Automated Deduction

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Outline

Substitution

- A substitution θ is a mapping from variables to terms such that the set $\{x \mid \theta(x) \neq x\}$ is finite.
- This set is called the domain of θ.
- Notation: $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, where $x_1, \ldots, x_n$ are pairwise different variables, denotes the substitution θ such that

$$\theta(x) = \begin{cases} t_i & \text{if } x = x_i; \\ x & \text{if } x \notin \{x_1, \ldots, x_n\}. \end{cases}$$

- Application of this substitution to an expression E: simultaneous replacement of x_i by t_i.
- Application of a substitution θ to E is denoted by $E\theta$.
- Since substitutions are functions, we can define their composition (written $\sigma\tau$ instead of $\tau \circ \sigma$). Note that we have $E(\sigma\tau) = (E\sigma)\tau$.

Example

Consider:

$$E = p(x, y, f(a))$$
$$\theta = \{x \mapsto b, y \mapsto x\}$$

What is $E\theta$?

Substitution composition

Suppose we have two substitutions

$$\theta_1 = \{x_1 \mapsto s_1, \ldots, x_m \mapsto s_m\} \text{ and}$$
$$\theta_2 = \{y_1 \mapsto t_1, \ldots, y_n \mapsto t_n\}.$$

How can we compute their composition $\theta_1\theta_2$?

Substitution composition

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How can we compute their composition $\theta_1\theta_2$?

The substitution $\theta_1\theta_2$ is obtained from the set:

$$\{x_1 \mapsto s_1\theta_2, \ldots, x_m \mapsto s_m\theta_2,$$
$$y_1 \mapsto t_1, \ldots, y_n \mapsto t_n\},$$

by deleting

- all $y_i \mapsto t_i$ with $y_i \in \{x_1, \ldots, x_m\}$,
- all $x_i \mapsto s_i\theta_2$ with $x_i = s_i\theta_2$.

Example

Consider:

$$\theta_1 = \{x \mapsto f(y), y \mapsto z\},$$
$$\theta_2 = \{x \mapsto a, y \mapsto b, z \mapsto y\}.$$

What is $\theta_1\theta_2$?

Instances, Ground

An instance of an expression (that is term, atom, literal, or clause) E is obtained by applying a substitution to E. Examples:

- some instances of the term $f(x, a, g(x))$ are:
  $f(x, a, g(x))$,
  $f(y, a, g(y))$,
  $f(a, a, g(a))$,
  $f(g(b), a, g(g(b)))$;
- but the term $f(b, a, g(c))$ is not an instance of this term.

Ground instance: instance with no variables.

Herbrand's Theorem

For a set of clauses S denote by S^* the set of ground instances of clauses in S.

Theorem Let Σ be a signature with at least one constant symbol and S be a set of (universal) clauses over Σ. The following conditions are equivalent.

1. S is unsatisfiable;
2. S^* is unsatisfiable;

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The only problem is that S^* can be infinite even if S is finite.

Lifting

Lifting is a technique for proving completeness theorems in the following way:

1. Prove completeness of the system for a set of ground clauses;
2. Lift the proof to the non-ground case.

Lifting, Example

Consider two (non-ground) clauses $p(x, a) \vee q_1(x)$ and $\neg p(y, z) \vee q_2(y, z)$. If the signature contains function symbols, then both clauses have infinite sets of instances:

$$\begin{array}{rcl} \{p(r, a) \vee q_1(r) & \mid & r \text{ is ground}\} \\ \{\neg p(s, t) \vee q_2(s, t) & \mid & s, t \text{ are ground}\} \end{array}$$

We can resolve such instances if and only if $r = s$ and $t = a$. Then we can apply the following inference

$$\frac{p(s, a) \vee q_1(s) \quad \neg p(s, a) \vee q_2(s, a)}{q_1(s) \vee q_2(s, a)} \text{ (BR)}$$

But there is an infinite number of such inferences.

Lifting, Idea

The idea is to represent an infinite number of ground inferences of the form

$$\frac{p(s,a) \vee q_1(s) \quad \neg p(s,a) \vee q_2(s,a)}{q_1(s) \vee q_2(s,a)} \text{ (BR)}$$

by a single non-ground inference

$$\frac{p(x,a) \vee q_1(x) \quad \neg p(y,z) \vee q_2(y,z)}{q_1(y) \vee q_2(y,a)} \text{ (BR)}$$

Is this always possible?

Yes!

$$\frac{p(x,a) \vee q_1(x) \quad \neg p(y,z) \vee q_2(y,z)}{q_1(y) \vee q_2(y,a)} \text{ (BR)}$$

Note that the substitution $\{x \mapsto y, z \mapsto a\}$ is a solution of the "equation" $p(x,a) = p(y,z)$.

What should we lift?

- Ordering $\succ$;
- Selection function σ;
- Calculus $\mathbb{S}\mathrm{up}_{\succ,\sigma}$.

Most importantly, for the lifting to work we should be able to solve equations $s = t$ between terms and between atoms. This can be done using most general unifiers.