Automated Deduction - SS 2019

Problem Set 3
Due May 23, 2019, 10am

Problem 3.1. (10 points) Apply the unification algorithm and show the most general unifier of the following formulas:

(a) $p(f(x), f(y), y)$ and $p(y, z, f(x))$;

Solution:

$$\begin{array}{lll}
E = \{p\big(f(x), f(y), y\big) = p\big(y, z, f(x)\big)\} & \Longrightarrow & \\
E = \{f(x) = y, \quad f(y) = z, \quad y = f(x)\} & \Longrightarrow & \\
E = \{y = f(x), \quad f(y) = z, \quad y = f(x)\} & \Longrightarrow & {\scriptstyle y \to f(x)} \\
E = \{y = f(x), \quad f(f(x)) = z, \quad f(x) = f(x)\} & \Longrightarrow & \\
E = \{y = f(x), \quad z = f(f(x)), \quad f(x) = f(x)\} & \Longrightarrow & {\scriptstyle z \to f(f(x))} \\
E = \{y = f(x), \quad z = f(f(x)), \quad f(x) = f(x)\} & \Longrightarrow & \\
E = \{y = f(x), \quad z = f(f(x))\} & \Longrightarrow & {\scriptstyle Success}
\end{array}$$

The mgu in this case is $\{y \to f(x), z \to f(f(x))\}$.

(b) $p(g(x, y), g(y, g(z, b)))$ and $p(z, g(a, x))$.

Solution:

$$\begin{array}{lll}
E = \{p\big(g(x, y), g(y, g(z, b))\big) = p\big(z, g(a, x)\big)\} & \Longrightarrow & \\
E = \{g(x, y) = z, \quad g(y, g(z, b)) = g(a, x)\} & \Longrightarrow & \\
E = \{z = g(x, y), \quad g(y, g(z, b)) = g(a, x)\} & \Longrightarrow & {\scriptstyle z \to g(x,y)} \\
E = \{z = g(x, y), \quad g(y, g(g(x, y), b)) = g(a, x)\} & \Longrightarrow & \\
E = \{z = g(x, y), \quad y = a, \quad g(g(x, y), b) = x\} & \Longrightarrow & {\scriptstyle y \to a} \\
E = \{z = g(x, a), \quad y = a, \quad g(g(x, a), b) = x\} & \Longrightarrow & \\
E = \{z = g(x, a), \quad y = a, \quad x = g(g(x, a), b)\} & \Longrightarrow & {\scriptstyle Failure}
\end{array}$$

Thus, $p(g(x, y), g(y, g(z, b)))$ and $p(z, g(a, x))$ are not unifiable.

Note: x, y, z denote variables, f, g are function symbols, p is a predicate symbol and a, b are constants.

Problem 3.2. (10 points) Consider an ordering $\succ$ on ground non-equality atoms that is total and well-founded. We denote the literal ordering induced by $\succ$ also by $\succ$. Let C and D be ground clauses without equality literals. Let A and B respectively denote the maximal atoms of C and D wrt $\succ$.
Assume that A and B are syntactically the same atoms. Assume also that A occurs negatively in C but only positively in D. Show that $C \succ_{bag} D$.

Solution:
Since B and A are syntactically the same atoms, we have that A is the maximal atom of D wrt $\succ$. We know that A occurs only positively in D. By using properties of the induced literal ordering $\succ$, we conclude that $\neg A \succ A \succ \neg D_j \succ D_j$ for every atom D_j of D different than A. Hence, by properties of the bag extension ordering of $\succ$, we have $\neg A \succ_{bag} D$.

By assumption, A is the maximal atom of C wrt $\succ$ and A occurs only negatively in C. As $\neg A \succ A$, we thus conclude that $\neg A \succ \neg C_i \succ C_i$, where C_i is an atom of C different than A. That is $\neg A$ is the maximal literal of C.

As $\neg A \succ_{bag} D$ and $\neg A$ is the maximal literal of C, by properties of the bag extension ordering of $\succ$, we finally conclude that $C \succ_{bag} D$.

Problem 3.3. (10 points) Let Σ be a signature containing only function symbols such that Σ contains at least one constant. Let $\gg$ be a precedence relation on Σ and $w : \Sigma \to \mathbb{N}$ be a weight function compatible with $\gg$. Consider the (ground) Knuth-Bendix order $\succ_{KB}$ induced by $\gg$ and w on the set of ground terms of Σ. Describe the set of ground terms that have the minimal weight wrt $\succ_{KB}$.

Solution:
Every ground term of Σ that has the minimal weight wrt $\succ_{KB}$ is either:
– a constant $c \in \Sigma$ such that c has the minimal weight among the constants of Σ,
– or a term $f^n(c)$, with $n \neq 0$, where $c \in \Sigma$ is a constant of the minimal weight among the constants of Σ and $w(f) = 0$.

Problem 3.3. (20 points) Consider the following set S of clauses:

$$\begin{array}{l} \neg p(z,a) \vee \neg p(z,x) \vee \neg p(x,z) \\ p(y,a) \vee p(y,f(y)) \\ p(w,a) \vee p(f(w),w) \end{array}$$

where p is a predicate symbol, f is a function symbol, x, y, z, w are variables and a is a constant.
Give a refutation proof of S by using the non-ground binary resolution inference system $\mathbb{BR}$. For each newly derived clause, label the clauses from which it was derived by which inference rule and indicate most general unifiers.

Solution:
For simplicity, we name the given clauses by numbers:

$$\begin{array}{ll} (1) & \neg p(z,a) \vee \neg p(z,x) \vee \neg p(x,z) \\ (2) & p(y,a) \vee p(y,f(y)) \\ (3) & p(w,a) \vee p(f(w),w) \end{array}$$

By negative factoring on (1), with the mgu $\{x \to a\}$, we get:

$$(4) \quad \neg p(z,a) \vee \neg p(a,z)$$

By negative factoring on (4), with the mgu $\{z \to a\}$, we get:

$$(5) \quad \neg p(a,a)$$

By resolution on (5) and (2), with the mgu $\{y \to a\}$, we get:

$$(6) \quad p(a,f(a))$$

By resolution on (4) and (6), with the mgu $\{z \to f(a)\}$, we get:

$$(7) \quad \neg p(f(a),a)$$

By resolution on (3) and (7), with the mgu $\{w \to a\}$, we get:

$$(8) \quad p(a,a)$$

By resolution on (5) and (8), we finally obtain the empty clause:

$$(9) \quad \square$$

Hence, our input set S of clauses (1), (2) and (3) is unsatisfiable.

Problem 3.4. (20 points) Consider the KBO ordering $\succ$ generated by the precedence $f \gg a \gg b \gg c$ and the weight function that assigns weight 1 to each symbol from $\{f, a, b, c\}$. Let σ be a well-behaved selection function w.r.t. $\succ$. Consider the set S of ground formulas:

$$\begin{array}{l} a = b \lor a = c \\ f(a) \neq f(b) \\ b = c \end{array}$$

Apply saturation on S using an inference process based on the ground superposition calculus $\text{Sup}_{\succ,\sigma}$ (including the inference rules of ground binary resolution with selection).
Show that S is unsatisfiable by finding a refutation of S such that during saturation only 4 new clauses are generated. Give details on what literals are selected and which terms are maximal.

Solution:
For simplicity, we again name the given clauses by numbers:

$$\begin{array}{ll} (1) & a = b \lor a = c \\ (2) & f(a) \neq f(b) \\ (3) & b = c \end{array}$$

According to the considered KBO ordering $\succ$, the well-behaved selection function σ w.r.t. $\succ$ selects $a = b$ in clause (1), with $a \succ b$. In the unit clause (2), the literal $f(a) \neq f(b)$ is selected, with $f(a) \succ f(b)$. Further, $b = c$ is selected in clause (3), with $b \succ c$.
By equality factoring over (1), we get

$$(4) \quad a = b \lor b \neq c.$$

Note that $b \neq c$ can always be selected in (4), with $b \succ c$. Then by binary resolution over (3) and (4), we obtain:

$$(5) \quad a = b,$$

and $a = b$ is selected in (5), with $a \succ b$. By superposition over (2) and (5), we get:

$$(6) \quad f(b) \neq f(b),$$

and then by equality resolution over (6) we derive the empty clause:

$$(7) \quad \square.$$

Hence, we conclude that the input set S of clauses (1), (2) and (3) is unsatisfiable. Note that the above refutation proof derived only 4 new clauses from S, including the empty clause $\square$.