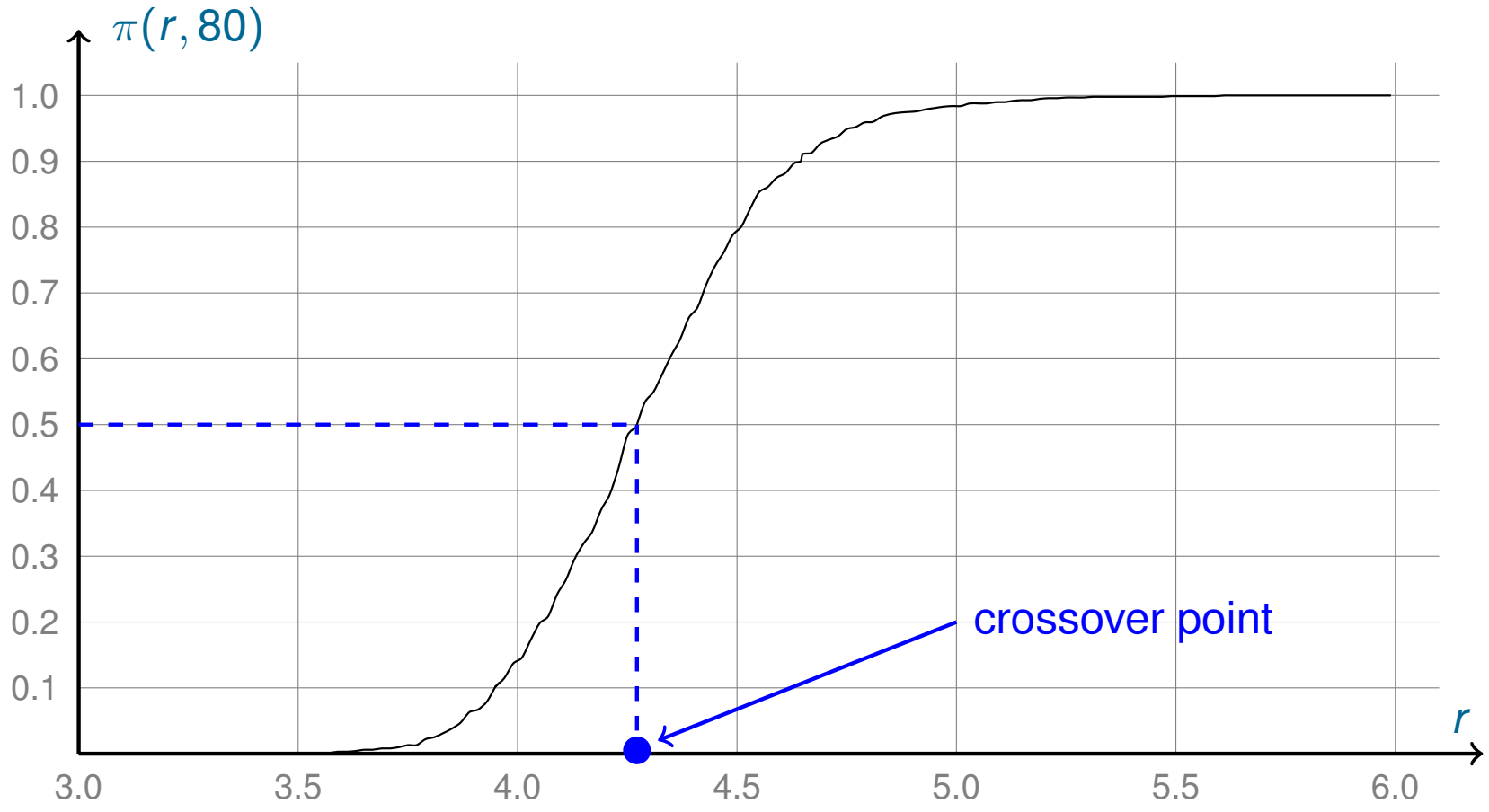
We will generate random instances of 23-SAT with 5-variables.

You will bet on whether the resuting set of clauses is satisfiable or not.

- What will you bet on if we generate 5 clauses?
- What will you bet on if we generate 100 clauses?
- What will you bet on if we generate 15 clauses?

Probability of obtaining an unsatisfiable set of 3-SAT

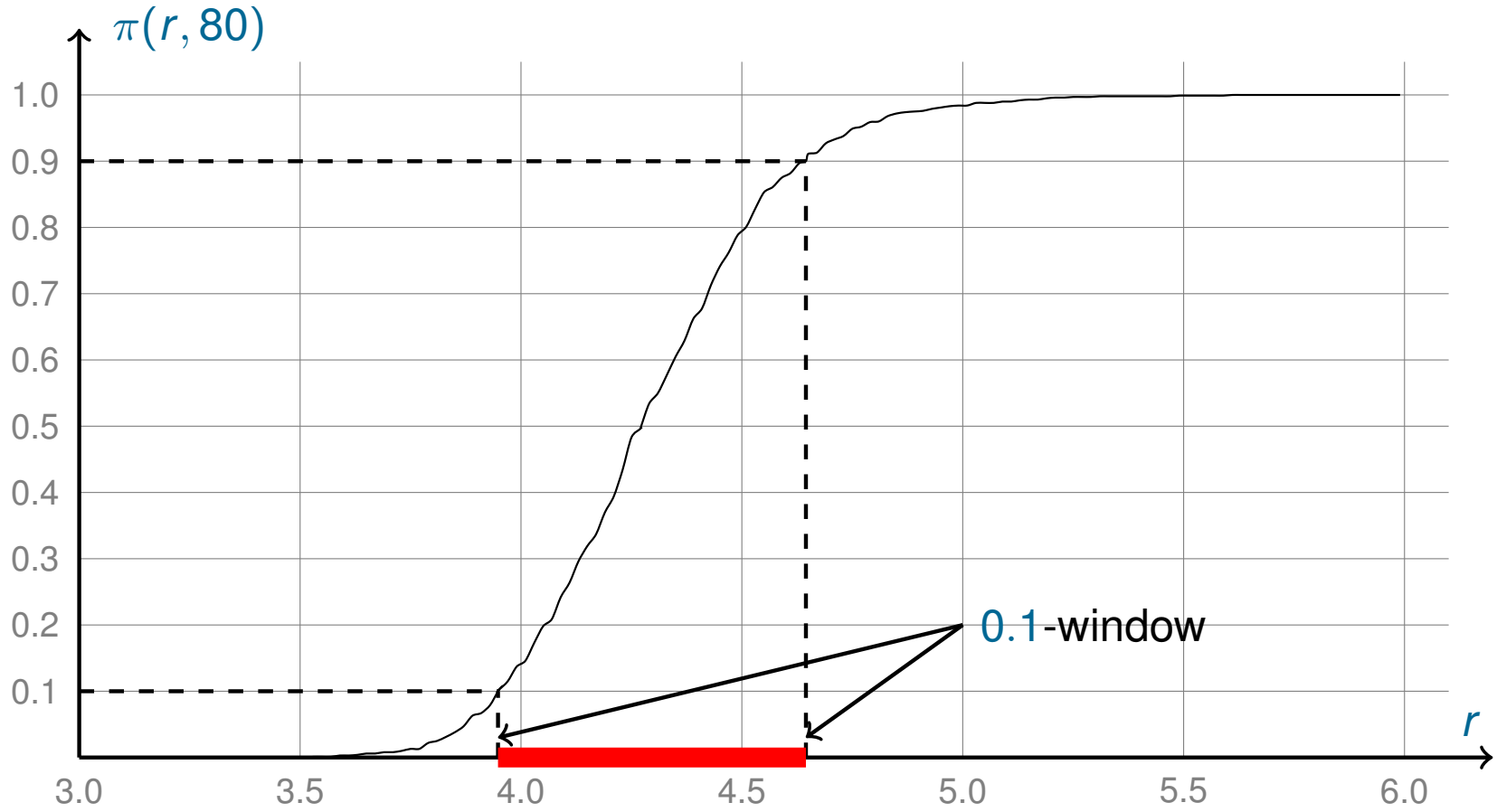
Crossover point: the value of r at which the probability crosses 0.5.



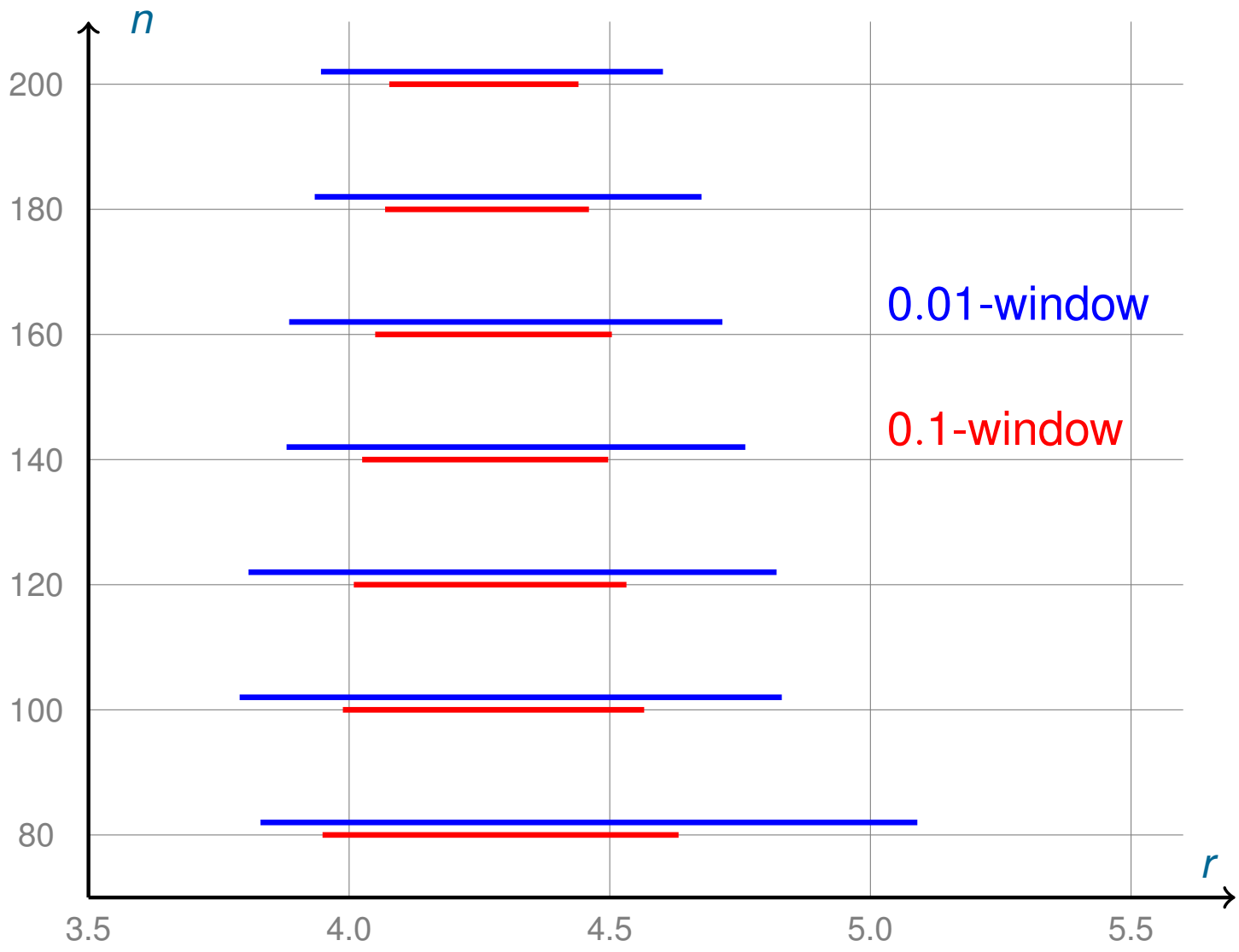
ϵ-window

Take a (small) number $0 < \epsilon < 0.5$. ϵ-window is the interval of values of r where the probability is between ϵ and $1 - \epsilon$.

For example, take $\epsilon = 0.1$.

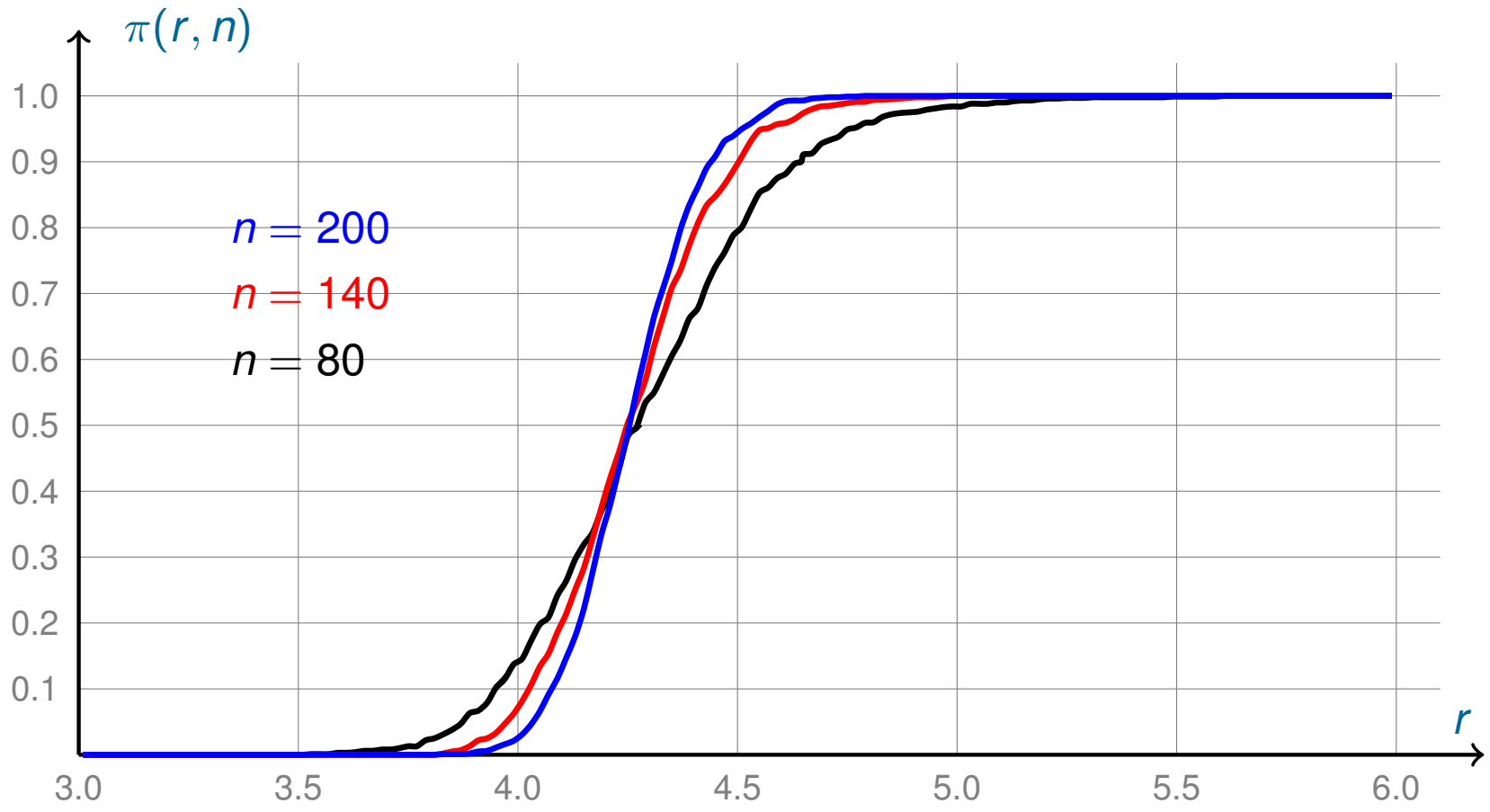


Scaling Window Effect

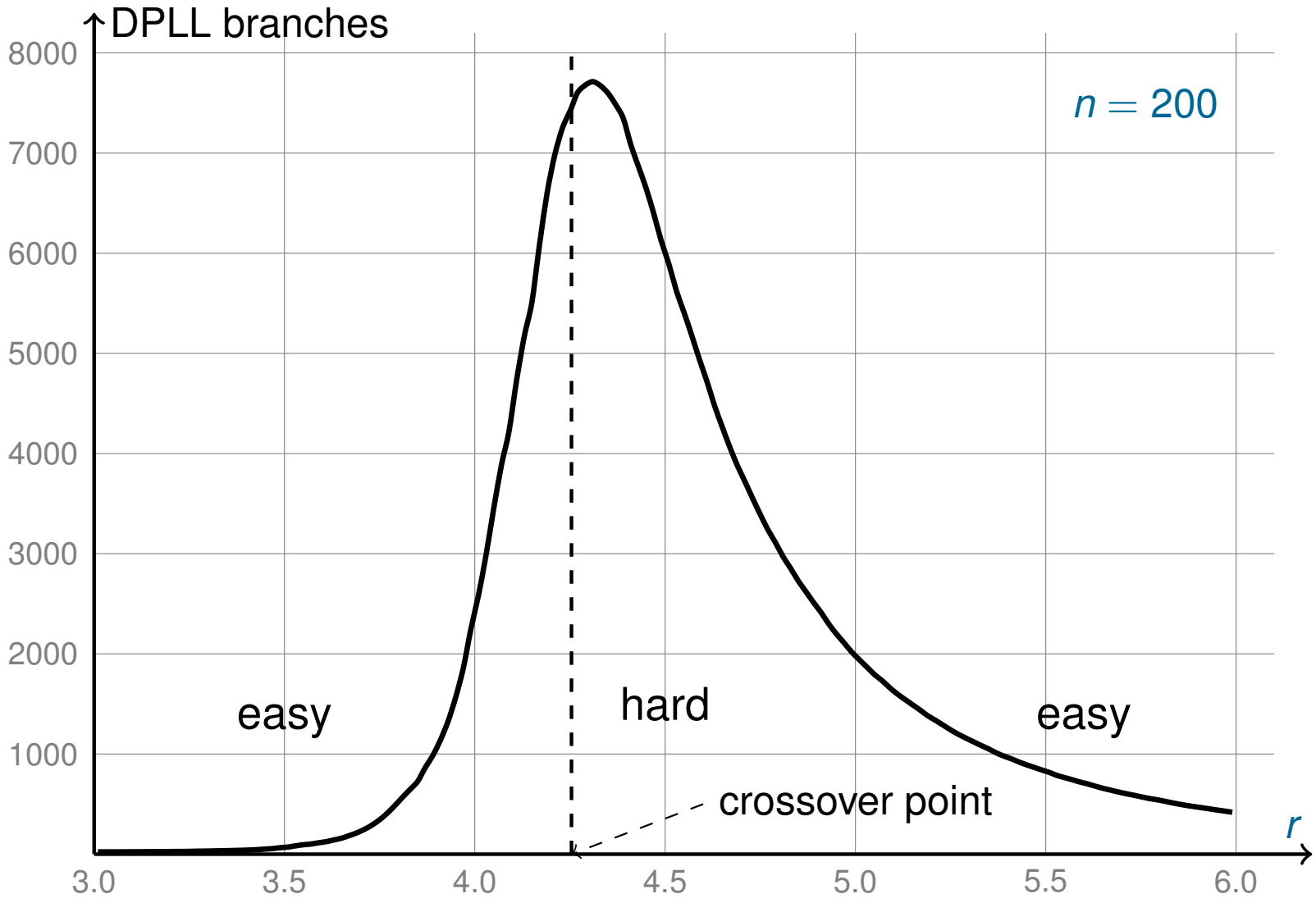


Conjecture: for $n \to \infty$ every ϵ-window "degenerates into a point".

Sharp Phase Transition of $\pi(r, n)$



Easy-Hard-Easy Pattern



Experiments show that the crossover point for 3-SAT is 4.25.

Satisfiability Algorithm that Cannot Establish Unsatisfiability

procedure *CHAOS*(*S*)
input: set of clauses *S*
output: interpretation *I* such that $I \models S$ or *don't know*
parameters: positive integer *MAX-TRIES*
begin
  repeat *MAX-TRIES* times
    I := random interpretation
    if $I \models S$ **then return** I
  return *don't know*
end

SAT as a Decision Problem

Decision problem: any problem in any infinite domain, that has a yes-no answer. Each element of this domain is called an instance of this problem.

Example: solvability of systems of linear inequalities over integers.

- an instance is a system of linear integer inequalities;
- an answer is yes if it the instance has a solution.

SAT is a decision problem:

- an instance is a finite set of clauses.
- it has a yes-no answer: yes (satisfiable) or no (unsatisfiable).

Witness for an instance $\mathcal{I}$: any data D such that, given D, one can check in polymonial time (in the size of D) that D is a yes-answer of $\mathcal{I}$.

Satisfiability has short witnessess: interpretations.

Unsatisfiability has no polynomial-size witnessess, unless $NP = co - NP$.

Randomised Algorithms for SAT

- Choose a random interpretation.
- If this interpretation is not a model, repeatedly choose a variable and change its value in the interpretation (flip the variable).

The flipped variables are chosen using heuristics or randomly, or both.

$$flip(I,p)(q) = \begin{cases} I(q), & \text{if } p \neq q; \\ 1, & \text{if } p = q \text{ and } I(p) = 0; \\ 0, & \text{if } p = q \text{ and } I(p) = 1. \end{cases}$$

In other words, the interpretation $flip(I,p)$ is obtained from I by changing its value on p.

GSAT

procedure *GSAT*(S)
input: set of clauses S
output: interpretation I such that $I \models S$ or *don't know*
parameters: integers *MAX-TRIES*, *MAX-FLIPS*
begin
  repeat *MAX-TRIES* times
    I := random interpretation
    if $I \models S$ **then return** I
    repeat *MAX-FLIPS* times
      p := a variable such that $flip(I, p)$ satisfies
      p := the maximal number of clauses in S
      $I = flip(I, p)$
      if $I \models S$ **then return** I
  return *don't know*
end

GSAT example

10		10		01
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

flip no.	interpretation			satisfied clauses				candidates for flipping	flipped variable
	p_1	p_2	p_3		p_1	p_2	p_3		
1	0	0	1	4	3	4	4	p_2, p_3	p_2
2	0	1	1	4	3	4	4	p_2, p_3	p_3
3	0	1	0	4	5	4	4	p_1	p_1
	1	1	0	5					

WSAT

procedure *WSAT*(S)
input: set of clauses S
output: interpretation I such that $I \models S$ or *don't know*
parameters: integers *MAX-TRIES*, *MAX-FLIPS*
begin
  repeat *MAX-TRIES* times
    I := random interpretation
    if $I \models S$ **then return** I
    repeat *MAX-FLIPS* times
      randomly select a clause $C \in S$ such that $I \not\models C$
      randomly select a variable p in C
      $I = flip(I, p)$
      if $I \models S$ **then return** I
  return *don't know*
end

WSAT example

10		10		01
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

flip no.	interpretation			unsatisfied clauses	candidates for flipping	flipped variable
	p_1	p_2	p_3			
1	0	0	1	$p_1 \vee p_2$	p_1, p_2	p_1
2	1	0	1	$\neg p_1 \vee \neg p_3$ $\neg p_1 \vee p_2$	p_1, p_2, p_3	p_2
3	1	1	1	$\neg p_2 \vee \neg p_3$ $\neg p_1 \vee \neg p_3$	p_1, p_2, p_3	p_3
	1	1	0			

Outline

Horn clauses

A clause is called Horn if it contains at most one positive literal.
Examples:

$$p_1$$
$$\neg p_1 \vee p_2$$
$$\neg p_1 \vee \neg p_2 \vee p_3$$
$$\neg p_3 \vee \neg p_4$$

The following clauses are non-Horn:

$$p_1 \vee p_2$$
$$p_1 \vee \neg p_2 \vee p_3$$

Satisfiability of Horn clauses can be decided by unit propagation

Example:

$$p_1$$
$$\neg p_1 \vee p_2$$
$$\neg p_1 \vee \neg p_2 \vee p_3$$
$$\neg p_3 \vee \neg p_4$$

Model: $\{p_1 \mapsto 1, p_2 \mapsto 1, p_3 \mapsto 1, p_4 \mapsto 0\}$
Note that deleting a literal from a Horn clause gives a Horn clause.
Therefore, unit propagation applied to a set C of Horn clauses gives a set C' of Horn clauses.

Two cases:

1. C' contains $\square$. Then, C' (and hence C) is unsatisfiable.
2. C' does not contain $\square$. Then:
   - Each clause in C' has at least two literals.
   - Hence each clause in C' contains at least one negative literal;
   - Hence setting all variables in C' to 0 satisfies C'.