Uniform Solving for $\omega$-regular Games

Quasipolynomial Computation of Nested Fixpoints (TACAS 2021)

Daniel Hausmann
Gothenburg University, Sweden

Lutz Schröder
University Erlangen-Nuremberg, Germany

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Parity game winning regions are nested fixpoints over powerset lattice.

Recent breakthrough result: solving parity games is in QP.

Idea: Adapt QP parity game solving algorithms to compute general nested fixpoints, obtain same results for more general games / logics.

Main contribution:
QP algorithm for computing nested fixpoints over arbitrary finite lattices.
Finite lattice \((L, \sqsubseteq)\) with basis \(B\) of size \(n = |B|\)

**Nested Fixpoints over \(L\)**

For monotone function \(f : L^d \rightarrow L\) (w.l.o.g. \(d\) even), put

\[
\text{NFP } f := \text{GFP } X_d \cdot \text{LFP } X_{d-1} \ldots \text{LFP } X_1.f(X_1, \ldots, X_d)
\]

(Our results actually hold for fixpoint equation systems)
Parity game $G = (V_\exists \cup V_\forall, E, \Omega)$, $n$ nodes, $d$ priorities

One-step game function $f_{PG} : (2^n)^d \to 2^n$:

$$(X_1, \ldots, X_d) \mapsto (V_\exists \cap \Diamond X_\Omega) \cup (V_\forall \cap \Box X_\Omega)$$

**Theorem [Walukiewicz, 1996]**

Player $\exists$ wins $v$ if and only if $v \in \text{NFP } f_{PG}$. 
Energy parity game $G = (V_\exists \cup V_\forall, E, \Omega, w), w : E \to \mathbb{Z}$

- Bound on histories $c = n \cdot d \cdot w_{\text{max}}$ [Chatterjee, Doyen, 2012]

One-step game function $f_{\text{EPG}} : (c^n)^d \to (c^n)$:

$$(X_1, \ldots, X_d) \mapsto (V_\exists \sqinter \Diamond_E X_\Omega) \sqcup (V_\forall \sqinter \Box_E X_\Omega)$$

**Theorem** [Amram, Maoz, Pistiner, Ringert, 2020]

Player $\exists$ wins $v$ with initial credit $c_0$ if and only if $(\text{NFP } f_{\text{EPG}})(v) \leq c_0$. 
Progress measure algorithm for computing NFP $f$ ($f : L^d \rightarrow L$)
Progress is measured using nodes in universal tree

**Main Contribution: Theorem**
The progress measure algorithm computes NFP $f$.

**Corollary [Czerwinski et al. 2018]**
Nested fixpoints over finite lattices can be computed with quasipolynomially many iterations.
Results:

- Quasipolynomial solving of fixpoint equations by universal trees
- Highly general quasipolynomial progress measure algorithm for
  - Parity games / model checking $\mu$-calculus
  - Energy parity games / model checking energy $\mu$-calculus
  - Mean pay-off parity games
  - Stochastic parity games (both qualitative and quantitative)
- Typical runtime: $O((hn)^{\log d})$ (notable exception: stochastic games)

Ongoing work:

- Implement algorithm to obtain generic game solver
- Does this work for all games with finite-history winning strategies?