Automated Deduction

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Outline

Unification and Lifting

Substitution

- A substitution θ is a mapping from variables to terms such that the set $\{x \mid \theta(x) \neq x\}$ is finite.
- This set is called the domain of θ.
- Notation: $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, where $x_1, \ldots, x_n$ are pairwise different variables, denotes the substitution θ such that

$$\theta(x) = \begin{cases} t_i & \text{if } x = x_i; \\ x & \text{if } x \notin \{x_1, \ldots, x_n\}. \end{cases}$$

- Application of this substitution to an expression E: simultaneous replacement of x_i by t_i.
- Application of a substitution θ to E is denoted by $E\theta$.
- Since substitutions are functions, we can define their composition (written $\sigma\tau$ instead of $\tau \circ \sigma$). Note that we have $E(\sigma\tau) = (E\sigma)\tau$.

Example

Consider:

$$E = p(x, y, f(a))$$
$$\theta = \{x \mapsto b, y \mapsto x\}$$

What is $E\theta$?

Substitution composition

Suppose we have two substitutions

$$\theta_1 = \{x_1 \mapsto s_1, \ldots, x_m \mapsto s_m\} \text{ and}$$
$$\theta_2 = \{y_1 \mapsto t_1, \ldots, y_n \mapsto t_n\}.$$

How can we compute their composition $\theta_1\theta_2$?

The substitution $\theta_1\theta_2$ is obtained from the set:

$$\{x_1 \mapsto s_1\theta_2, \ldots, x_m \mapsto s_m\theta_2,$$
$$y_1 \mapsto t_1, \ldots, y_n \mapsto t_n\},$$

by deleting

- all $y_i \mapsto t_i$ with $y_i \in \{x_1, \ldots, x_m\}$,
- all $x_i \mapsto s_i\theta_2$ with $x_i = s_i\theta_2$.

Example

Consider:

$$\theta_1 = \{x \mapsto f(y), y \mapsto z\},$$
$$\theta_2 = \{x \mapsto a, y \mapsto b, z \mapsto y\}.$$

What is $\theta_1\theta_2$?

Instances, Ground

An instance of an expression (that is term, atom, literal, or clause) E is obtained by applying a substitution to E. Examples:

- some instances of the term $f(x, a, g(x))$ are:
  $f(x, a, g(x))$,
  $f(y, a, g(y))$,
  $f(a, a, g(a))$,
  $f(g(b), a, g(g(b)))$;
- but the term $f(b, a, g(c))$ is not an instance of this term.

Ground instance: instance with no variables.

Herbrand's Theorem

For a set of clauses S denote by S^* the set of ground instances of clauses in S.

Theorem Let Σ be a signature with at least one constant symbol and S be a set of (universal) clauses over Σ. The following conditions are equivalent.

1. S is unsatisfiable;
2. S^* is unsatisfiable;

By compactness of first-order logic the last condition is equivalent to

3. there exists a finite unsatisfiable set of ground instances of clauses in S.

The theorem reduces the problem of checking unsatisfiability of sets of arbitrary clauses to checking unsatisfiability of sets of ground clauses ...

The only problem is that S^* can be infinite even if S is finite.

Lifting

Lifting is a technique for proving completeness theorems in the following way:

1. Prove completeness of the system for a set of ground clauses;
2. Lift the proof to the non-ground case.

Lifting, Example

Consider two (non-ground) clauses $p(x, a) \vee q_1(x)$ and $\neg p(y, z) \vee q_2(y, z)$. If the signature contains function symbols, then both clauses have infinite sets of instances:

$$\begin{array}{rcl} \{p(r, a) \vee q_1(r) & \mid & r \text{ is ground}\} \\ \{\neg p(s, t) \vee q_2(s, t) & \mid & s, t \text{ are ground}\} \end{array}$$

We can resolve such instances if and only if $r = s$ and $t = a$. Then we can apply the following inference

$$\frac{p(s, a) \vee q_1(s) \quad \neg p(s, a) \vee q_2(s, a)}{q_1(s) \vee q_2(s, a)} \text{ (BR)}$$

But there is an infinite number of such inferences.

Lifting, Idea

The idea is to represent an infinite number of ground inferences of the form

$$\frac{p(s,a) \lor q_1(s) \quad \neg p(s,a) \lor q_2(s,a)}{q_1(s) \lor q_2(s,a)} \text{ (BR)}$$

by a single non-ground inference

$$\frac{p(x,a) \lor q_1(x) \quad \neg p(y,z) \lor q_2(y,z)}{q_1(y) \lor q_2(y,a)} \text{ (BR)}$$

Is this always possible?

Yes!

$$\frac{p(x,a) \lor q_1(x) \quad \neg p(y,z) \lor q_2(y,z)}{q_1(y) \lor q_2(y,a)} \text{ (BR)}$$

Note that the substitution $\{x \mapsto y, z \mapsto a\}$ is a solution of the "equation" $p(x,a) = p(y,z)$.

What should we lift?

- Ordering $\succ$;
- Selection function σ;
- Calculus $\mathbb{S}\mathrm{up}_{\succ,\sigma}$.

Most importantly, for the lifting to work we should be able to solve equations $s = t$ between terms and between atoms. This can be done using most general unifiers.