

Automated Deduction

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Decision Procedures for Propositional Satisfiability

More sophisticated decision procedures:

- ▶ Splitting;
- ▶ DPLL;
- ▶ (BDDs – binary decision diagrams);
- ▶ (resolution).

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We next look at the DPLL method, which uses the splitting algorithm.

(DPLL: Davis-Putnam-Logemann-Loveland, 1962)

DPLL works on a restricted set of formulas, we need to first look at the **needed normal forms** and **normal form conversions**.

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In other words, p and $\neg p$ are complementary.

- ▶ **Clause**: a disjunction $L_1 \vee \dots \vee L_n$, $n \geq 0$ of literals.
 - ▶ **Empty clause**, denoted by \square : $n = 0$ (the empty clause is **false** in every interpretation).
 - ▶ **Unit clause**: $n = 1$.
 - ▶ **Horn clause**: a clause with at most one positive literal.

CNF

- ▶ A formula A is in **conjunctive normal form**, or simply **CNF**, if it is either \top , or \perp , or a conjunction of disjunctions of literals:

$$A = \bigwedge_i \bigvee_j L_{i,j}.$$

(In other words, A is a conjunction of clauses.)

- ▶ A formula B is called a **conjunctive normal form of a formula A** if B is equivalent to A and B is in conjunctive normal form.

Satisfiability on CNF

- ▶ An interpretation I satisfies a formula in CNF

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if and only if it satisfies every clause

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- ▶ An interpretation I satisfies a clause

$$L_1 \vee \dots \vee L_k$$

if and only if it satisfies at least one literal L_m in this clause.

CNF transformation

$$\begin{aligned}A \leftrightarrow B &\Rightarrow (\neg A \vee B) \wedge (\neg B \vee A), \\A \rightarrow B &\Rightarrow \neg A \vee B, \\ \neg(A \wedge B) &\Rightarrow \neg A \vee \neg B, \\ \neg(A \vee B) &\Rightarrow \neg A \wedge \neg B, \\ \neg\neg A &\Rightarrow A, \\ (A_1 \wedge \dots \wedge A_m) \vee B_1 \vee \dots \vee B_n &\Rightarrow (A_1 \vee B_1 \vee \dots \vee B_n) \wedge \\ &\quad \dots \wedge \\ &\quad (A_m \vee B_1 \vee \dots \vee B_n).\end{aligned}$$

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A formula to which no rewrite rule is applicable

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- ▶ contains no \leftrightarrow ;
- ▶ contains no \rightarrow ;
- ▶ may only contain \neg applied to atoms;
- ▶ cannot contain \wedge in the scope of \vee ;
- ▶ (hence) is in CNF.

CNF, example

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow$$

$$A \leftrightarrow B \Rightarrow (\neg A \vee B) \wedge (\neg B \vee A),$$

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$$(A_1 \wedge \dots \wedge A_m) \vee B_1 \vee \dots \vee B_n \Rightarrow \begin{array}{l} (A_1 \vee B_1 \vee \dots \vee B_n) \\ \dots \\ (A_m \vee B_1 \vee \dots \vee B_n). \end{array} \quad \begin{array}{l} \wedge \\ \wedge \end{array}$$

CNF, example

$$\begin{aligned} &\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow \\ &\neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow \\ &\neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow \\ &(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow \\ &(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow \\ &(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge \neg r \Rightarrow \\ &(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow \\ &(p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow \\ &(p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \end{aligned}$$

$$A \leftrightarrow B \Rightarrow (\neg A \vee B) \wedge (\neg B \vee A),$$

$$A \rightarrow B \Rightarrow \neg A \vee B,$$

$$\neg(A \wedge B) \Rightarrow \neg A \vee \neg B,$$

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$$\neg\neg A \Rightarrow A,$$

$$\begin{aligned} (A_1 \wedge \dots \wedge A_m) \vee B_1 \vee \dots \vee B_n &\Rightarrow (A_1 \vee B_1 \vee \dots \vee B_n) \quad \wedge \\ &\dots \quad \wedge \\ &(A_m \vee B_1 \vee \dots \vee B_n). \end{aligned}$$

CNF, example

$$\begin{aligned} & \neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow \\ & \neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow \\ & \neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge \neg r \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow \\ & (p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow \\ & (p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \end{aligned}$$

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CNF, example

$$\begin{aligned} & \neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow \\ & \neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow \\ & \neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge \neg r \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow \\ & (p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow \\ & (p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \\ & (\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \end{aligned}$$

$$A \leftrightarrow B \Rightarrow (\neg A \vee B) \wedge (\neg B \vee A),$$

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CNF and satisfiability

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow$$

...

$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

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Therefore, the formula

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has the same models as the set consisting of four clauses

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The CNF transformation reduces the satisfiability problem for formulas to the satisfiability problem for sets of clauses.

Problem with CNF

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))).$$

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$$\begin{aligned} p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) &\Rightarrow \\ (\neg p_1 \vee (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))) &\wedge \\ (p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))) & \end{aligned}$$

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Compute the CNF of

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If we continue, the formula will **grow exponentially**.

CNF is exponential

There are formulas for which the **shortest CNF has an exponential size**.

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There are formulas for which the **shortest CNF has an exponential size**.

Is there any way to **avoid exponential blowup**?

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Using so-called **naming** or **definition introduction**.

- ▶ Take a non-trivial subformula A .
- ▶ Introduce a new **name** n for it. A name is a new propositional variable.

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Using so-called **naming** or **definition introduction**.

- ▶ Take a non-trivial subformula A .
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- ▶ Add a formula stating that n is equivalent to A (**definition for n**).

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$$
$$n \leftrightarrow (p_5 \leftrightarrow p_6)$$

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$$\begin{aligned} p_1 &\leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \\ n &\leftrightarrow (p_5 \leftrightarrow p_6) \end{aligned}$$

- ▶ Replace the subformula by its name:

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The new set of two formulas has the same models as the original one **if we restrict ourselves to the original set of variables** $\{p_1, \dots, p_6\}$.
But this set is **not equivalent** to the original formula.

After several steps

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$$

$$p_1 \leftrightarrow (p_2 \leftrightarrow n_3);$$

$$n_3 \leftrightarrow (p_3 \leftrightarrow n_4);$$

$$n_4 \leftrightarrow (p_4 \leftrightarrow n_5);$$

$$n_5 \leftrightarrow (p_5 \leftrightarrow p_6).$$

After several steps

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$$

$$p_1 \leftrightarrow (p_2 \leftrightarrow n_3);$$

$$n_3 \leftrightarrow (p_3 \leftrightarrow n_4);$$

$$n_4 \leftrightarrow (p_4 \leftrightarrow n_5);$$

$$n_5 \leftrightarrow (p_5 \leftrightarrow p_6).$$

The conversion of the **original formula** to CNF introduces **32 copies** of p_6 .

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$$n_5 \leftrightarrow (p_5 \leftrightarrow p_6).$$

The conversion of the **original formula** to CNF introduces **32 copies** of p_6 .

The conversion of the **new set of formulas** to CNF introduces **4 copies** of p_6 .

Definitional Translations

Lemma [Naming]

Let S be a set of formulas and A a formula. Let n be a boolean variable not occurring in S , nor in A .

Then S is satisfiable if and only if so is the set of formulas $S \cup \{n \leftrightarrow A\}$.

Clausal Form

- ▶ **Clausal form of a formula A** : a set of clauses which is satisfiable if and only if A is satisfiable.

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We can require even more: that A and its clausal form have the same models **in the language of A** .

Using clausal normal forms instead of conjunctive normal forms we can convert any formula to a set of clauses in **almost linear time**.

Definitional Clause Form Transformation

This algorithm converts a formula A into a set of clauses S such that S is a **clausal normal form of A** .

If A has the form $C_1 \wedge \dots \wedge C_n$, where $n \geq 1$ and each C_i is a clause, then $S \stackrel{\text{def}}{=} \{C_1, \dots, C_n\}$.

Otherwise, introduce a name for each subformula B of A such that B is not a literal and use this name instead of the formula.

Example

subformula	definition	clauses
$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		

Converting a formula to clausal form.

Example

subformula	definition	clauses
$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		
$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$		
$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$		
$p \rightarrow q$		
$p \wedge q \rightarrow r$		
$p \wedge q$		
$p \rightarrow r$		

Take all subformulas that are not literals.

Example

	subformula	definition	clauses
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$		
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$		
n_4	$p \rightarrow q$		
n_5	$p \wedge q \rightarrow r$		
n_6	$p \wedge q$		
n_7	$p \rightarrow r$		

Introduce names for these formulas.

Example

	subformula	definition	clauses
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \leftrightarrow \neg n_2$	
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$n_2 \leftrightarrow (n_3 \rightarrow n_7)$	
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	
n_4	$p \rightarrow q$	$n_4 \leftrightarrow (p \rightarrow q)$	
n_5	$p \wedge q \rightarrow r$	$n_5 \leftrightarrow (n_6 \rightarrow r)$	
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	
n_7	$p \rightarrow r$	$n_7 \leftrightarrow (p \rightarrow r)$	

Introduce definitions.

Example

	subformula	definition	clauses
			n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$n_2 \leftrightarrow (n_3 \rightarrow n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \leftrightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \leftrightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$n_7 \leftrightarrow (p \rightarrow r)$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Convert the resulting formula to CNF using the standard algorithm.

Optimised Definitional Clause Form Transformation

If we introduce a name n for a subformula B and the occurrence of the subformula B in formula A is either only positive or negative, then an **implication is used instead of equivalence**.

Optimised Definitional Clause Form Transformation

If we introduce a name n for a subformula B and the occurrence of the subformula B in formula A is either only positive or negative, then an **implication is used instead of equivalence**.

- ▶ if B has only positive occurrences in A , then $n \rightarrow B$ is used instead of $n \leftrightarrow B$;
- ▶ if B has only negative occurrences in A , then $B \rightarrow n$ is used instead of $n \leftrightarrow B$;

Example

	subformula	definition	clauses
			n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \rightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$(n_3 \rightarrow n_7) \rightarrow n_2$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \rightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \rightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \rightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$(p \wedge q) \rightarrow n_6$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$(p \rightarrow r) \rightarrow n_7$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Example

	subformula	definition	clauses
			n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \rightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$(n_3 \rightarrow n_7) \rightarrow n_2$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \rightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \rightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \rightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$(p \wedge q) \rightarrow n_6$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$(p \rightarrow r) \rightarrow n_7$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

All clauses shown in the **red color** are not generated by the optimised transformation.

Example

	subformula	definition	clauses
			n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \rightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$(n_3 \rightarrow n_7) \rightarrow n_2$	$n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \rightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$
n_4	$p \rightarrow q$	$n_4 \rightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$
n_5	$p \wedge q \rightarrow r$	$n_5 \rightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$
n_6	$p \wedge q$	$(p \wedge q) \rightarrow n_6$	$\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$(p \rightarrow r) \rightarrow n_7$	$p \vee n_7$ $\neg r \vee n_7$

The optimised transformation gives fewer clauses.

Satisfiability-checking for sets of clauses

The CNF transformation of

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$$

gives the set of four clauses:

$$\neg p \vee q$$

$$\neg p \vee \neg q \vee r$$

$$p$$

$$\neg r$$

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The CNF transformation of

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$$

gives the set of four clauses:

$$\begin{aligned} &\neg p \vee q \\ &\neg p \vee \neg q \vee r \\ &p \\ &\neg r \end{aligned}$$

Every interpretation that satisfies this set of clauses **must** assign **1** to p and **0** to r , so **we do not have to guess values of these variables**.

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Every interpretation that satisfies this set of clauses **must** assign **1** to p and **0** to r , so **we do not have to guess values of these variables**.

In fact, we can do even better and establish unsatisfiability **without any guessing**.

Searching for satisfiability

{

$\neg p \vee q$
 $\neg p \vee \neg q \vee r$
 p
 $\neg r$

Searching for satisfiability

{

$\neg p \vee q$

$\neg p \vee \neg q \vee r$

p

$\neg r$

Searching for satisfiability

$$\{p \mapsto 1 \quad \}$$

$$\neg p \vee q$$

$$\neg p \vee \neg q \vee r$$

$$p$$

$$\neg r$$

Searching for satisfiability

$\{p \leftrightarrow 1 \quad \quad \quad \}$

$\neg p \vee q$

$\neg p \vee \neg q \vee r$

p

$\neg r$

Searching for satisfiability

$$\{p \mapsto 1 \quad \}$$

$$q$$
$$\neg q \vee r$$

$$\neg r$$

Searching for satisfiability

$$\{p \mapsto 1 \quad \}$$

$$q$$
$$\neg q \vee r$$

$$\neg r$$

Searching for satisfiability

$$\{p \mapsto 1, r \mapsto 0\}$$

$$q$$
$$\neg q \vee r$$

$$\neg r$$

Searching for satisfiability

$$\{p \mapsto 1, r \mapsto 0\}$$

$$q$$
$$\neg q \vee r$$

$$\neg r$$

Searching for satisfiability

$\{p \mapsto 1, r \mapsto 0\}$

q

$\neg q$

Searching for satisfiability

$\{p \mapsto 1, r \mapsto 0\}$

q

$\neg q$

Searching for satisfiability

$$\{p \mapsto 1, r \mapsto 0, q \mapsto 1\}$$

$$q$$

$$\neg q$$

Searching for satisfiability

$$\{p \mapsto 1, r \mapsto 0, q \mapsto 1\}$$

q

$\neg q$

Searching for satisfiability

$$\{p \mapsto 1, r \mapsto 0, q \mapsto 1\}$$



This set of clauses is unsatisfiable.

Unit propagation

Let S be a set of clauses. **Unit propagation**: repeatedly performing the following transformation: if S contains a unit clause, i.e. a clause consisting of one literal L , then

1. remove from S every clause of the form $L \vee C'$;
2. replace in S every clause of the form $\bar{L} \vee C'$ by the clause C' .

Unit Propagation, Example

 n_1 $\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$ $\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$ $\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$ $\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$ $\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$ $\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$ $\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Unit Propagation, Example

n_1

$\neg n_1 \vee \neg n_2$

$n_1 \vee n_2$

$\neg n_2 \vee \neg n_3 \vee n_7$

$n_3 \vee n_2$

$\neg n_7 \vee n_2$

$\neg n_3 \vee n_4$

$\neg n_3 \vee n_5$

$\neg n_4 \vee \neg n_5 \vee n_3$

$\neg n_4 \vee \neg p \vee q$

$p \vee n_4$

$\neg q \vee n_4$

$\neg n_5 \vee \neg n_6 \vee r$

$n_6 \vee n_5$

$\neg r \vee n_5$

$\neg n_6 \vee p$

$\neg n_6 \vee q$

$\neg p \vee \neg q \vee n_6$

$\neg n_7 \vee \neg p \vee r$

$p \vee n_7$

$\neg r \vee n_7$

Unit Propagation, Example

$$\begin{array}{l} \neg n_2 \\ \neg n_2 \vee \neg n_3 \vee n_7 \\ n_3 \vee n_2 \\ \neg n_7 \vee n_2 \\ \neg n_3 \vee n_4 \\ \neg n_3 \vee n_5 \\ \neg n_4 \vee \neg n_5 \vee n_3 \\ \neg n_4 \vee \neg p \vee q \\ p \vee n_4 \end{array} \qquad \begin{array}{l} \neg q \vee n_4 \\ \neg n_5 \vee \neg n_6 \vee r \\ n_6 \vee n_5 \\ \neg r \vee n_5 \\ \neg n_6 \vee p \\ \neg n_6 \vee q \\ \neg p \vee \neg q \vee n_6 \\ \neg n_7 \vee \neg p \vee r \\ p \vee n_7 \\ \neg r \vee n_7 \end{array}$$

Unit Propagation, Example

$$\neg n_2$$

$$\neg n_2 \vee \neg n_3 \vee n_7$$

$$n_3 \vee n_2$$

$$\neg n_7 \vee n_2$$

$$\neg n_3 \vee n_4$$

$$\neg n_3 \vee n_5$$

$$\neg n_4 \vee \neg n_5 \vee n_3$$

$$\neg n_4 \vee \neg p \vee q$$

$$p \vee n_4$$

$$\neg q \vee n_4$$

$$\neg n_5 \vee \neg n_6 \vee r$$

$$n_6 \vee n_5$$

$$\neg r \vee n_5$$

$$\neg n_6 \vee p$$

$$\neg n_6 \vee q$$

$$\neg p \vee \neg q \vee n_6$$

$$\neg n_7 \vee \neg p \vee r$$

$$p \vee n_7$$

$$\neg r \vee n_7$$

Unit Propagation, Example

n_3
 $\neg n_7$
 $\neg n_3 \vee n_4$
 $\neg n_3 \vee n_5$
 $\neg n_4 \vee \neg n_5 \vee n_3$
 $\neg n_4 \vee \neg p \vee q$
 $p \vee n_4$

$\neg q \vee n_4$
 $\neg n_5 \vee \neg n_6 \vee r$
 $n_6 \vee n_5$
 $\neg r \vee n_5$
 $\neg n_6 \vee p$
 $\neg n_6 \vee q$
 $\neg p \vee \neg q \vee n_6$
 $\neg n_7 \vee \neg p \vee r$
 $p \vee n_7$
 $\neg r \vee n_7$

Unit Propagation, Example

n_3
 $\neg n_7$
 $\neg n_3 \vee n_4$
 $\neg n_3 \vee n_5$
 $\neg n_4 \vee \neg n_5 \vee n_3$
 $\neg n_4 \vee \neg p \vee q$
 $p \vee n_4$

$\neg q \vee n_4$
 $\neg n_5 \vee \neg n_6 \vee r$
 $n_6 \vee n_5$
 $\neg r \vee n_5$
 $\neg n_6 \vee p$
 $\neg n_6 \vee q$
 $\neg p \vee \neg q \vee n_6$
 $\neg n_7 \vee \neg p \vee r$
 $p \vee n_7$
 $\neg r \vee n_7$

Unit Propagation, Example

$$\begin{array}{l} n_4 \\ n_5 \\ \neg n_4 \vee \neg p \vee q \\ p \vee n_4 \end{array} \qquad \begin{array}{l} \neg q \vee n_4 \\ \neg n_5 \vee \neg n_6 \vee r \\ n_6 \vee n_5 \\ \neg r \vee n_5 \\ \neg n_6 \vee p \\ \neg n_6 \vee q \\ \neg p \vee \neg q \vee n_6 \\ \\ p \\ \neg r \end{array}$$

Unit Propagation, Example

$$\begin{array}{l} n_4 \\ n_5 \\ \neg n_4 \vee \neg p \vee q \\ p \vee n_4 \end{array} \qquad \begin{array}{l} \neg q \vee n_4 \\ \neg n_5 \vee \neg n_6 \vee r \\ n_6 \vee n_5 \\ \neg r \vee n_5 \\ \neg n_6 \vee p \\ \neg n_6 \vee q \\ \neg p \vee \neg q \vee n_6 \\ \\ p \\ \neg r \end{array}$$

Unit Propagation, Example

$$\neg n_6$$

$$\neg n_6 \vee q$$

$$\neg q \vee n_6$$

$$q$$

Unit Propagation, Example

$$\neg n_6$$

$$\neg n_6 \vee q$$

$$\neg q \vee n_6$$

$$q$$

Unit Propagation, Example



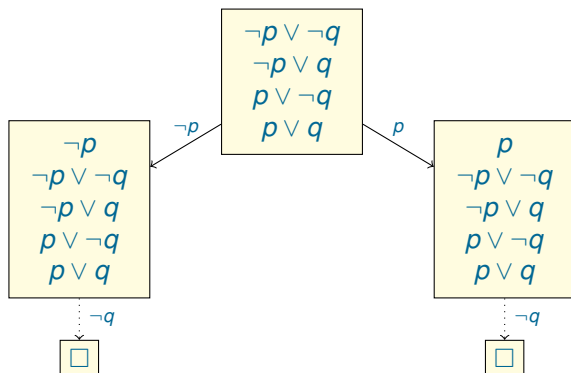
We established unsatisfiability of this set of clauses **in a completely deterministic way**, by unit propagation.

DPLL = splitting + unit propagation

```
procedure DPLL(S)  
input: set of clauses S  
output: satisfiable or unsatisfiable  
parameters: function select_literal  
begin  
  S := propagate(S)  
  if S is empty then return satisfiable  
  if S contains  $\square$  then return unsatisfiable  
  L := select_literal(S)  
  if DPLL( $S \cup \{L\}$ ) = satisfiable  
    then return satisfiable  
    else return DPLL( $S \cup \{\bar{L}\}$ )  
end
```

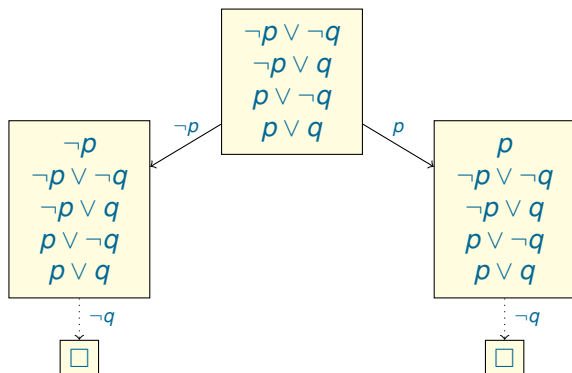
DPLL. Example 1

Can be illustrated using **DPLL trees** (similar to splitting trees).



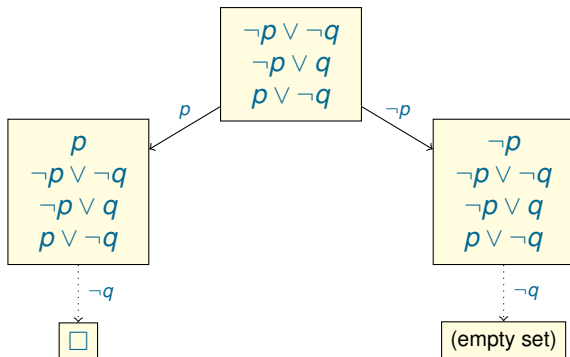
DPLL. Example 1

Can be illustrated using DPLL trees (similar to splitting trees).



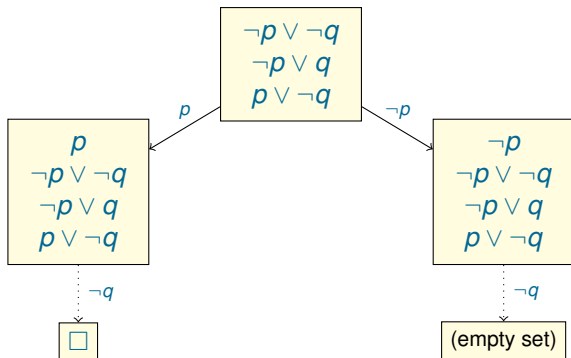
Since all branches end up in a set containing the **empty clause**, the initial set of clauses is **unsatisfiable**.

DPLL. Example 2



The set of clauses is **satisfiable**.

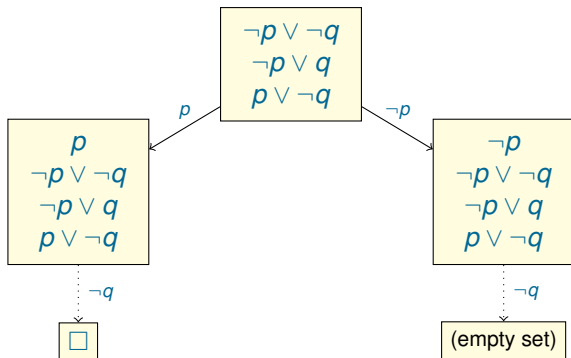
DPLL. Example 2



The set of clauses is **satisfiable**.

The model can be obtained by collecting all **selected literals** and **literals used in unit propagation** on the branch resulting in the empty set.

DPLL. Example 2



The set of clauses is **satisfiable**.

The model can be obtained by collecting all **selected literals** and **literals used in unit propagation** on the branch resulting in the empty set.

This DPLL tree gives us the model $\{p \mapsto 0, q \mapsto 0\}$.

Two optimisations

Any clause $p \vee \neg p \vee C$ is a **tautology**. Tautologies can be removed.

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Any clause $p \vee \neg p \vee C$ is a **tautology**. Tautologies can be removed.
A literal L in S is called **pure** if S contains no clauses of the form $\bar{L} \vee C$

All clauses containing a pure literal **can be satisfied** by assigning a suitable truth value to the variable of this literal.

Hence, clauses containing pure literals can be removed, too.

Pure literals: example

$$\neg p_2 \vee \neg p_3$$

$$p_1 \vee \neg p_2$$

$$\neg p_1 \vee p_2 \vee \neg p_3$$

$$\neg p_1 \vee \neg p_3$$

$$p_1 \vee p_2$$

$$\neg p_1 \vee \neg p_2 \vee \neg p_3$$

Pure literals: example

$$\begin{aligned} &\neg p_2 \vee \neg p_3 \\ &p_1 \vee \neg p_2 \\ &\neg p_1 \vee p_2 \vee \neg p_3 \\ &\neg p_1 \vee \neg p_3 \\ &p_1 \vee p_2 \\ &\neg p_1 \vee \neg p_2 \vee \neg p_3 \end{aligned}$$

The literal $\neg p_3$ is pure in this set. We can remove all clauses containing this literal.

Pure literals: example

$$p_1 \vee \neg p_2$$

$$p_1 \vee p_2$$

The literal $\neg p_3$ is pure in this set. We can remove all clauses containing this literal.

Pure literals: example

$$p_1 \vee \neg p_2$$

$$p_1 \vee p_2$$

The literal $\neg p_3$ is pure in this set. We can remove all clauses containing this literal.

Now the literal p_1 is pure in this set. We can remove all clauses containing this literal.

Pure literals: example

The literal $\neg p_3$ is pure in this set. We can remove all clauses containing this literal.

Now the literal p_1 is pure in this set. We can remove all clauses containing this literal.

We obtain the empty set of clauses.

Pure literals: example

$$\begin{aligned} & \neg p_2 \vee \neg p_3 \\ & p_1 \vee \neg p_2 \\ & \neg p_1 \vee p_2 \vee \neg p_3 \\ & \neg p_1 \vee \neg p_3 \\ & p_1 \vee p_2 \\ & \neg p_1 \vee \neg p_2 \vee \neg p_3 \end{aligned}$$

The literal $\neg p_3$ is pure in this set. We can remove all clauses containing this literal.

Now the literal p_1 is pure in this set. We can remove all clauses containing this literal.

We obtain the empty set of clauses.

This gives us two models:

$$\begin{aligned} & \{p_1 \mapsto 1, p_2 \mapsto 0, p_3 \mapsto 0\} \\ & \{p_1 \mapsto 1, p_2 \mapsto 1, p_3 \mapsto 0\} \end{aligned}$$

Running a SAT solver

Very simple but efficient SAT solver: MiniSat,
<http://minisat.se/>.

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$$\begin{aligned} & p_1 \\ & \neg p_1 \vee p_2 \\ \neg p_1 \vee & \neg p_2 \vee p_3 \\ & \neg p_2 \vee \neg p_3 \end{aligned}$$

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$$\begin{aligned} & p_1 \\ & \neg p_1 \vee p_2 \\ & \neg p_1 \vee \neg p_2 \vee p_3 \\ & \neg p_2 \vee \neg p_3 \end{aligned}$$

DIMACS input format:

```
p cnf 3 4
1 0
-1 2 0
-1 -2 3 0
-2 -3 0
```

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3 variables, **4** clauses.

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DIMACS input format:

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p cnf 3 4
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-1 -2 3 0
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```

3 variables, **4** clauses.

$\neg p_1 \vee \neg p_2 \vee p_3$