Satisfiability Checking for the Coalgebraic μ-Calculus

Daniel Hausmann
daniel.hausmann@fau.de

Chair for Theoretical Computer Science, Friedrich-Alexander-Universität Erlangen-Nürnberg

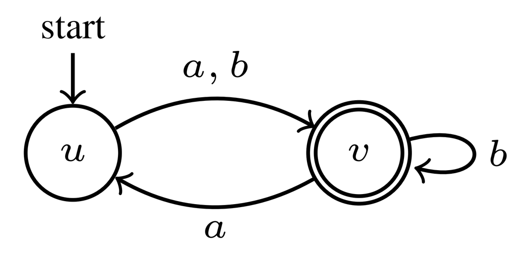
Automata on Infinite Words

Co-Büchi automata (CBA):	From some point on, only accepting states are visited
Büchi automata (BA):	Some accepting state is visited infinitely often
Parity automata (PA):	Highest *priority* that is visited infinitely often is even

Limit-stationary CBA:	Accepting SCCs are single states
Limit-linear CBA:	Accepting SCCs are linear
Limit-deterministic BA/PA:	Accepting SCCs are deterministic

Example Automata

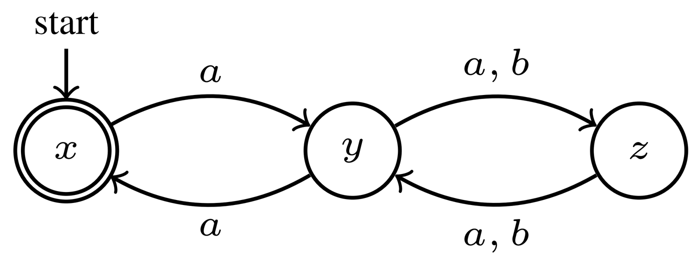
CBA:



limit-stationary

accepts: $(a+b)^*b^\omega$

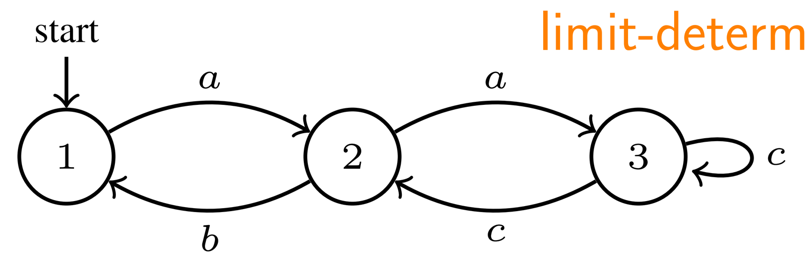
BA:



accepts: $(a((a+b)(a+b))^*a)^\omega$

PA:

limit-deterministic



accepts: $a((ac^+)+ba)^*(ba)^\omega$

Determinization Methods

Input: Automaton A of size n. **Output:** Equivalent deterministic automaton B.

Type of A	Method	Type of B	Size of B
limit-stationary CBA	focusing method	DCBA	$n \cdot 2^n$
limit-linear CBA	adaptive focusing method	DCBA	$n^2 \cdot 2^n$
CBA	Miyana/Hayashi	DCBA	3^n
limit-deterministic BA or PA	permutation method	DPA	$n!$ or $(n^2)!$
BA or PA	Safra/Piterman	DPA	$(n!)^2$ or $((n^2)!)!$

The Coalgebraic μ-Calculus

Syntax: $\varphi, \psi := \bot \mid \top \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid X \mid \heartsuit\varphi \mid \nu X.\,\varphi \mid \mu X.\,\varphi$ $\quad X \in V$ *fixpoint variables*, $\heartsuit \in \Lambda$ *modal operators*, e.g. $\Lambda = \{\Diamond, \Box\}$

Semantics: Use T-*coalgebras* as models, e.g. for $T = \mathcal{P}$ (powerset), models are Kripke frames (W, R); have e.g. $x \models \Diamond\varphi \Leftrightarrow \exists y \in R(x).\, y \models \varphi$.
Fixpoint operators iterate the argument formula finitely (μX) or infinitely (νX) often, using X to iterate.

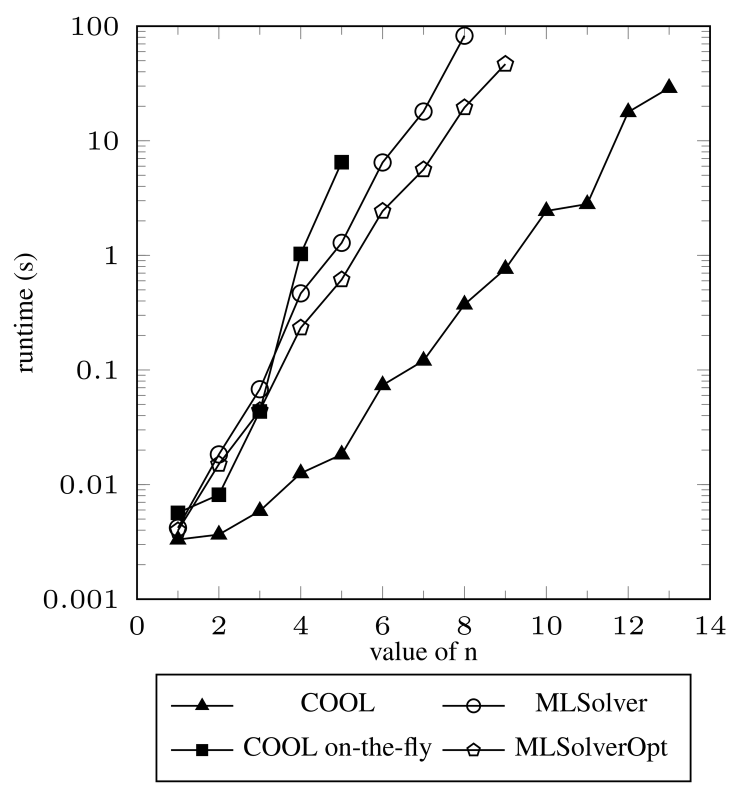
Satisfiability Checking for the Coalgebraic μ-Calculus

Input: Fixpoint formula φ. Decide satisfiability of φ by solving *parity game* G played over determinized *tracking automaton* A(φ).

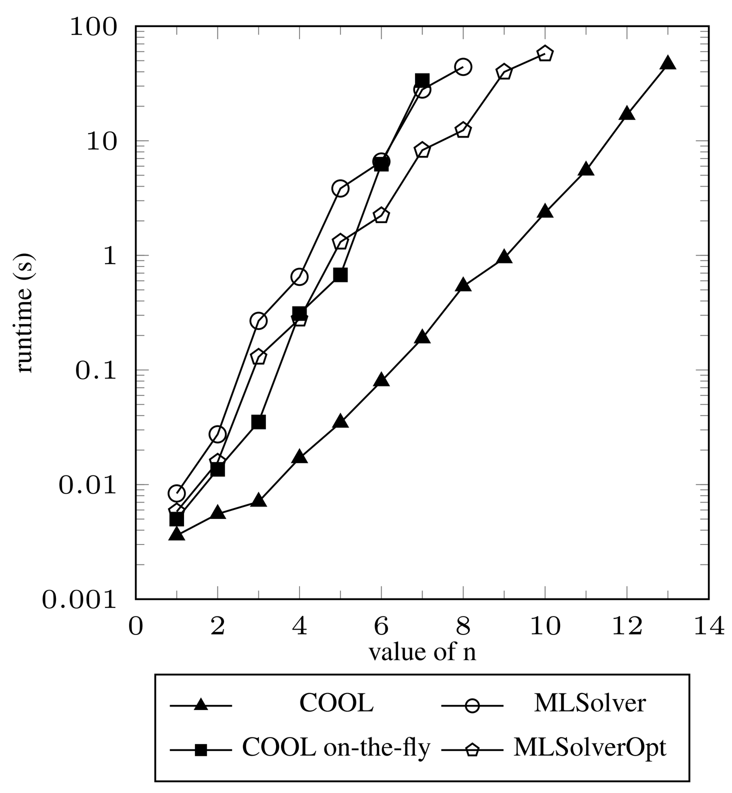
Syntactic shape of φ	Example formula	Intuition for example formula	Type of A(φ)	Size of G
depth-1 linear	$\mu X.\,\psi \vee \Diamond X$	"a state satisying ψ is reachable"	limit-stationary CBA	$n \cdot 2^n$
linear	$\mu X.\,\psi \vee \Diamond\Diamond X$	"ψ is reachable by an even number of steps"	limit-linear CBA	$n^2 \cdot 2^n$
alternation-free	$\nu X.\,\psi \wedge \Diamond X \wedge \Box X$	"all paths are infinite and ψ holds everywhere"	CBA	3^n
aconjunctive	$\nu X.\mu Y.\,(\psi \wedge \Box X) \vee \Box Y$	"on all paths, ψ holds infinitely often"	limit-deterministic PA	$(n^2)!$
no restriction	$\nu X.\mu Y.\,\psi \wedge \Box X \wedge (\chi \vee \Diamond Y)$	"ψ holds everywhere and is χ is always reachable"	PA	$((n^2)!)^2$

Implementation as part of COOL (Coalgebraic Ontology Logic Reasoner)

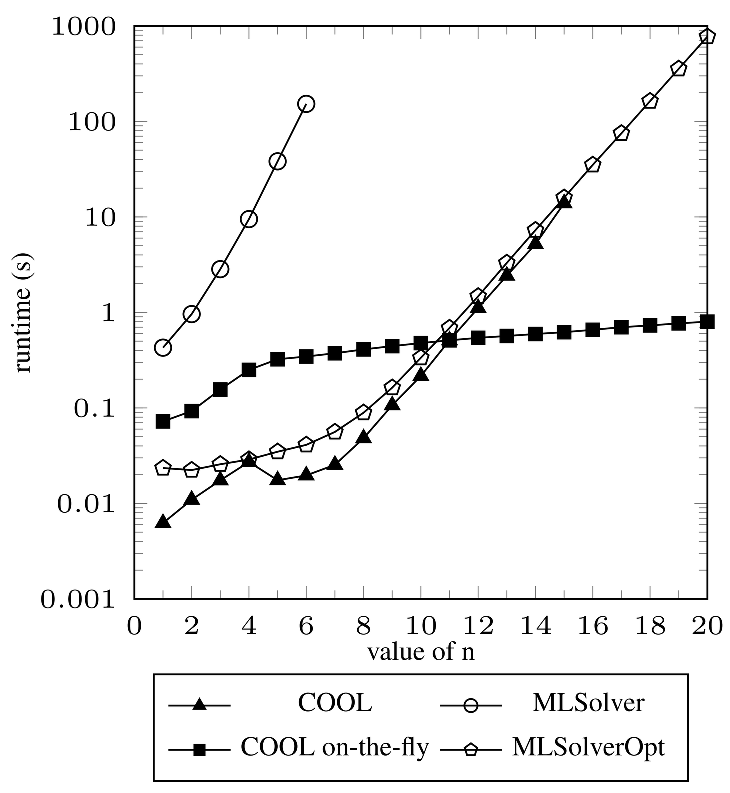
Solves satisfiability games *on-the-fly* and in *coalgebraic* generality; `https://www8.cs.fau.de/research:software:cool`



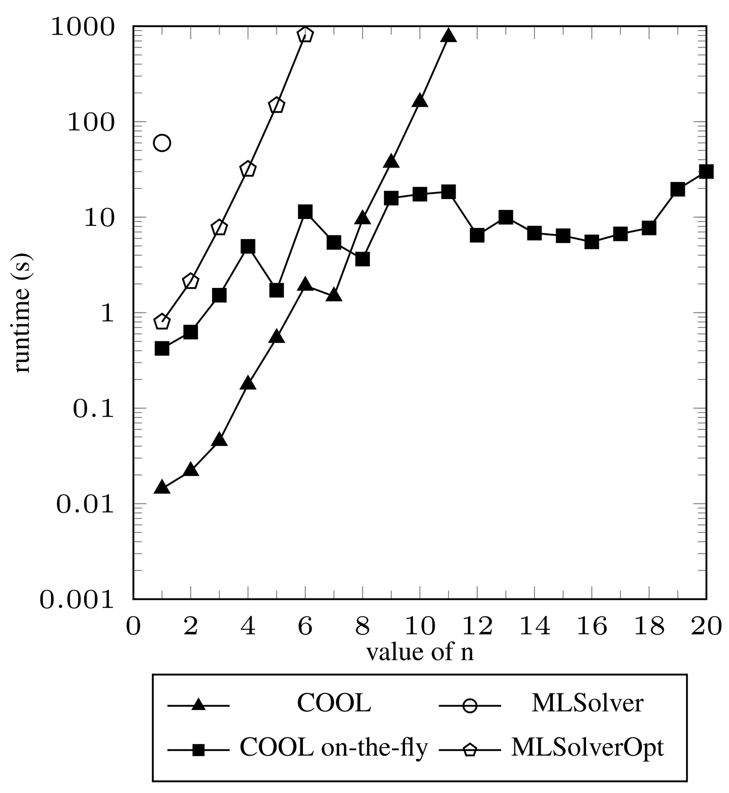
$\varphi_{\mathsf{aut}}(n) \to (\varphi_{\mathsf{ne}}(n) \leftrightarrow \bigvee_{i \text{ even}} \mu X.\nu Y.\mu Z.\, \theta_\Diamond(i))$



$\varphi_{\mathsf{game}}(n) \to (\varphi_{\mathsf{win}}(n) \to \bigwedge_{i \text{ odd}} \nu X.\mu Y.\nu Z.\, \varphi_{\mathsf{strat}}(\theta_\heartsuit(i)))$



early-ac$(n,4,2)$



early-ac$_{gc}(n,4,2)$

where $\theta_\heartsuit(i) = (q_i \wedge \heartsuit Y) \vee \bigvee_{i<j\leq n}(q_j \wedge \heartsuit X) \vee \bigvee_{1\leq j\leq i}(q_j \wedge \heartsuit Z)$ and $\varphi_{\mathsf{strat}}(\psi_\heartsuit) = (q_e \wedge \psi_\Diamond) \vee (q_a \wedge \psi_\Box)$

The central results of this work have been published in the following papers:

Daniel Hausmann and Lutz Schröder. Global caching for the flat coalgebraic μ-calculus. In Temporal Representation and Reasoning, TIME 2015, pages 121–143. IEEE Computer Society, 2015.

Daniel Hausmann, Lutz Schröder, and Christoph Egger. Global caching for the alternation-free coalgebraic μ-calculus. In Concurrency Theory, CONCUR 2016, volume 59 of LIPIcs, pages 34:1–34:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

Daniel Hausmann, Lutz Schröder, and Hans-Peter Deifel. Permutation games for the weakly aconjunctive μ-calculus. In Tools and Algorithms for the Construction and Analysis of Systems, TACAS 2018, volume 10806 of LNCS, pages 361–378. Springer, 2018.