Optimal Satisfiability Checking for the Coalgebraic μ-Calculus

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Coalgebraic One-Step Satisfiability

Set-endofunctor T, set Λ of (unary) modal operators.

For each $\heartsuit \in \Lambda$, assume T-*predicate lifting*, that is, family

$$(\llbracket\heartsuit\rrbracket_X : \mathcal{P}(X) \to \mathcal{P}(TX))_{X \in \mathbf{Set}}$$

of functions, satisfying a naturality requirement.

Given set A, put $\Lambda(A) = \{\heartsuit a \mid \heartsuit \in \Lambda, a \in A\}$

One-step satisfiability problem

Let $v \subseteq \Lambda(A)$ and $U \subseteq \mathcal{P}(A)$ with $a \neq b$ whenever $\heartsuit_1 a, \heartsuit_2 b \in v$. Put

$$\llbracket v\rrbracket_1 = \bigcap_{\heartsuit a \in v} \llbracket\heartsuit\rrbracket_U \{u \in U \mid a \in u\}$$

One-step satisfiability problem: Do we have $T(U) \cap \llbracket v\rrbracket_1 \neq \emptyset$?

Denote time to solve problem by $t(|v|, |U|)$ with $|v| \leq |A|$, $|U| \leq 2^{|A|}$.

Coalgebraic One-Step Satisfiability, example

Basic modal logic: $T = \mathcal{P}$, $\Lambda = \{\Diamond, \Box\}$,

$$[\![\Diamond]\!]_X(A) = \{B \in \mathcal{P}(X) \mid A \cap B \neq \emptyset\}$$
$$[\![\Box]\!]_X(A) = \{B \in \mathcal{P}(X) \mid B \subseteq A\}$$

Example

$A = \{b, c, d\}$, $v = \{\Diamond b, \Diamond c, \Box d\} \subseteq \Lambda(A)$, $U = \{x, y\} \subseteq \mathcal{P}(A)$,
$x = \{b, d\}$, $y = \{c, d\}$

Do we have $\mathcal{P}(U) \cap [\![v]\!]_1 \neq \emptyset$?

In general: $t(|v|, |U|) \in \mathcal{O}(|v|^2 \cdot |U|)$, i.e. problem is in P.

Coalgebraic One-Step Satisfiability, example ctd.

Graded modal logic: *bag functor* $T = \mathcal{B}$, $\mathcal{B}(X) = \{\theta : X \to \mathbb{N} \cup \infty\}$, $\Lambda = \{\langle k \rangle, [k] \mid k \in \mathbb{N}\}$, predicate liftings

$$[\![\langle k \rangle]\!]_X(A) = \{\theta \in \mathcal{B}(X) \mid \theta(A) > k\}$$
$$[\![[k]]\!]_X(A) = \{\theta \in \mathcal{B}(X) \mid \theta(X \setminus A) \leq k\},$$

where $\theta(A) = \sum_{a \in A} \theta(a)$.

Example

$A = \{b, c, d\}$, $v = \{\langle 2 \rangle b, \langle 1 \rangle c, [1]d\} \subseteq \Lambda(A)$, $U = \{x, y, z\} \subseteq \mathcal{P}(A)$, $x = \{b, d\}$, $y = \{c\}$, $z = \{b\}$

Do we have $\mathcal{B}(U) \cap [\![v]\!]_1 \neq \emptyset$?

In general: $t(|v|, |U|) \in \mathcal{O}((2b + 2)^{|v|})$ where b is maximal grade in v, i.e. problem is in P [Kupferman, Sattler, Vardi, 2002].

The Coalgebraic μ-Calculus [Cirstea et al., 2009]

Assume set $\mathbf{V}$ of fixpoint variables.

Syntax:

$\phi, \psi := \top \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid X \mid \heartsuit\phi \mid \mu X.\phi \mid \nu X.\phi \qquad \heartsuit \in \Lambda, X \in \mathbf{V}$

Assume monotonicity of predicate liftings ($A \subseteq B \Rightarrow [\![\heartsuit]\!]A \subseteq [\![\heartsuit]\!]B$).

Semantics:

Models: *T-Coalgebras* $(W, \xi : W \to TW)$, extension of formulas:

$$[\![X]\!]_\sigma = \sigma(X) \qquad [\![\heartsuit\phi]\!]_\sigma = \xi^{-1}[[\![\heartsuit]\!]_W [\![\phi]\!]_\sigma]$$

$$[\![\mu X.\,\phi]\!]_\sigma = \mathsf{LFP}([\![\phi]\!]_\sigma^X) \qquad [\![\nu X.\,\phi]\!]_\sigma = \mathsf{GFP}([\![\phi]\!]_\sigma^X)$$

where $\sigma : \mathbf{V} \to \mathcal{P}(W)$, where $[\![\phi]\!]_\sigma^X(A) = [\![\phi]\!]_{\sigma[X \mapsto A]}$ for $A \subseteq W$ and where $(\sigma[X \mapsto A])(X) = A$, $(\sigma[X \mapsto A])(Y) = \sigma(Y)$ for $X \neq Y$.

Observe: $\xi(x) \in TW \cap \bigcap_{\heartsuit\psi \in I(x)} [\![\heartsuit]\!]_W [\![\psi]\!]$ where $I(x) = \{\heartsuit\psi \mid x \in [\![\heartsuit\psi]\!]\}$.

Main Result

Theorem

If the one-step satisfiability problem for a coalgebraic logic is in P, then the satisfiability problem of the μ-calculus over this logic is in $\textsc{ExpTime}$.

Some Complexity Results on Satisfiability

Previous work in the coalgebraic setting:

- [Cirstea et al. 2009]: Relying on suitable sets of one-step rules
- [Fontaine, Leal, Venema, 2010]: One-step satisfiability games

μ-calculus	one-step rules	one-step games	here
standard	ExpTime	2-ExpTime	ExpTime
alternating-time	ExpTime	2-ExpTime	ExpTime
probabilistic	ExpTime	2-ExpTime	ExpTime
graded	–	2-ExpTime	ExpTime
Presburger	–	2-ExpTime	ExpTime
probabilistic with polynomials	–	2-ExpTime	ExpTime
...	...	...	...

New examples

The Presburger μ-calculus

$T = \mathcal{B}$, $\Lambda = \{L_{a_1,\dots,a_m,b}, M_{a_1,\dots,a_m,b} \mid m, a_1, \dots a_m, b \in \mathbb{N}\}$,

$$[\![L_{a_1,\dots,a_m,b}]\!]_X(A_1,\dots,A_m) = \{\theta \in \mathcal{G}(X) \mid \sum_{1 \le i \le m} a_i \cdot \theta(A_i) > b\}$$
$$[\![M_{a_1,\dots,a_m,b}]\!]_X(A_1,\dots,A_m) = \{\theta \in \mathcal{G}(X) \mid \sum_{1 \le i \le m} a_i \cdot \theta(X \setminus A_i) \le b\}$$

The probabilistic μ-calculus with polynomial inequalities

$T = \mathcal{D}$, $\Lambda = \{L_{p,b}, M_{p,b} \mid b, m \in \mathbb{N}, p \in \mathbb{Q}_{>0}[X_1,\dots,X_m]\}$,

$$[\![L_{p,b}]\!]_X(A_1,\dots,A_m) = \{d \in \mathcal{D}(X) \mid p(d(A_1),\dots,d(A_m)) > b\}$$
$$[\![M_{p,b}]\!]_X(A_1,\dots,A_m) = \{d \in \mathcal{D}(X) \mid p(d(X \setminus A_1),\dots,d(X \setminus A_m)) \le b\}$$

Both one-step satisfiability problems are in P [Kupke, Pattinson, Schröder, 2015].

Corollary

The satisfiability problems of the following μ-calculi are in ExpTime*:*

- *the relational μ-calculus ($T = \mathcal{P}$),*
- *the alternating-time μ-calculus (concurrent game frame functor),*
- *with graded transition systems as models ($T = \mathcal{B}$):*
  - *the graded μ-calculus,*
  - *the Presburger μ-calculus,*
  - *the graded μ-calculus with polynomial inequalities*
- *with Markov chains as models ($T = \mathcal{D}$):*
  - *the (two-valued) probabilistic μ-calculus,*
  - *the (two-valued) probabilistic μ-calculus with polynomial inequalities*

Tracking Automata

Fix target formula χ, let $\mathbf{F}$ denote the *Fischer-Ladner closure* of χ.

Definition

Put selections $= \mathcal{P}(\Lambda(\mathbf{F}))$, $\Sigma =$ selections $\cup (\mathbf{F} \times \{0,1\})$. *Tracking automaton* for χ is nondeterministic parity automaton $A_\chi = (\mathbf{F}, \Sigma, \Delta, \chi, \alpha)$. Transition relation: for $\rho \in$ selections, $\Delta(\psi, \rho) = \{\psi_1 \in \mathbf{F} \mid \psi \in \rho \cap \Lambda(\{\psi_1\})\}$ and for $(\phi, b) \in \mathbf{F} \times \{0,1\}$,

$$\begin{aligned}\Delta(\psi,(\phi,b)) = &\{\phi_b \mid \psi = \phi = \phi_1 \vee \phi_2\} \cup \\ &\{\phi_i \mid \psi = \phi = \phi_1 \wedge \phi_2, i \in \{0,1\}\} \cup \\ &\{\phi_1[X \mapsto \eta X.\phi_1] \mid \psi = \phi = \eta X.\phi_1\}\end{aligned}$$

Priority function α assigns even numbers to least fixpoints, odd numbers to greatest fixpoints, according to *alternation depth*.

Tracking automaton A_χ accepts words that encode *bad branches*, i.e. those on which some least fixpoint is unfolded indefinitely; put $L(\mathsf{A}_\chi) =: \mathsf{BadBranch}$.

Determinize A_χ (e.g. through Büchi automata, using Safra/Piterman method) and complement it. Obtain deterministic parity automaton $\mathsf{B}_\chi = (D, \Sigma, \delta, q_0, \beta)$ with

$$L(\mathsf{B}_\chi) = \overline{L(\mathsf{A}_\chi)} = \overline{\mathsf{BadBranch}} =: \mathsf{GoodBranch},$$

with $|D| \leq \mathcal{O}(((nk)!)^2)$ where $n := |\chi|$ and k is alternation depth of χ and with $j := 2nk$ priorities. Have labeling function $l : D \to \mathcal{P}(\mathbf{F})$. For $v \in$ prestates, fix non-modal $\psi_v \in l(v)$.

Propagation

One-step propagation

For sets $G \subseteq D$ and $\mathbf{X} = X_1, \ldots, X_j \subseteq G^j$, we put

$$
\begin{aligned}
f(\mathbf{X}) =& \{v \in \text{prestates} \mid \exists b \in \{0,1\}.\, \delta(v,(\psi_v,b)) \in X_{\beta(v,(\psi_v,b))}\} \cup \\
& \{v \in \text{states} \mid T(\textstyle\bigcup_{1\leq i\leq j} X_i(v)) \cap [\![l(v)]\!]_1 \neq \emptyset\} \\
g(\mathbf{X}) =& \{v \in \text{prestates} \mid \forall b \in \{0,1\}.\, \delta(v,(\psi_v,b)) \notin X_{\beta(v,(\psi_v,b))}\} \cup \\
& \{v \in \text{states} \mid T(\textstyle\bigcup_{1\leq i\leq j} X_i(v)) \cap [\![l(v)]\!]_1 = \emptyset\},
\end{aligned}
$$

where $\beta(v,(\psi_v,b))$ abbreviates $\beta(v,(\psi_v,b),\delta(v,(\psi_v,b)))$ and where

$$X_i(v) = \{l(u) \in X_i \mid \exists \sigma \in \text{selections}.\, \delta(v,\sigma) = \{u\}, \beta(v,\sigma,u) = i\}.$$

Propagation for states is an instance of the one-step satisfiability problem.

Propagation, ctd.

Propagation

Given set $G \subseteq D$, put

$$\mathbf{E}_G = \eta_j X_j. \ldots \eta_2 X_2.\eta_1 X_1.f(\mathbf{X}) \qquad \mathbf{A}_G = \overline{\eta_j} X_j \ldots \overline{\eta_2} X_2.\overline{\eta_1} X_1.g(\mathbf{X}),$$

where $\mathbf{X} = X_1, \ldots, X_j$ for $X_i \subseteq G$, where $\eta_i = \mu$ for odd i, $\eta_i = \nu$ for even i and where $\overline{\eta} = \mu$ if $\eta = \nu$ and $\overline{\eta} = \nu$ if $\eta = \mu$.

Algorithm (global caching)

Input: Formula χ. Initialize $U = \{v_0\}$ and $G = \emptyset$.

1. Expansion: Choose $u \in U$, remove it from U and add it to G. If u is a pre-state, then add $\{\delta(u, \sigma) \mid \sigma \in \{\psi_u\} \times \{0, 1\}\}$ to U. If u is a state, then add $\{\delta(u, \sigma) \mid \sigma \in \text{selections}\}$ to U.
2. Optional propagation: Compute $\mathbf{E}_G$ and/or $\mathbf{A}_G$. If $v_0 \in \mathbf{E}_G$, then return ‘satisfiable‘, if $v_0 \in \mathbf{A}_G$, then return ‘unsatisfiable‘.
3. If $U \neq \emptyset$, then continue with step 1.
4. Final propagation: Compute $\mathbf{E}_G$. If $v_0 \in \mathbf{E}_G$, then return ‘satisfiable‘, otherwise return ‘unsatisfiable‘.

Lemma

Given a target formula χ with $|\chi| = n$ and $\text{ad}(\chi) = k$, the algorithm terminates and runs in time $\mathcal{O}(((nk)!)^{4nk} \cdot t(n, 2^n))$.

Results

Theorem

The algorithm returns ‘satisfiable‘ if and only if χ is satisfiable.

Corollary

Satisfiable coalgebraic μ-calculus formulas have models of size $\mathcal{O}(((nk)!)^2)$. In all our examples, the branching degree in models is polynomial in n (polysize one-step model property).

Satisfiability games [Fontaine, Leal, Venema, 2010]

Let the branching degree in models be polynomial.

Small satisfiability games

Put $Y = \{U \subseteq \text{selections} \mid |U| \text{ is polynomial in } n\}$. *Small satisfiability game* for χ: parity game (V, E, γ) with $V = D \cup D \times Y$,

$$
\begin{aligned}
E(d) &= \{\delta(d, (\psi_d, b)) \mid b \in \{0,1\}\} && \text{for pre-states } d \in D \\
E(d) &= \{(d, U) \mid U \in Y\} && \text{for states } d \in D \\
E(d, U) &= \{\delta(d, \rho) \mid \rho \in U\} && \text{for } (d, U) \in D \times Y
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma(d, \delta(d, (\psi_d, b))) &= \beta(d, (\psi_d, b), \delta(d, (\psi_d, b))) \\
\gamma(d, (d, U)) &= 0 \\
\gamma((d, U), \delta(d, \rho)) &= \beta(d, \rho, \delta(d, \rho))
\end{aligned}
$$

Satisfiability games, ctd.

Theorem

Player Eloise wins the small satisfiability game for χ if and only if χ is satisfiable.

In contrast to [Fontaine, Leal, Venema, 2010], the games have size $|V| \in 2^{\mathcal{O}(p(n))}$ where p is some polynomial.

Corollary

Deciding the winner of small satisfiability games is in ExpTime.

However, to obtain small satisfiability games, we require polynomial branching which has to be shown independently, e.g. by our new one-step satisfiability method.

Conclusion

Results:

- Satisfiability of a coalgebraic μ-calculus is in $\textsc{ExpTime}$ if the one-step satisfiability problem of the base logic is in P. One-step rules no longer required.
- All currently known one-step satisfiability problems are in P. In particular, we cover graded and probabilistic μ-calculi with polynomial inequalities.
- Bound on model size $\mathcal{O}(((nk)!)^2)$ for *all* coalgebraic μ-calculi.

Future:

- More involved examples?
- Satisfiability checking for the hybrid coalgebraic μ-calculus.

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