# Completeness of Superposition

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# Bag Extension of an Ordering

Bag = finite multiset.

Let $>$ be any ordering on a set $X$. The bag extension of $>$ is a binary relation $>^{bag}$, on bags over $X$, defined as the smallest transitive relation on bags such that

$$\{x, y_1, \ldots, y_n\} >^{bag} \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$$
$$\text{if } x > x_i \text{ for all } i \in \{1 \ldots m\},$$

where $m \geq 0$.

Idea: a bag becomes smaller if we replace an element by any finite number of smaller elements.

The following results are known about the bag extensions of orderings:

1. $>^{bag}$ is an ordering;
2. If $>$ is total, then so is $>^{bag}$;
3. If $>$ is well-founded, then so is $>^{bag}$.

# Clause Orderings

From now on consider clauses also as bags of literals. Note:

- we have an ordering $\succ$ for comparing literals;
- a clause is a bag of literals.

Hence

- we can compare clauses using the bag extension $\succ^{bag}$ of $\succ$.

For simpicity we denote the multiset ordering also by $\succ$.

# Redundancy

A clause $C \in S$ is called redundant in $S$ if it is a logical consequence of clauses in $S$ strictly smaller than $C$.

# Inference Process with Redundancy

Let $\mathbb{I}$ be an inference system. Consider an inference process with two kinds of step $S_i \Rightarrow S_{i+1}$:

1. Adding the conclusion of an $\mathbb{I}$-inference with premises in $S_i$.
2. Deletion of a clause redundant in $S_i$, that is

$$S_{i+1} = S_i - \{C\},$$

   where $C$ is redundant in $S_i$.

# Fairness: Persistent Clauses and Limit

Consider an inference process

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

A clause $C$ is called persistent if

$$\exists i \forall j \geq i (C \in S_j).$$

The limit $S_\omega$ of the inference process is the set of all persistent clauses:

$$S_\omega = \bigcup_{i=0,1,\ldots} \bigcap_{j \geq i} S_j.$$

# Fairness

The process is called $\mathbb{I}$-fair if every inference with persistent premises in $S_\omega$ has been applied, that is, if

$$\frac{C_1 \quad \ldots \quad C_n}{C}$$

is an inference in $\mathbb{I}$ and $\{C_1, \ldots, C_n\} \subseteq S_\omega$, then $C \in S_i$ for some $i$.

# Ground Superposition Inference System $\mathbb{S}\mathrm{up}_{\succ,\sigma}$

Superposition: (right and left)

$$\frac{\underline{l \simeq r} \vee C \quad \underline{s[l] \simeq t} \vee D}{s[r] \simeq t \vee C \vee D}\ \text{(Sup)}, \quad \frac{\underline{l \simeq r} \vee C \quad \underline{s[l] \not\simeq t} \vee D}{s[r] \not\simeq t \vee C \vee D}\ \text{(Sup)},$$

where (i) $l \succ r$;
(ii) $l \simeq r$ is the greatest literal in $l \simeq r \vee C$
(iii) $s[l] \succ t$;
(iv) $s[l] \simeq t$ is the greatest literal in $s[l] \simeq t \vee D$ (only for superposition-right)

Equality Resolution:

$$\frac{\underline{s \not\simeq s} \vee C}{C}\ \text{(ER)},$$

Equality Factoring:

$$\frac{\underline{s \simeq t} \vee s \simeq t' \vee C}{s \simeq t \vee t \not\simeq t' \vee C}\ \text{(EF)},$$

where (i) $s \succ t$;
(ii) $s \simeq t$ is maximal in $s \simeq t \vee s \simeq t' \vee C$

# Completeness of $\mathbb{S}\mathrm{up}_{\succ,\sigma}$

**Completeness Theorem.** Let $\succ$ be a simplification ordering and $\sigma$ a well-behaved selection function. Let also

1. $S_0$ be a set of clauses;
2. $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be a fair $\mathbb{S}\mathrm{up}_{\succ,\sigma}$-inference process.

Then $S_0$ is unsatisfiable if and only if $\square \in S_i$ for some $i$.

# Saturation up to Redundancy

A set $S$ of clauses is called saturated up to redundancy if for every $\mathbb{I}$-inference

$$\frac{C_1 \quad \ldots \quad C_n}{C}$$

with premises in $S$, either

1. $C \in S$; or
2. $C$ is redundant w.r.t. $S$, that is, $S_{\prec C} \models C$.

# Proof of Completeness

A trace of a clause $C$: a set of clauses $\{C_1, \ldots, C_n\} \subseteq S_\omega$ such that

1. $C \succ C_i$ for all $i = 1, \ldots, n$;
2. $C_1, \ldots, C_n \models C$.

**Lemma.** Every removed clause has a trace.
**Lemma.** The limit $S_\omega$ is saturated up to redundancy.
**Lemma.** The limit $S_\omega$ is logically equivalent to the initial set $S_0$.
**Lemma.** A set $S$ of clauses saturated up to redundancy is unsatisfiable if and only if $\square \in S$.

(Proof on the board)

Interestingly, only the last lemma uses rules of $\mathbb{S}\mathrm{up}_{\succ,\sigma}$.

# Rewrite Rule Systems

- A rewrite rule is an expression $l \to r$.
- A rewrite rule system $R$ is a set of rewrite rules.
- We say that a rewrite rule $l \to r$ rewrites $s[l]$ into $s[r]$.
- We write $s \to_R t$ if some rewrite rule in $R$ rewrites $s$ into $t$. The relation $\to_R^*$ is the reflexive and transitive closure of $\to_R$. In other words, we have $s \to_R^* t$, if there exists a sequence of terms $t_0, \ldots, t_n$ such that, $n \geq 0$, $s = t_0$, $t_n = t$ and we have

$$t_0 \to_R t_1 \to_R \cdots \to_R t_n.$$

In this case we also say that $s$ rewrites into $t$ in $n$ steps using $R$.

# Rewrite Rule Systems

- $t$ is irreducible w.r.t. $R$ is no rule in $R$ rewrites $t$.
- $R$ is called terminating if there is no infinite sequence

$$t_0 \to_R t_1 \to_R t_2 \to_R \cdots$$

- $t$ is called a normal form of $s$ w.r.t. $R$ if $s \to_R^* t$ and $t$ is irredicible w.r.t. $R$.
- If $R$ is terminating, then every term has a normal form
- $R$ is called canonical if it is terminating and every term has a unique normal form.
- $R$ is called non-overlapping if for every two different rules $l_1 \to r_1$ and $l_2 \to r_2$ in $R$, $l_1$ is not a subterm of $l_2$.
- If $R$ is terminating and non-overlapping, then it is canonical

# Congruence

- A congruence relation is a relation satisfying equality axioms (reflexivity, symmetry, transitivity and congruence);
- Any canonical system $R$ defines a congruence relation, denoted by $=_R$ as follows: $s =_R t$ iff $s$ and $t$ have the same normal form.

# Model Construction

**Lemma.** A set $S$ of clauses saturated up to redundancy is unsatisfiable if and only if $\square \in S$.

Let set $S$ be saturated up to redundancy and $\square \notin S$. We will build a congruence relation satisfying all clauses in $S$.

We will build this relation as $=_R$ for some canonical rewrite system $R$, which will be built step by step by induction on $\succ$. $R$ can be growing during the model construction. At each step, we denote by $I_R$ the interpretation in which the equality is defined as $=_R$.

[Step $l$], where $l$ is a ground term. We assume that $R$ was built for all terms $t \prec l$. Then, if

1. $l$ is irreducible w.r.t. $R$;
2. $S$ contains a clause $\underline{l \simeq r} \vee C$ such that (i) $I_R \not\models C$ (ii) $l \succ r$; (iii) $l \simeq r$ is the greatest literal in $l \simeq r \vee C$

then take any such clause and add $l \rightarrow r$ to $R$

# Model Construction

We claim:

1. $R$ is canonical;
2. $I_R$ satisfies all clauses in $S$.

(Proof on the board)

# Proof

Note that:

- $R$ is non-overlapping by construction;
- $R$ is terminating (because $\succ$ is monotonic and well-founded)

These two properties imply that $R$ is canonical.

# Proof

Some general properties of the model construction:

- If $l$ is irreducible after step $l$, then it will be irreducible after
- The normal form of a term $l$ does not change after step $l$
- If $C$ is a clause in which $l$ is the greatest term, then the truth value of $C$ in $I_R$ does not change after step $l$ (this follows from the previous property)

# Proof

Now we prove that $I_R$ satisfies all clauses in $S$. Suppose it does not. Then there is a clause $F \in S$ such that $I_R \not\models F$. Note that $F$ is non-empty, since $S$ does not contain the empty clause.

Since $\succ$ is well-founded on clauses, the set of all clauses in $S$, which are false in $I_R$ contains the least element. Denote this clause by $F$.

We will now show, by contradiction, that $S$ contains a clause smaller than $F$ and false in $I_R$. To prove this, we consider several cases, depending on which literal(s) are selected in $F$.

# Proof Case

Suppose that $F$ has a negative selected literal. Then $F$ has the form $\underline{s \not\simeq t} \vee D$.
Consider the case when $s$ coincides with $t$, then $F$ has the form $\underline{s \not\simeq s} \vee D$. Consider the equality resolution inference

$$\frac{\underline{s \not\simeq s} \vee D}{D}$$

Since $F$ is persistent and the process is fair, this inference was applied at some step, so $D$ belongs to some search space $S_i$. Note that $F \succ D$ and $D \vdash F$. The last property implies that $I_R \not\models F$.

If $D \in S$, that is, $D$ is persistent, than we are done, since we have found a clause in $S$ that is false in $I_R$ and smaller than $D$.

If $D \notin S$, then it was removed and so has a trace $D_1, \ldots, D_n$. Since $D_1, \ldots, D_n \vdash D$ and $D$ is false, then there exists some $D_i$ such that $D_i$ is false too. Note that we have $D_i \prec D \prec F$, so again we have found a clause smaller than $F$ and false in $I_R$.

# Proof Case

Consider the case when $F$ has the form $\underline{s \not\simeq t} \vee D$ and $s$ does not coincide with $t$. W.l.o.g. assume $s \succ t$.

Consider the case when $s$ is reducible, then $R$ contains some rule $l \to r$ and $F$ has the form $\underline{s[l] \not\simeq t} \vee D$. By construction, $S$ contains a clause $\underline{l \simeq r} \vee C$, such that $l$ is greater than any term $r, C$ and $I_R \not\models C$.

Consider the superposition inference

$$\frac{\underline{l \simeq r} \vee C \quad \underline{s[l] \not\simeq t} \vee D}{s[r] \not\simeq t \vee C \vee D}$$

Similar to the previous case we can prove that the conclusion $s[r] \not\simeq t \vee C \vee D$ is smaller than $F$ and false.

Again, to conclude we consider two cases: when $s[r] \not\simeq t \vee C \vee D$ is persistent and when it was removed. The proof is exactly as in the previous case.

# Proof Case

The next case is when $F$ has the form $\underline{s \not\simeq t} \vee D$, $s \succ t$ and $s$ is irreducible.
Since $F$ is false in $I_R$, we have $I_R \models s = t$. But then by our construction $s$ must be reducible because we would add some rule $s \to l$ at the step $s$.

# Other Cases

Try yourself . . .

# $\mathbb{S}\mathrm{up}_{\succ,\sigma}$ with Predicates

Superposition: (right and left)

$$\frac{\underline{l \simeq r} \vee C \quad \underline{A[l]} \vee D}{A[r] \vee C \vee D} \text{ (Sup)}, \quad \frac{\underline{l \simeq r} \vee C \quad \underline{\neg A[l]} \vee D}{\neg A[r] \vee C \vee D} \text{ (Sup)},$$

where (i) $l \succ r$;
(ii) $l \simeq r$ is the greatest literal in $l \simeq r \vee C$
(iii) $A[l]$ is the greatest literal in $A[l] \vee D$ (only for superposition-right)

Binary Resolution:

$$\frac{\underline{A} \vee C \quad \underline{\neg A} \vee D}{C \vee D} \text{ (ER)},$$

where (i) $A$ is maximal in $A \vee C$

Equality Factoring:

$$\frac{\underline{A} \vee \underline{A} \vee C}{A \vee C} \text{ (Fact)},$$

where (i) $A$ is maximal in $A \vee A \vee C$

# Arbitrary Predicates

First, build the congruence as before. We use induction on ground atoms instead of ground terms. We will again build an interpretation $I_R$ step by step. Initially, all ground non-equality atoms are false in $I_R$. Then we will make some of them true.

Take a ground atom $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are irreducible. We assume that for all atoms smaller than $P(t_1, \ldots, t_n)$, we have already defined whether they are true. We make $P(t_1, \ldots, t_n)$ true if

- $S$ contains a clause $\underline{P(t_1, \ldots, t_n)} \vee C$ such that (i) $I_R \not\models C$ and (ii) $P(t_1, \ldots, t_n)$ is the greatest literal in $l \simeq r \vee C$,

# Saturation up to Redundancy and Satisfiability Checking

**Lemma.** A set $S$ of clauses saturated up to redundancy is unsatisfiable if and only if $\square \in S$.

Therefore, if we built a set saturated up to redundancy, then the initial set $S_0$ is satisfiable. This is a powerful way of checking redundancy: one can even check satisfiability of formulas having only infinite models.

The only problem with this characterisation is that there is no obvious way to build a model of $S_0$ out of a saturated set.