Game-Based Local Model Checking for the Coalgebraic μ-Calculus

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CONCUR 2019 – August 30 2019

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Model Checking for μ-Calculi

- Model checking for the μ-calculus = solving parity games.
- Coalgebraic μ-calculus [Cîrstea et al., 2011] instantiates to e.g. standard, graded, probabilistic, alternating-time μ-calculi.
- Model checking coalgebraic μ-calculus can be reduced to solving parity games, incurring exponential blowup [Cîrstea et al., 2011].

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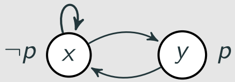
- Model checking coalgebraic μ-calculus can be reduced to solving parity games, incurring exponential blowup [Cîrstea et al., 2011].

We show:

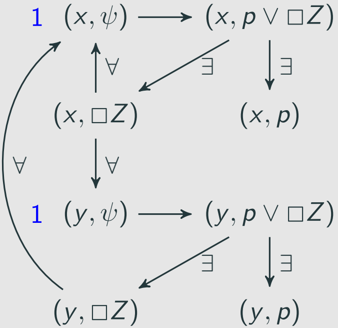
- For the monotone, alternating-time and graded (unary coding of grades) μ-calculi, exponential blowup can be avoided.
- Model checking for the coalgebraic μ-calculus is in $\mathrm{NP} \cap \mathrm{co\text{-}NP}$.
- Fixpoint iteration algorithm for parity games can be adapted to solve coalgebraic parity games, yielding bound $\mathcal{O}(p \cdot n^{\frac{d}{2}})$.

Model Checking for the μ-Calculus, example



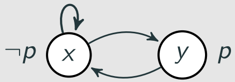
$$\psi := \mu Z.\, p \vee \Box Z$$

yields
parity game
$\rightsquigarrow$



Player Eloise wins node (x, ψ) if and only if ψ is satisfied at x.

Model Checking for the μ-Calculus, example



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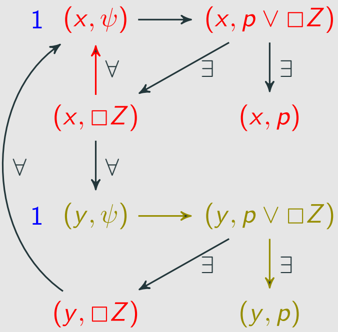

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The Coalgebraic μ-Calculus [Cîrstea et al., 2011]

Set **V** of fixpoint variables, set Λ of modalities, closed under duals.

Syntax:

$\phi, \psi := \top \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid X \mid \heartsuit\psi \mid \mu X.\psi \mid \nu X.\psi \qquad \heartsuit \in \Lambda, X \in \mathbf{V}$

Set-endofunctor T, *predicate lifting*[1] for $\heartsuit \in \Lambda$: natural transformation

$$[\![\heartsuit]\!] : \mathcal{Q} \to \mathcal{Q} \circ T^{op}$$

Assume monotonicity of predicate liftings ($A \subseteq B \Rightarrow [\![\heartsuit]\!]A \subseteq [\![\heartsuit]\!]B$)

[1] [Pattinson, 2007]

The Coalgebraic μ-Calculus [Cîrstea et al., 2011]

Semantics:

Models: *T-coalgebras* $(C, \xi : C \to TC)$, extension of formulas:

$$[\![X]\!]_\sigma = \sigma(X) \qquad [\![\heartsuit\psi]\!]_\sigma = \xi^{-1}[[\![\heartsuit]\!][\![\psi]\!]_\sigma]$$

$$[\![\mu X.\,\psi]\!]_\sigma = \mathsf{LFP}([\![\psi]\!]_\sigma^X) \qquad [\![\nu X.\,\psi]\!]_\sigma = \mathsf{GFP}([\![\psi]\!]_\sigma^X)$$

where $\sigma : \mathbf{V} \to \mathcal{P}(C)$, where $[\![\psi]\!]_\sigma^X(A) = [\![\psi]\!]_{\sigma[X \mapsto A]}$ for $A \subseteq C$ and where $(\sigma[X \mapsto A])(X) = A$, $(\sigma[X \mapsto A])(Y) = \sigma(Y)$ for $X \neq Y$.

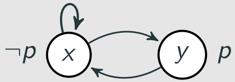
Hence $x \in [\![\heartsuit\psi]\!]$ if and only if $\xi(x) \in [\![\heartsuit]\!][\![\psi]\!]$.

Instances of the Coalgebraic μ-Calculus

- $T = \mathcal{P}$: transition systems $(C, \xi : C \to \mathcal{P}(C))$
  - modalities: $\Diamond, \Box$
  - standard μ-calculus, e.g. $\mu X.\, \psi \vee \Diamond X$
- $T = \mathcal{B}$ (bag functor): graded transition systems $(C, \xi : C \to \mathcal{B}(C))$
  - modalities: $\langle g \rangle$, $[g]$, $g \in \mathbb{N}$
  - graded μ-calculus[2], e.g. $\mu X.\, \psi \vee \langle 1 \rangle X$
- $T = \mathcal{G}$: concurrent game frames
  - Set N of agents, modalities $[D]$,$\langle D \rangle$, $D \subseteq N$
  - alternating-time μ-calculus[3], e.g. $\nu X.\, \psi \wedge [D]X$
- $T = \mathcal{D}$: Markov chains
  - modalities $\langle p \rangle$,$[p]$, $p \in \mathbb{Q} \cap [0,1]$
  - (two-valued) probabilistic μ-calculus, e.g. $\nu X.\, \psi \wedge \langle 0.5 \rangle X$

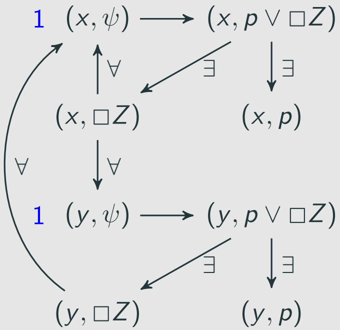
[2] [Kupferman et al.,2002]
[3] [Alur et al., 2002]

Model Checking for the Coalgebraic μ-Calculus, example

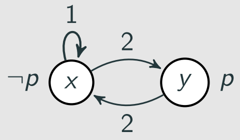


$\psi := \mu Z.\, p \vee \Box Z$

yields game $\rightsquigarrow$



Model Checking for the Coalgebraic μ-Calculus, example



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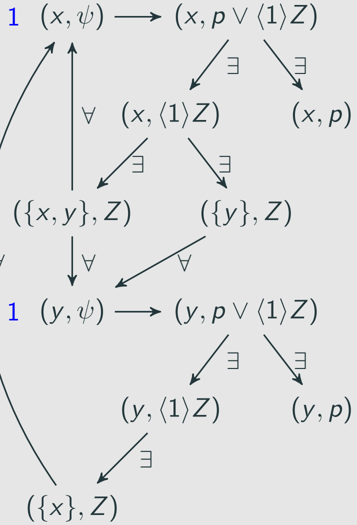

Cîrstea et al., 2011:

$E(v, \heartsuit\psi) = \{(U, \psi) \mid \xi(v) \in [\![\heartsuit]\!](U)\}$

$E(U, \psi) = \{(u, \psi) \mid u \in U\}$

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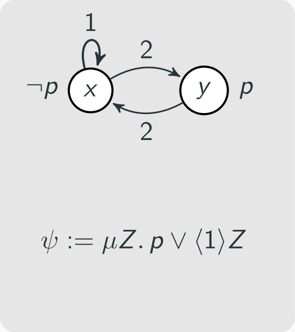

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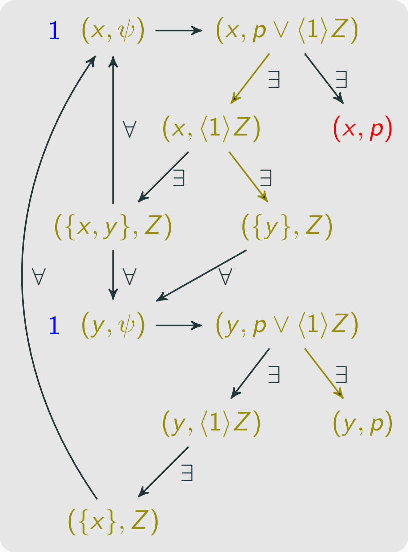

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exponentially many nodes! ☹

Containment in NP $\cap$ co-NP

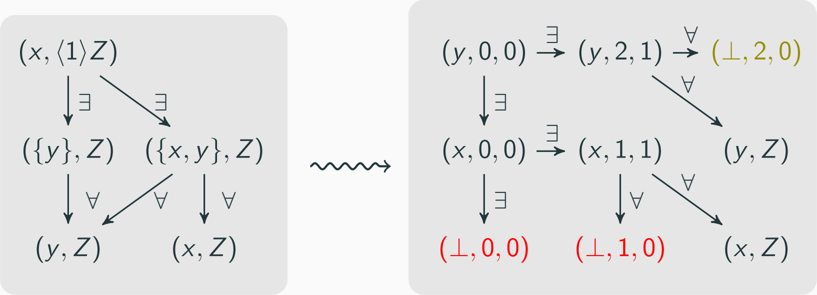
Theorem

If modalities for a coalgebraic logic can be evaluated in P, the model checking problem of the μ-calculus over this logic is in $\mathrm{NP} \cap \mathrm{co}\text{-}\mathrm{NP}$.

Proof: Logic is closed under negation, hence containment in NP suffices. Guess *polynomial*-sized witness for Eloise winning exponential-size game; verify witness in polynomial time by checking that all paths are even and that modalities are satisfied within witness.

Faster Model Checking for some Coalgebraic μ-Calculi

For some logics, smaller modal one-step games exist, e.g.



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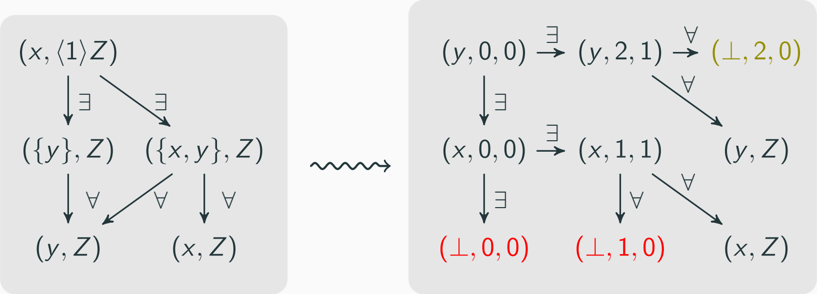

Theorem

The model checking problem of the serial, alternating-time and graded (with grades coded in unary) μ-calculi is in QP.

Main Result

Theorem

If the modalities of a coalgebraic logic can be evaluated in time p, the model checking problem of the μ-calculus over this logic can be solved in time $\mathcal{O}(p \cdot n^{\frac{d}{2}})$, ($d$ alternation depth, $n = |\chi| \cdot |C|$).

Proof: By reduction to computing a nested fixpoint.

Corollary

- The model checking problem of the probabilistic μ-calculus can be solved in time $\mathcal{O}((\text{size}(\chi))^2 \cdot n^{\frac{d}{2}+4})$.
- The model checking problem of the graded (with grades coded binary) μ-calculus can be solved in time $\mathcal{O}(\text{size}(\chi) \cdot n^{\frac{d}{2}+2})$.

Introducing: Coalgebraic Parity Games

Definition - Coalgebraic parity game:

T-coalgebra $(C, \xi : C \to TC)$ with mappings $\Omega : C \to \mathbb{N}$, $m : C \to \Lambda$.

Eloise *wins* node $c \in C$ if there is *even* graph (D, R) on C s.t.

$$\text{for all } d \in D,\ \xi(d) \in [\![m(d)]\!]R(d).$$

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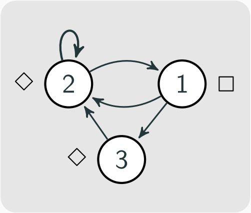
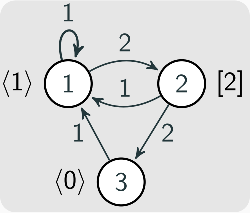
- $T = \mathcal{P}$: parity game for T is graph $(C, \xi : C \to \mathcal{P}(C))$ with priority map Ω and node ownership map $m : C \to \{\Diamond, \Box\}$.
- $T = \mathcal{D}$: parity game for T is Markov chain $(C, \xi : C \to \mathcal{D}(C))$ with priority map Ω and map $m : C \to \{\langle p \rangle, [p] \mid p \in \mathbb{Q} \cap [0, 1]\}$.

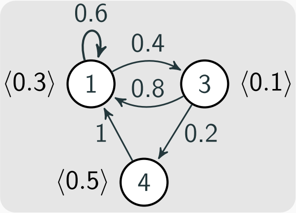
Coalgebraic Parity Games, examples



$T = \mathcal{P}$: standard

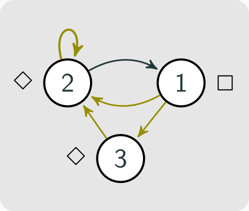


$T = \mathcal{B}$: graded



$T = \mathcal{D}$: probabilistic

Coalgebraic Parity Games, examples, strategies



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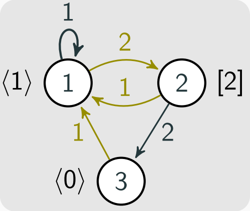

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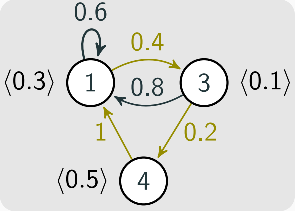

$T = \mathcal{D}$: probabilistic

Solving Coalgebraic Parity Games

Compute winning regions in coalgebraic parity games by fixpoint iteration:

Define $f : C \times \mathsf{Cl}(\psi)$ by

$$\begin{aligned} f(X_0, \dots, X_k) = & \{(v, \psi) \in V_\exists \mid \exists i. \Omega(v, \psi) = i, E(v, \psi) \cap X_i \neq \emptyset\} \cup \\ & \{(v, \psi) \in V_\forall \mid \exists i. \Omega(v, \psi) = i, E(v, \psi) \subseteq X_i\} \cup \\ & \{(v, \heartsuit\psi) \mid \xi(v) \in [\![\heartsuit]\!] X_0\} \end{aligned}$$

$$\text{Put } \mathsf{win}_\exists = \eta_k X_k . \eta_{k-1} X_{k-1} . \dots \eta_0 X_0 . f(X_0, \dots, X_k)$$

(η_i = GFP if i even, η_i = LFP if i odd.)

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(η_i = GFP if i even, η_i = LFP if i odd.)

Theorem:

We have $x \in [\![\psi]\!]$ if and only if $(x,\psi) \in \mathsf{win}_\exists$.

Enables local model checking: Start with initial node, expand nodes step by step, compute $\mathsf{win}_\exists$ (and dual set $\mathsf{win}_\forall$) at any point (partial game).

Conclusion

Results:

- Model checking problem of a coalgebraic μ-calculus is in $\mathrm{NP} \cap \mathrm{co\text{-}NP}$ if modalities can be evaluated in polynomial time.
- For the serial, alternating-time and graded (with grades coded in unary) μ-calculi, model checking is in QP.
- Reduction to coalgebraic parity games; solving these by fixpoint iteration yields time bound $\mathcal{O}(p \cdot n^{\frac{d}{2}})$.
- Implementation as part of COOL[4] solves coalgebraic parity games.

Current/future work:

- Compute nested fixpoints in quasipolynomial time.[5]
- Use Zielonka's algorithm to compute nested fixpoints.

[4] https://www8.cs.fau.de/research:software:cool

[5] https://arxiv.org/abs/1907.07020

References i

C. Cîrstea, C. Kupke, and D. Pattinson.
EXPTIME tableaux for the coalgebraic μ-calculus.
In *Computer Science Logic, CSL 2009*, volume 5771 of *LNCS*, pages 179–193. Springer, 2009.

G. Fontaine, R. A. Leal, and Y. Venema.
Automata for coalgebras: An approach using predicate liftings.
In S. Abramsky, C. Gavoille, C. Kirchner, F. Meyer auf der Heide, and P. G. Spirakis, editors, *ICALP (2)*, volume 6199 of *Lecture Notes in Computer Science*. Springer, 2010.

References ii

D. Hausmann and L. Schröder.
Global caching for the flat coalgebraic μ-calculus.
In *Temporal Representation and Reasoning, TIME 2015*, pages 121–143. IEEE Computer Society, 2015.

D. Hausmann, L. Schröder, and H.-P. Deifel.
Permutation games for the weakly aconjunctive μ-calculus.
In *Tools and Algorithms for the Construction and Analysis of Systems, TACAS 2018*, volume 10205–10206 of *LNCS*. Springer, to appear.

D. Hausmann, L. Schröder, and C. Egger.
Global caching for the alternation-free coalgebraic μ-calculus.
In *Concurrency Theory, CONCUR 2016*, volume 59 of *LIPIcs*, pages 34:1–34:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

References iii

D. Kozen.
A finite model theorem for the propositional μ-calculus.
Studia Logica, 47:233–241, 1988.

N. Piterman.
From nondeterministic büchi and streett automata to deterministic parity automata.
Logical Methods in Computer Science, 3(3):5, 2007.

S. Safra.
On the complexity of omega-automata.
In *Foundations of Computer Science, FOCS 1988*, pages 319–327. IEEE Computer Society, 1988.