Automated Deduction

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Outline

Inference Process

Inference process: sequence of sets of formulas $S_0, S_1, \ldots$, denoted by

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

$(S_i \Rightarrow S_{i+1})$ is a step of this process.

We say that this step is an $\mathbb{I}$-step if

1. there exists an inference

$$\frac{F_1 \quad \ldots \quad F_n}{F}$$

   in $\mathbb{I}$ such that $\{F_1, \ldots, F_n\} \subseteq S_i$;
2. $S_{i+1} = S_i \cup \{F\}$.

An $\mathbb{I}$-inference process is an inference process whose every step is an $\mathbb{I}$-step.

Property

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an $\mathbb{I}$-inference process and a formula F belongs to some S_i. Then S_i is derivable in $\mathbb{I}$ from S_0. In particular, every S_i is a subset of the $\mathbb{I}$-closure of S_0.

Limit of a Process

The limit of an inference process $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ is the set of formulas $\bigcup_i S_i$.

In other words, the limit is the set of all derived formulas.

Suppose that we have an infinite inference process such that S_0 is unsatisfiable and we use the binary resolution inference system.

Question: does completeness imply that the limit of the process contains the empty clause?

Fairness

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an inference process with the limit S_ω. The process is called fair if for every $\mathbb{I}$-inference

$$\frac{F_1 \quad \ldots \quad F_n}{F} \ ,$$

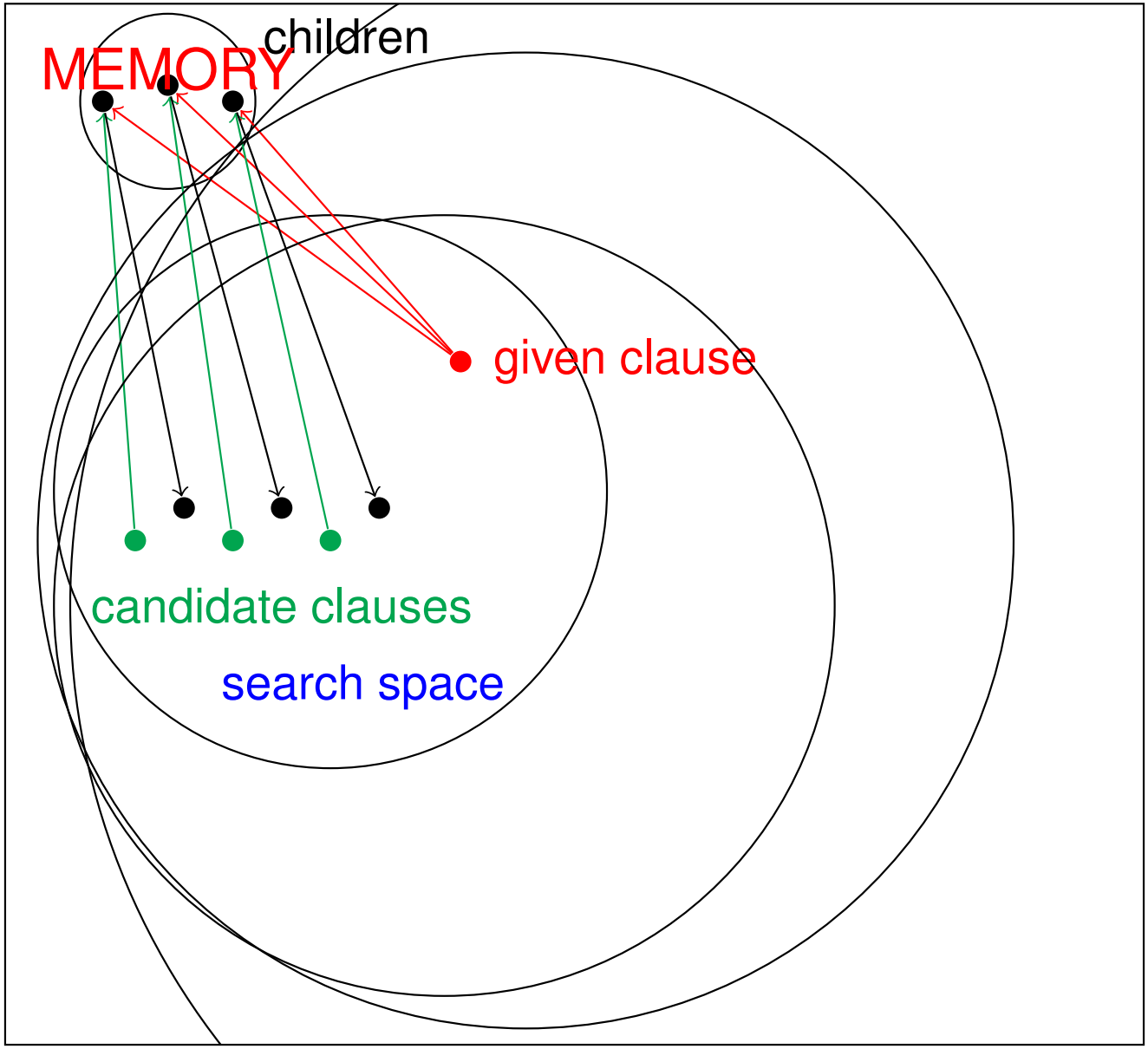
if $\{F_1, \ldots, F_n\} \subseteq S_\omega$, then there exists i such that $F \in S_i$.

Completeness, reformulated

Theorem Let $\mathbb{I}$ be an inference system. The following conditions are equivalent.

1. $\mathbb{I}$ is complete.
2. For every unsatisfiable set of formulas S_0 and any fair $\mathbb{I}$-inference process with the initial set S_0, the limit of this inference process contains $\square$.

Fair Saturation Algorithms: Inference Selection by Clause Selection



Saturation Algorithm

A saturation algorithm tries to saturate a set of clauses with respect to a given inference system.
In theory there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run forever, but without generating $\square$. In this case the input set of clauses is satisfiable.

Saturation Algorithm in Practice

In practice there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run until we run out of resources, but without generating $\square$. In this case it is unknown whether the input set is unsatisfiable.

Outline

Subsumption and Tautology Deletion

A clause is a propositional tautology if it is of the form $p \vee \neg p \vee C$, that is, it contains a pair of complementary literals.
There are also equational tautologies, for example $a \neq b \vee b \neq c \vee f(c, c) = f(a, a)$.

A clause C subsumes any clause $C \vee D$, where D is non-empty.

It was known since 1965 that subsumed clauses and propositional tautologies can be removed from the search space.

Problem

How can we prove that completeness is preserved if we remove subsumed clauses and tautologies from the search space?

Solution: general theory of redundancy.

Bag Extension of an Ordering

Bag = finite multiset.
Let $>$ be any (strict) ordering on a set X. The bag extension of $>$ is a binary relation $>^{bag}$, on bags over X, defined as the smallest transitive relation on bags such that

$$\{x, y_1, \ldots, y_n\} >^{bag} \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$$
$$\text{if } x > x_i \text{ for all } i \in \{1 \ldots m\},$$

where $m \geq 0$.
Idea: a bag becomes smaller if we replace an element by any finite number of smaller elements.
The following results are known about the bag extensions of orderings:

1. $>^{bag}$ is an ordering;
2. If $>$ is total, then so is $>^{bag}$;
3. If $>$ is well-founded, then so is $>^{bag}$.

Clause Orderings

From now on consider clauses also as bags of literals. Note:

- we have an ordering $\succ$ for comparing literals;
- a clause is a bag of literals.

Hence

- we can compare clauses using the bag extension $\succ^{bag}$ of $\succ$.

For simpicity we denote the multiset ordering also by $\succ$.

Redundancy

A clause $C \in S$ is called redundant in S if it is a logical consequence of clauses in S strictly smaller than C.

Examples

A tautology $p \vee \neg p \vee C$ is a logical consequence of the empty set of formulas:

$$\models p \vee \neg p \vee C,$$

therefore it is redundant.
We know that C subsumes $C \vee D$. Note

$$C \vee D \succ C$$
$$C \models C \vee D$$

therefore subsumed clauses are redundant.

If $\square \in S$, then all non-empty other clauses in S are redundant.

Redundant Clauses Can be Removed

In $\mathbb{BR}_\sigma$ (and in all calculi we will consider later) redundant clauses can be removed from the search space.

Inference Process with Redundancy

Let $\mathbb{I}$ be an inference system. Consider an inference process with two kinds of step $S_i \Rightarrow S_{i+1}$:

1. Adding the conclusion of an $\mathbb{I}$-inference with premises in S_i.
2. Deletion of a clause redundant in S_i, that is

$$S_{i+1} = S_i - \{C\},$$

where C is redundant in S_i.

Fairness: Persistent Clauses and Limit

Consider an inference process

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

A clause C is called persistent if

$$\exists i \forall j \geq i (C \in S_j).$$

The limit S_ω of the inference process is the set of all persistent clauses:

$$S_\omega = \bigcup_{i=0,1,\ldots} \bigcap_{j \geq i} S_j.$$

Fairness

The process is called $\mathbb{I}$-fair if every inference with persistent premises in S_ω has been applied, that is, if

$$\frac{C_1 \quad \ldots \quad C_n}{C}$$

is an inference in $\mathbb{I}$ and $\{C_1, \ldots, C_n\} \subseteq S_\omega$, then $C \in S_i$ for some i.

Completeness of $\mathbb{BR}_\sigma$

Completeness Theorem. Let $\succ$ be a well-founded ordering and σ a well-behaved selection function. Let also

1. S_0 be a set of clauses;
2. $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be a fair $\mathbb{BR}_\sigma$-inference process.

Then S_0 is unsatisfiable if and only if $\square \in S_i$ for some i.

Saturation up to Redundancy

A set S of clauses is called saturated up to redundancy if for every $\mathbb{I}$-inference

$$\frac{C_1 \quad \ldots \quad C_n}{C}$$

with premises in S, either

1. $C \in S$; or
2. C is redundant w.r.t. S, that is, $S_{\prec C} \models C$.

Saturation up to Redundancy and Satisfiability Checking

Lemma. A set S of clauses saturated up to redundancy is unsatisfiable if and only if $\square \in S$.

Therefore, if we built a set saturated up to redundancy, then the initial set S_0 is satisfiable. This is a powerful way of checking redundancy: one can even check satisfiability of formulas having only infinite models.

The only problem with this characterisation is that there is no obvious way to build a model of S_0 out of a saturated set.

Binary Resolution with Selection

One of the key properties to satisfy this lemma is the following: the conclusion of every rule is strictly smaller that the rightmost premise of this rule.

- Binary resolution,

$$\frac{\underline{p} \vee C_1 \quad \underline{\neg p} \vee C_2}{C_1 \vee C_2} \text{ (BR).}$$

- Positive factoring,

$$\frac{\underline{p} \vee \underline{p} \vee C}{p \vee C} \text{ (Fact).}$$