NP Reasoning in the Monotone μ-Calculus

Daniel Hausmann and Lutz Schröder

University Erlangen-Nuremberg, Germany

Overview

Complexities of satisfiability checking for some basic modal logics:

• **K** / $\mathcal{ALC}$	PSPACE
• **K** + global axioms (universal modality)	EXPTIME
• modal μ-calculus	EXPTIME
• monotone modal logic	NP [Vardi, 1989]
• monotone modal logic + global axioms	NP [here]
• alternation-free monotone μ-calculus + global axioms	NP [here]

The Monotone μ-Calculus

Fix sets At, Act, Var of *atoms*, *actions* and *fixpoint variables*.

Syntax:

$$\phi, \psi ::= \bot \mid \top \mid p \mid \phi \wedge \psi \mid \phi \vee \psi \mid [a]\phi \mid \langle a \rangle \phi \mid X \mid \nu X.\phi \mid \mu X.\phi$$

$(p \in \mathsf{At},\ a \in \mathsf{Act},\ X \in \mathsf{Var})$

Semantics:

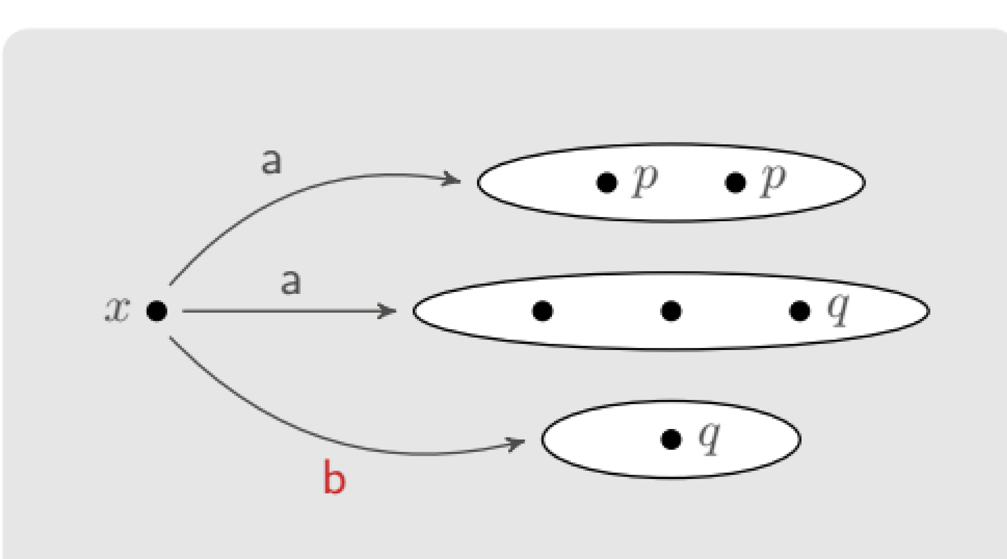
Formulae are interpreted over *neighbourhood structures* $M = (W, N, I)$ with $N : \mathsf{Act} \times W \to \mathcal{P}(\mathcal{P}(W))$, $I : \mathsf{At} \to \mathcal{P}(W)$, valuation $\sigma : \mathsf{Var} \to \mathcal{P}(W)$:

$$[\![[a]\phi]\!]_\sigma = \{w \in W \mid \forall S \in N(a,w).\, S \cap [\![\phi]\!]_\sigma \neq \emptyset\}$$
$$[\![\langle a \rangle\phi]\!]_\sigma = \{w \in W \mid \exists S \in N(a,w).\, S \subseteq [\![\phi]\!]_\sigma\}$$
$$[\![\mu X.\phi]\!]_\sigma = \bigcap\{Z \subseteq W \mid [\![\phi]\!]^X_\sigma(Z) \subseteq Z\}$$
$$[\![\nu X.\phi]\!]_\sigma = \bigcup\{Z \subseteq W \mid Z \subseteq [\![\phi]\!]^X_\sigma(Z)\}$$

where $[\![X]\!]_\sigma = \sigma(X)$ and $[\![\phi]\!]^X_\sigma(Z) = [\![\phi]\!]_{\sigma[X \mapsto Z]}$ for $Z \subseteq W$.

Closure $\mathsf{cl}(\phi)$ of ϕ: subformulae of ϕ with fixpoints unfolded at most once
Deferrals $\mathsf{dfr} \subseteq \mathsf{cl}(\phi)$: formulae originating from least fixpoints

Example of a neighbourhood structure and some (un)satisfied formulae:



$x \in [\![\langle a \rangle p]\!]$

$x \in [\![[a](p \vee q)]\!]$

$x \notin [\![\langle b \rangle p]\!]$

Cannot express e.g. "p holds in every successor state"
"p holds in at least one successor state"

Tableaux

Pre-tableaux are graphs whose nodes are labelled with sets of formulae, transitions according to tableaux rules:

$$(\bot)\ \frac{\Gamma, \bot}{} \qquad (\lightning)\ \frac{\Gamma, p, \neg p}{}$$

$$(\wedge)\ \frac{\Gamma, \phi_0 \wedge \phi_1}{\Gamma, \phi_0, \phi_1} \qquad (\vee)\ \frac{\Gamma, \phi_0 \vee \phi_1}{\Gamma, \phi_0 \qquad \Gamma, \phi_1}$$

$$(\langle a \rangle)\ \frac{\Gamma, \langle a \rangle \phi_0, [a]\phi_1}{\phi_0, \phi_1} \qquad (\eta)\ \frac{\Gamma, \eta X.\, \phi_0}{\Gamma, \phi_0[\eta X.\, \phi_0 / X]}$$

where $a \in \mathsf{Act}$, $p \in \mathsf{At}$, $X \in \mathsf{Var}$, $\eta \in \{\mu, \nu\}$

For alphabet Σ identifying rule applications, tracking functions

$$\gamma : \mathsf{cl}(\phi) \times \Sigma \to \mathcal{P}(\mathsf{cl}(\phi)) \qquad \delta : \mathsf{dfr} \times \Sigma \to \mathcal{P}(\mathsf{dfr}),$$

track (least fixpoint) formulae along tableaux rules.

Tableaux are pre-tableaux in which all δ-traces are finite.

Satisfiability Games

Two-player Büchi games with $\mathcal{O}(|\phi|^2)$ Eloise-nodes ($|V_\exists| \leq |\phi|^2$):

states: set of saturated sets of formulae, Σ_p : propositional rules

$U = \{\Psi \subseteq \mathsf{cl}(\phi) \mid 2 \geq |\Psi|\}$ $\quad V_\exists = U^2$ $\quad V_\forall = \mathsf{states}^2$ $\quad F = \{(\Psi, \emptyset) \in V_\exists\}$

Node	Moves
$(\Psi, \Phi) \in V_\exists$	$\{(\gamma(\Psi, w), \delta(\Phi, w)) \in V_\forall \mid w \in (\Sigma_p)^*, \lvert w \rvert \leq 3n\}$
$(\Gamma, \Phi) \in V_\forall$	$\{(\{\phi_0, \phi_1\}, \Phi') \in V_\exists \mid \{\langle a \rangle \phi_0, [a]\phi_1\} \subseteq \Gamma\}$, if $\Phi \neq \emptyset$, then $\Phi' = \delta(\Phi, (\langle a \rangle \phi_0, [a]\phi_1))$, if $\Phi = \emptyset$, then $\Phi' = \{\phi_0, \phi_1\}$ }

- Propositional reasoning condensed into single Eloise-moves
- Modal steps track at most two formulae
- Implicit: economic variant of Miyano/Hayashi Co-Büchi automata determinization

Theorem 1

Formula ϕ is satisfiable $\Leftrightarrow$ There is a tableau for ϕ.

Theorem 2

There is a tableau for ϕ $\Leftrightarrow$ Eloise wins satisfiability game.

Main Result

The satisfiability problem for the alternation-free monotone μ-calculus with global assumptions is NP-complete.

Proof sketch:

Formula ϕ is satisfiable $\Leftrightarrow$ There is a tableau for ϕ $\Leftrightarrow$ Eloise wins the satisfiability game for ϕ

Example Logics

Logics that embed into the monotone μ-calculus:

- Epistemic Logic
  $\langle a \rangle \phi$ – "Agent a knows ϕ"
- Concurrent PDL (CPDL), [Peleg, 1987]
  $\langle \alpha \rangle \phi$ – "There is execution of program α in parallel, nondeterministic system s.t. all end states satisfy ϕ"
- Game Logic, [Parikh, 1983]
  $\langle \alpha \rangle \phi$ – "Player Angel has strategy to achieve ϕ in game α"

$\rightsquigarrow$ The satisfiability problems of (the alternation-free fragments of) these logics is NP-complete.

Future Work

How about the full monotone μ-calculus / full Game Logic? (i.e., is assumption of alternation-freeness mandatory?)

Further Information

Extended version of IJCAR 2020 paper:
https://arxiv.org/abs/2002.05075

25-minute talk at IJCAR 2020:
Video: https://www8.cs.fau.de/ext/daniel/monotone.mp4
Slides: https://www8.cs.fau.de/ext/daniel/monotone.pdf