Overview

- Winning regions in parity games are nested fixpoints over powerset lattices
- Parity games can be solved in quasipolynomial time [Calude et al., 2017]
- Solutions of quantitative and probabilistic games are nested fixpoints over quantitative and probabilistic lattices, respectively

We show: Quasipolynomial methods from parity games can be used to compute nested fixpoints over arbitrary finite lattices

Nested Fixpoints over Finite Lattices

Fix finite lattice $(L, \sqsubseteq)$ with basis $B$ of size $n = |B|$, join $\sqcup$, meet $\sqcap$.

For $d$ monotone functions $f_i : L^d \rightarrow L$, system of equations consists of $d$ equations of the form

$$X_i = \eta_i f_i(X_1, \ldots, X_d)$$

where $\eta_i \in \{\text{LFP, GFP}\}$.

For a partial valuation $\sigma : \{1,\ldots,d\} \rightarrow L$, inductively define

$$[X_i]^\sigma = \eta_i X_i, f_i^\sigma,$$

where the function $f_i^\sigma$ is given by

$$f_i^\sigma(A) = f_i(\text{ev}(\sigma', 1), \ldots, \text{ev}(\sigma', i-1), A, \text{ev}(\sigma', i+1), \ldots, \text{ev}(\sigma', d))$$

for $A \in L$, where $\sigma' = \sigma[i \mapsto A]$ and

$$\text{ev}(\sigma, j) = \begin{cases} \sigma(j) & j \in \text{dom}(\sigma) \\ [X_i]^\sigma & j \notin \text{dom}(\sigma) \end{cases},$$

and where

$$\text{GFP} g = \sqcup \{ V \subseteq L | V \sqsubseteq g(V) \} \quad \text{LFP} g = \cap \{ V \subseteq L | g(V) \subseteq V \}.$$  

Solution for variable $X_i$ is $[X_i]^\text{dom}(\sigma) = \emptyset$.

Canonical equation system for single function $f : L^d \rightarrow L$

$$X_i = \eta_i X_{i-1}, \quad i > 1$$

$$X_1 = \text{LFP } f(X_1, \ldots, X_d),$$

where $\eta_i = \text{LFP}$ if $i$ odd, $\eta_i = \text{GFP}$ if $i$ even.

Nested fixpoint of $f : L^d \rightarrow L$ is solution of canonical equation system:

$$\text{NFP } f := [X_d]$$

A Generic Progress Measure Algorithm

Fix $(n, d)$-universal tree $(T, \delta : T \times \{1, \ldots, d\} \rightarrow T, \leq)$, least node w.r.t. $\leq = t_{\min}$

Measure: $\mu : B \rightarrow T \cup \{\ast\}$; define function Lift on measures by

$$(\text{Lift}(\mu))(v) = \min\{ t \in T | v \subseteq f(U_t^{(1)}, \ldots, U_t^{(d)}) \}$$

where $\min(\emptyset) = \ast$ and

$$U_t^e = \{ u \in B | \mu(u) < \delta(t, i) \}$$

Our algorithm:
1. Initialize $\mu(v) = t_{\min}$ for all $v \in B$
2. If Lift(\mu) \neq \mu, then put $\mu := \text{Lift}(\mu)$ and go to 2; else go to 3.
3. Return $A = \{ v \in B | \mu(v) = \ast \}$

Theorem: We have $v \in A$ if and only if $v \notin \text{NFP } f$.

There are $(n, d)$-universal trees of size quasipolynomial in $n, d$ [Czerwiński, Daviaud, Fijalkow, Jurdzinski, Łazić, Parys, 2018].

Corollary: The number of iterations required to compute nested fixpoints over $L$ is quasipolynomial in $n$ and $d$.

Examples of Nested Fixpoints

- Parity games:
  $$G = (V \cup V_e, E, \sigma, \Omega), \text{ } n \text{ nodes, } d \text{ priorities}$$
  One-step game function $f_{\text{EPG}} : (2^n)^d \rightarrow 2^n$.
  $$(X_1, \ldots, X_d) \mapsto (V_\cap \Omega \cup X_0) \cup (V_\cap \square X_0)$$
  Theorem [Walukiewicz, 1996]: Player $\exists$ wins $v$ if and only if $v \in \text{NFP } f_{\text{EPG}}$

- Energy parity games:
  $$G = (V \cup V_e, E, \Omega, w), \text{ } w : E \rightarrow \mathbb{Z}$$
  Lemma [Chatterjee, Doyen, 2012]: Memory of winning histories bounded by $c = n \cdot d \cdot w_{\max}$
  One-step game function $f_{\text{EPG}} : (c^n)^d \rightarrow c^n$.
  $$(X_1, \ldots, X_d) \mapsto (V_\cap \Omega \cup X_0) \cup (V_\cap \square X_0)$$
  Theorem [Amram et al., 2020]: Player $\exists$ wins $v$ with initial credit $c_0$ if and only if $(\text{NFP } f_{\text{EPG}}(v) \leq c_0$.

- Stochastic parity games:
  $$G = (V_\cup V_e, \forall \cup \bigcup, E, \Omega), E[V_e] \subseteq \text{Dist}(V)$$
  Lemma [Chatterjee, Henzinger, 2008]: Approximation of values bound $\lambda$ exponential in $n$.
  $$f_{\text{EPG}} : (\lambda^n)^d \rightarrow \lambda^n$$
  $$(X_1, \ldots, X_d) \mapsto (V_\cap \forall \cup X_0) \cup (V_\cap \square X_0)$$
  Theorem [Chatterjee, Henzinger, 2008]: Player $\exists$ wins $v$ with probability $\geq p_0$ if $(\text{NFP } f_{\text{EPG}}(v) \geq p_0$.

- Mean pay-off parity games and weighted parity games: by reduction to energy parity games

Summary of Results

Progress measure algorithm recovers known complexities:

<table>
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<tr>
<th>type of games</th>
<th>game solving</th>
<th>model checking</th>
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<tr>
<td>parity</td>
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<td>energy parity</td>
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<tr>
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If $d \leq \log n$, NFP $f$ can be computed with polynomially many iterations of $f$.

Further applications in graded $\mu$-calculus, (two-valued) probabilistic $\mu$-calculus, alternating-time $\mu$-calculus, lattice $\mu$-calculus

Future Work

- Implementation of generic fixpoint solver underway
- Identify further examples (e.g. Streett games?)

Further Information

- Extended version of TACAS 2021 paper: https://arxiv.org/abs/1907.07020
- 25-minute talk at TACAS 2021: Video: https://www8.cs.fau.de/ext/daniel/qfpf.mp4