Computing Nested Fixpoints in Quasipolynomial Time

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Why Nested Fixpoints?

- Model checking for the μ-calculus = solving parity games.
- Satisfiability checking for the μ-calculus by solving parity games.
- Winning regions of parity games are nested fixpoints.
- Model checking and satisfiability checking for generalized μ-calculi (graded, probabilistic, alternating-time) by nested fixpoints.
- Synthesis for linear-time logics (e.g. LTL).
- Computing generalized fair bisimulations.
- Type checking for inductive-coinductive types.

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We show:

- Nested fixpoints stabilize after quasipolynomially many iterations.
- The problem of computing nested fixpoints is in $\mathrm{NP} \cap \mathrm{co}\text{-}\mathrm{NP}$.
- Zielonka's algorithm can be adapted to compute nested fixpoints.

Fixpoints of Set-Functions

Function $\alpha : \mathcal{P}(U)^k \to \mathcal{P}(U)$ is monotone if for all $U_i \subseteq V_i$, $1 \leq i \leq k$,

$$\alpha(U_1, \ldots, U_k) \subseteq \alpha(V_1, \ldots, V_k)$$

Extremal Fixpoints, Nested Fixpoints

Let $f : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\alpha : \mathcal{P}(U)^k \to \mathcal{P}(U)$ be monotone functions.

$$\mathsf{LFP}\, f = \bigcap \{Z \subseteq U \mid f(Z) \subseteq Z\}$$

$$\mathsf{GFP}\, f = \bigcup \{Z \subseteq U \mid Z \subseteq f(Z)\}$$

$$\mathsf{NFP}\, \alpha = \eta_k X_k . \eta_{k-1} X_{k-1} . \ldots . \eta_1 X_1 . \alpha(X_1, \ldots, X_k),$$

where $\eta_i = \mathsf{LFP}$ if i is odd, $\eta_i = \mathsf{GFP}$ if i is even.

Nested Fixpoints and Parity Games

Parity game $(V = V_\exists \cupdot V_\forall, E \subseteq V \times V, \Omega)$ with k priorities. Define:

$$\begin{aligned} \Omega_i =& \{v \in V \mid \Omega(v) = i\} \\ \Diamond U =& \{v \in V \mid E(v) \cap U \neq \emptyset\} \\ \Box U =& \{v \in V \mid E(v) \subseteq U\} \end{aligned}$$

$$\alpha_{\mathsf{PG}}(X_1, \ldots, X_k) = (V_\exists \cap (\bigcup_{1 \leq i \leq k} \Omega_i \cap \Diamond X_i)) \cup (V_\forall \cap (\bigcup_{1 \leq i \leq k} \Omega_i \cap \Box X_i))$$

Theorem (e.g. [Dawar,Grädel,2008],[Bruse,Falk,Lange,2014])

$$\mathsf{win}_\exists = \mathsf{NFP}\, \alpha_{\mathsf{PG}}$$

A Tool: Fixpoint Parity Games (Venema, König et al.)

Fixpoint Parity Game for NFP α

Parity game (V, E, Ω), nodes: $V = U \cup \mathcal{P}(U)^k \cup \mathcal{P}(U) \times \{1, \ldots, k\}$

node	priority	owner	moves to
$u \in U$	0	$\exists$	$\{\mathbf{U} \in \mathcal{P}(U)^k \mid u \in \alpha(\mathbf{U})\}$
$\mathbf{U}$	0	$\forall$	$\{(U_j, j) \mid 1 \leq j \leq k\}$
(U, j)	j	$\forall$	$\{v \mid v \in U\}$

where $\mathbf{U} = (U_1, \ldots, U_k) \in \mathcal{P}(U)^k$.

Theorem [König et al. 2019]

Eloise wins node u if and only if $u \in$ NFP α.

Problem: exponential size

- still useful for showing *history-freeness* for nested fixpoints.

History-freeness for Nested Fixpoints

History-free witnesses

Even graph $S \subseteq U \times \{1, \dots, k\} \times U$ s.t. for all $(u, p, u') \in S$,

$$u \in \alpha(S_1(u), \dots, S_k(u)),$$

where $S_i(u) = \{v \mid (u, i, v) \in S\}$.

Note: $|S| \in \mathcal{O}(|U|^2)$

Lemma

There is a history-free witness mentioning u if and only if $u \in \mathsf{NFP}\,\alpha$.

Containment in NP $\cap$ co-NP

Theorem

If $\alpha(X_1, \ldots, X_k)$ can, for all $X_1, \ldots, X_n$, be computed in polynomial time, the problem of computing NFP α is in $\mathrm{NP} \cap \mathrm{co\text{-}NP}$.

Proof: Each State is contained in NFP or in dual nested fixpoint, hence containment in NP suffices. Guess *polynomial*-sized history-free witness for Eloise winning exponential-sized game. Verify witness in polynomial time: check that all paths are even and verify compatibility with α.

Parity Games in Quasipolynomial Time [Calude et al.,2017]

Idea: Annotate nodes with quasipolynomial histories ("statistics")

$$\overline{o} = (o_{\lceil \log n \rceil + 1}, \ldots, o_0) \qquad 1 \leq o_i \leq k$$

Define $\overline{o}@i = (o'_{\lceil \log n \rceil + 1}, \ldots, o'_0)$ as follows:

- i even: pick greatest j s.t. $i > o_j > 0$. If no such j exists, then $j = *$.
- i odd: pick greatest j s.t.
  a) $i > o_j > 0$ or
  b) o_j even for all $j' < j$, $o_{j'}$ odd (and if $o_j > 0$, $i < o_j$).
- If $j = *$, then $\overline{o}@i = \overline{o}$. Otherwise, $o'_{j'} = o_{j'}$ for $j' > j$, $o'_j = i$ and $o'_{j'} = 0$ for $j' < j$.

Move from $(v, \overline{o})$ to $(w, \overline{o}@\Omega(w))$ if move from v to w exists in original game. Solve safety game of quasipolynomial size $n \cdot k^{\lceil \log n \rceil + 2}$.

Quasipolynomial Approximation

Use Calude et al.'s quasipolynomial histories to compute nested fixpoint:

Put $\mathrm{hi} = \{(o_{\lceil \log n \rceil + 1}, \dots, o_0) \mid 1 \leq o_i \leq k\}$ having $|hi| \leq k^{\lceil \log n \rceil + 2}$ and define $\gamma : \mathcal{P}(U \times \mathrm{hi}) \to \mathcal{P}(U \times \mathrm{hi})$ by

$$\gamma(Y) = \{(v, \overline{o}) \in (U \times \mathrm{hi}) \mid v \in \alpha(Y^{\overline{o}@1}, \dots, Y^{\overline{o}@k})\}$$

where

$$Y^{\overline{o}'} = \begin{cases} \emptyset & \text{leftmost digit in } \overline{o}' \text{ is not } 0 \\ \{u \in U \mid (u, \overline{o}') \in Y\} & \text{otherwise.} \end{cases}$$

Main Theorem:

Let $\alpha : \mathcal{P}(U)^k \to \mathcal{P}(U)$ be monotone. Then $\mathrm{NFP}\,\alpha = \pi_1[\mathrm{GFP}\,\gamma]$.

Zielonka's Algorithm for Solving Parity Games

Define

$$\mathrm{Attr}_{\exists}^{\mathsf{PG}}(G,F) = \mu X.\,G \cap (F \cup \alpha_{PG}(X,\dots,X))$$
$$\mathrm{Attr}_{\forall}^{\mathsf{PG}}(G,F) = \mu X.\,G \cap (F \cup \overline{\alpha_{PG}}(X,\dots,X))$$

Algorithm: Solve parity game (G, E, Ω) [Zielonka]

1: **procedure** $\mathrm{SOLVE}_{\exists}(G, i)$ ▷ i even
2: $N_i := \{v \in G \mid \Omega(v) = i\}$; ▷ maximal priority nodes
3: $H := G \setminus \mathrm{Attr}_{\exists}^{\mathsf{PG}}(G, N_i)$; ▷ exclude Eloise-attractor of N_i
4: $W_{\forall} := \mathrm{SOLVE}_{\forall}(H, i-1)$; ▷ solve smaller game
5: $G := G \setminus \mathrm{Attr}_{\forall}^{\mathsf{PG}}(G, W_{\forall})$; ▷ remove Abelard-attractor of $W_{\forall}$
6: **if** $W_{\forall} \neq \emptyset$ **then** GOTO 2:
7: **else** RETURN G.

Zielonka's Algorithm for Computing Nested Fixpoints

Define

$$\mathsf{Attr}_{\exists}(G,F) = \mu X.\, G \cap (F \cup \alpha(X,\ldots,X))$$
$$\mathsf{Attr}_{\forall}(G,F) = \mu X.\, G \cap (F \cup \overline{\alpha}(X,\ldots,X))$$

Algorithm: Compute NFP α

```
procedure SOLVE_∃(G, i)                              ▷ i even
    N_i := {v ∈ G | Ω(v) = i};                       ▷ maximal priority nodes
    H := G \ Attr_∃(G, N_i);                         ▷ exclude Eloise-attractor of N_i
    W_∀ := SOLVE_∀(H, i − 1);                        ▷ compute smaller fixpoint
    G := G \ Attr_∀(G, W_∀);                         ▷ remove Abelard-attractor of W_∀
    if W_∀ ≠ ∅ then GOTO 2:
    else RETURN G.
```

The Fixpoint Law behind Zielonka's Algorithm

NFP α as a system of equations:

$$\begin{aligned} X_i &=_{\mathsf{LFP}} X_{i-1} && i > 1, i \text{ odd} \\ X_i &=_{\mathsf{GFP}} X_{i-1} && i \text{ even} \\ X_1 &=_{\mathsf{GFP}} \alpha(X_1, \ldots, X_k) \end{aligned}$$

A second system of equations:

$$\begin{aligned} Y_i &=_{\mathsf{LFP}} (\Omega_{>}(i) \cup \alpha(Y_i, \ldots, Y_i) \cup Y_{i-1}) \cap (\Omega_{\leq}(i) \cup Y_{i+1}) && i \text{ odd} \\ Y_i &=_{\mathsf{GFP}} (\Omega_{\leq}(i) \cap \alpha(Y_i, \ldots, Y_i) \cap Y_{i-1}) \cup (\Omega_{>}(i) \cap Y_{i+1}) && i \text{ even} \end{aligned}$$

Theorem:

$X_k = Y_k$.

The Coalgebraic μ-Calculus [Cîrstea et al., 2011]

Set **V** of fixpoint variables, set Λ of modalities, closed under duals.

Syntax:

$$\phi, \psi := \top \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid X \mid \heartsuit\psi \mid \mu X.\psi \mid \nu X.\psi \qquad \heartsuit \in \Lambda, X \in \mathbf{V}$$

Set-endofunctor T, *predicate lifting*[1] for $\heartsuit \in \Lambda$: natural transformation

$$[\![\heartsuit]\!] : \mathcal{Q} \to \mathcal{Q} \circ T^{op}$$

E.g. for $T = \mathcal{P}$,

$$[\![\Diamond]\!](A) = \{B \in \mathcal{P}(C) \mid B \cap A \neq \emptyset\}$$
$$[\![\Box]\!](A) = \{B \in \mathcal{P}(C) \mid B \subseteq A\}$$

[1] [Pattinson, 2007]

The Coalgebraic μ-Calculus [Cîrstea et al., 2011]

Assume monotonicity of predicate liftings ($A \subseteq B \Rightarrow [\![\heartsuit]\!]A \subseteq [\![\heartsuit]\!]B$)

Semantics:

Models: *T-coalgebras* $(C, \xi : C \to TC)$, extension of formulas:

$$[\![X]\!]_\sigma = \sigma(X) \qquad [\![\heartsuit\psi]\!]_\sigma = \xi^{-1}[[\![\heartsuit]\!][\![\psi]\!]_\sigma]$$

$$[\![\mu X.\,\psi]\!]_\sigma = \mathsf{LFP}([\![\psi]\!]^X_\sigma) \qquad [\![\nu X.\,\psi]\!]_\sigma = \mathsf{GFP}([\![\psi]\!]^X_\sigma)$$

where $\sigma : \mathbf{V} \to \mathcal{P}(C)$, where $[\![\psi]\!]^X_\sigma(A) = [\![\psi]\!]_{\sigma[X \mapsto A]}$ for $A \subseteq C$ and where $(\sigma[X \mapsto A])(X) = A$, $(\sigma[X \mapsto A])(Y) = \sigma(Y)$ for $X \neq Y$.

Instances of the Coalgebraic μ-Calculus

- $T = \mathcal{P}$: transition systems $(C, \xi : C \to \mathcal{P}(C))$
  - modalities: $\Diamond$,$\Box$
  - standard μ-calculus, e.g. $\mu X.\, \psi \vee \Diamond X$
- $T = \mathcal{B}$ (bag functor): graded transition systems $(C, \xi : C \to \mathcal{B}(C))$
  - modalities: $\langle g \rangle$, $[g]$, $g \in \mathbb{N}$
  - graded μ-calculus[2], e.g. $\mu X.\, \psi \vee \langle 1 \rangle X$
- $T = \mathcal{G}$: concurrent game frames
  - Set N of agents, modalities $[D]$,$\langle D \rangle$, $D \subseteq N$
  - alternating-time μ-calculus[3], e.g. $\nu X.\, \psi \wedge [D]X$
- $T = \mathcal{D}$: Markov chains
  - modalities $\langle p \rangle$,$[p]$, $p \in \mathbb{Q} \cap [0, 1]$
  - (two-valued) probabilistic μ-calculus, e.g. $\nu X.\, \psi \wedge \langle 0.5 \rangle X$

[2] [Kupferman et al.,2002]
[3] [Alur et al., 2002]

Recent Results on the Coalgebraic μ-Calculus

- Reduce model checking [H,Schröder,CONCUR 2019] and satisfiability checking [H,Schröder,FoSSaCS 2019] for the coalgebraic μ-calculus to computing nested fixpoints.

Corollary

Model checking for coalgebraic μ-calculi is in QP and in $\mathrm{NP} \cap \mathrm{Co\text{-}NP}$.

Corollary

Satisfiability checking for coalgebraic μ-calculi can be done in time $\mathcal{O}(2^{nk \log n})$ (down from $\mathcal{O}(2^{n^2 k^2 \log n})$).

Introducing: Coalgebraic Parity Games

Definition - Coalgebraic parity game:

T-coalgebra $(C, \xi : C \to TC)$ with mappings $\Omega : C \to \mathbb{N}$, $m : C \to \Lambda$.

Eloise wins node $c \in C$ if there is even graph (D, R) on C s.t.

$$\text{for all } d \in D,\ \xi(d) \in [\![m(d)]\!]R(d).$$

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e.g.

- $T = \mathcal{P}$: parity game for T is graph $(C, \xi : C \to \mathcal{P}(C))$ with priority map Ω and node ownership map $m : C \to \{\Diamond, \Box\}$.

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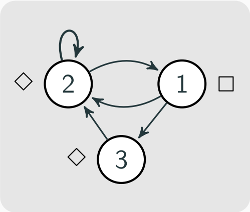
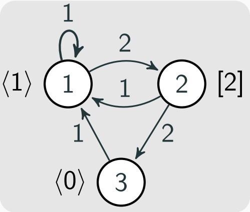
e.g.

- $T = \mathcal{P}$: parity game for T is graph $(C, \xi : C \to \mathcal{P}(C))$ with priority map Ω and node ownership map $m : C \to \{\Diamond, \Box\}$.
- $T = \mathcal{D}$: parity game for T is Markov chain $(C, \xi : C \to \mathcal{D}(C))$ with priority map Ω and map $m : C \to \{\langle p \rangle, [p] \mid p \in \mathbb{Q} \cap [0, 1]\}$.

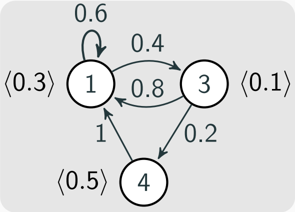
Coalgebraic Parity Games, examples



$T = \mathcal{P}$: standard

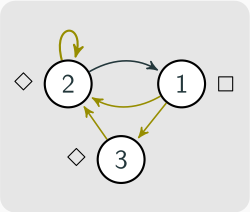


$T = \mathcal{B}$: graded

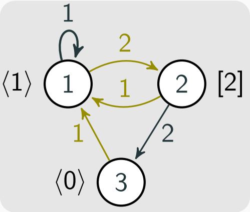


$T = \mathcal{D}$: probabilistic

Coalgebraic Parity Games, examples, strategies



$T = \mathcal{P}$: standard



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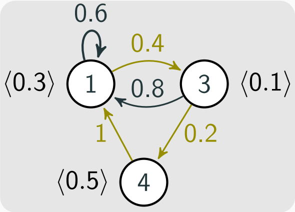

$T = \mathcal{D}$: probabilistic

Solving Coalgebraic Parity Games

Winning regions in coalgebraic parity games are nested fixpoints:

Given game (C, ξ, m, Ω), define $f : \mathcal{P}(C)^k \to \mathcal{P}(C)$ by

$$f(X_0, \dots, X_k) = \{v \mid \exists i, \heartsuit \in \Lambda.\, m(v) = \heartsuit, \Omega(v) = i \text{ and } \xi(v) \in [\![\heartsuit]\!] X_i\}$$

Solving Coalgebraic Parity Games

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Theorem [H,Schröder,CONCUR 2019]:

Player Eloise wins u in coalgebraic parity game if and only if $u \in \mathsf{NFP}\, f$.

Coalgebraic μ-calculus model checking = solving coalgebraic parity games.

Enables on-the-fly model checking: Start with initial node, expand nodes step by step, compute NFP f at any point (solving a partial game).

Conclusion

Results:

- Computing nested fixpoints by
  - (fixpoint iteration),
  - Calude et al.'s quasipolynomial algorithm
  - Zielonka's algorithm
- Computing nested fixpoints also is in $\mathrm{NP} \cap \mathrm{Co\text{-}NP}$.
- Reduction of satisfiability checking and model checking for the *coalgebraic* μ-calculus to computing nested fixpoints.

Future work:

- Computing fair bisimulations as nested fixpoints.
- Type checking for inductive-coinductive types by computing nested fixpoints.

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