Uniform Solving for ω-regular Games

Quasipolynomial Computation of Nested Fixpoints (TACAS 2021)

Daniel Hausmann and Lutz Schröder

Gothenburg University, Sweden and University Erlangen-Nuremberg, Germany

Overview

– Winning regions in parity games are nested fixpoints over powerset lattices

– Parity games can be solved in quasipolynomial time [Calude et al., 2017]

– Solutions of quantitative and probabilistic games are nested fixpoints over quantitative and probabilistic lattices, respectively

We show: Quasipolynomial methods from parity games can be used to compute nested fixpoints over *arbitrary finite lattices*

Nested Fixpoints over Finite Lattices

Fix finite lattice $(L, \sqsubseteq)$ with basis B of size $n = |B|$, join $\sqcup$, meet $\sqcap$.

For d monotone functions $f_i : L^d \to L$, *system of equations* consists of d equations of the form

$$X_i =_{\eta_i} f_i(X_1, \dots, X_d)$$

where $\eta_i \in \{\mathsf{LFP}, \mathsf{GFP}\}$.

For a partial valuation $\sigma : \{1, \dots, d\} \rightharpoonup L$, inductively define

$$[\![X_i]\!]^\sigma = \eta_i X_i . f_i^\sigma,$$

where the function f_i^σ is given by

$$f_i^\sigma(A) = f_i(\mathsf{ev}(\sigma', 1), \dots, \mathsf{ev}(\sigma', i-1), A, \mathsf{ev}(\sigma', i+1), \dots, \mathsf{ev}(\sigma', d))$$

for $A \in L$, where $\sigma' = \sigma[i \mapsto A]$ and

$$\mathsf{ev}(\sigma, j) = \begin{cases} \sigma(j) & j \in \mathsf{dom}(\sigma) \\ [\![X_j]\!]^\sigma & j \notin \mathsf{dom}(\sigma) \end{cases} \qquad (\sigma[i \mapsto A])(j) = \begin{cases} \sigma(j) & j \neq i \\ A & j = i \end{cases}$$

and where

$$\mathsf{GFP}\, g = \bigsqcup\{V \sqsubseteq L \mid V \sqsubseteq g(V)\} \qquad \mathsf{LFP}\, g = \bigsqcap\{V \sqsubseteq L \mid g(V) \sqsubseteq V\}.$$

Solution for variable X_i is $[\![X_i]\!]^\epsilon$ ($\mathsf{dom}(\epsilon) = \emptyset$).

Canonical equation system for single function $f : L^d \to L$:

$$X_i =_{\eta_i} X_{i-1} \qquad i > 1$$
$$X_1 =_{\mathsf{LFP}} f(X_1, \dots, X_d),$$

where $\eta_i = \mathsf{LFP}$ if i odd, $\eta_i = \mathsf{GFP}$ if i even.

Nested fixpoint of $f : L^d \to L$ is solution of canonical equation system:

$$\mathsf{NFP}\, f := [\![X_d]\!]$$

A Generic Progress Measure Algorithm

Fix (n, d)-*universal tree* $(T, \delta : T \times \{1, \dots, d\} \to T, \leq)$, least node w.r.t. $\leq$: t_{min}

Measure: $\mu : B \to T \cup \{\star\}$; define function Lift on measures by

$$(\mathsf{Lift}(\mu))(v) = \min\{t \in T \mid v \sqsubseteq f(U_1^{\mu,t}, \dots, U_d^{\mu,t})\}$$

where $\min(\emptyset) = \star$ and

$$U_i^{\mu,t} = \bigsqcup\{u \in B \mid \mu(u) \leq \delta(t, i)\}$$

Our algorithm:

1. Initialize $\mu(v) = t_{\mathsf{min}}$ for all $v \in B$
2. If $\mathsf{Lift}(\mu) \neq \mu$, then put $\mu := \mathsf{Lift}(\mu)$ and go to 2; else go to 3.
3. Return $\mathbb{A} = \{v \in B \mid \mu(v) \neq \star\}$

Theorem: We have $v \in \mathbb{A}$ if and only if $v \sqsubseteq \mathsf{NFP}\, f$.

There are (n, d)-universal trees of size quasipolynomial in n, d [Czerwiński, Daviaud, Fijalkow, Jurdziński, Lazić, Parys, 2018].

Corollary: The number of iterations required to compute nested fixpoints over L is quasipolynomial in n and d.

Examples of Nested Fixpoints

– **Parity games:**

$G = (V_\exists \uplus V_\forall, E, \Omega)$, n nodes, d priorities

One-step game function $f_{\mathsf{PG}} : (2^n)^d \to 2^n$:

$$(X_1, \dots, X_d) \mapsto (V_\exists \cap \Diamond X_\Omega) \cup (V_\forall \cap \Box X_\Omega)$$

Theorem [Walukiewicz, 1996]: Player $\exists$ wins v if and only if $v \in \mathsf{NFP}\, f_{\mathsf{PG}}$.

– **Energy parity games**:

$G = (V_\exists \uplus V_\forall, E, \Omega, w)$, $w : E \to \mathbb{Z}$

Lemma [Chatterjee, Doyen, 2012]: Memory of winning histories bounded by $c = n \cdot d \cdot w_{\max}$

One-step game function $f_{\mathsf{EPG}} : (c^n)^d \to c^n$:

$$(X_1, \dots, X_d) \mapsto (V_\exists \sqcap \Diamond_E X_\Omega) \sqcup (V_\forall \sqcap \Box_E X_\Omega)$$

Theorem [Amram et al., 2020]: Player $\exists$ wins v with initial credit c_0 if and only if $(\mathsf{NFP}\, f_{\mathsf{EPG}})(v) \leq c_0$.

– **Stochastic parity games:**

$G = (V_\exists \uplus V_\forall \uplus V_p, E, \Omega)$, $E[V_p] \subseteq \mathsf{Dist}(V)$

Lemma [Chatterjee, Henzinger, 2008]: Approximation of values bound λ exponential in n.

$$f_{\mathsf{EPG}} : \quad (\lambda^n)^k \to \lambda^n$$
$$(X_1, \dots, X_k) \mapsto (V_\exists \sqcap \Diamond_S X_\Omega) \sqcup (V_\forall \sqcap \Box_S X_\Omega)$$

Theorem [Chatterjee, Henzinger, 2008]: Player $\exists$ wins v with probability $\geq p_0$ iff $(\mathsf{NFP}\, f_{\mathsf{SPG}})(v) \geq p_0$.

– **Mean pay-off parity games** and **weighted parity games:** by reduction to energy parity games

Summary of Results

Progress measure algorithm recovers known complexities:

type of games	game solving	model checking
parity	QP	QP
energy parity	pseudo-QP	QP in c
mean pay-off parity	pseudo-QP	?

If $d \leq \log n$, $\mathsf{NFP}\, f$ can be computed with polynomially many iterations of f.

Further applications in graded μ-calculus, (two-valued) probabilistic μ-calculus, alternating-time μ-calculus, latticed μ-calculi

Future Work

– Implementation of generic fixpoint solver underway

– Identify further examples (e.g. Streett games?)

Further Information

– Extended version of TACAS 2021 paper: https://arxiv.org/abs/1907.07020

– 25-minute talk at TACAS 2021:
Video: https://www8.cs.fau.de/ext/daniel/qpfp.mp4
Slides: https://www8.cs.fau.de/ext/daniel/tacas21-qpfp.pdf