Optimal Satisfiability Checking for Arithmetic μ-Calculi

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Satisfiability Checking for the μ-Calculus

Satisfiability problem of μ-calculus is ExpTime-complete [Kozen, 1983]

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2. Determinize, complement A, obtain DPA B accepting *good paths*.

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Player Eloise wins the game if and only if ψ is satisfiable.

Satisfiability Checking for the Coalgebraic μ-Calculus

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Our approach (coalgebraic satisfiability games):

Input: Fixpoint formula ψ

1. Construct NPA A, tracking formulas through potential models and accepting *bad paths* that contain some unsatisfied μ-formula.
2. Determinize, complement A, obtain DPA B accepting *good paths*.
3. Solve coalgebraic game over B, relying on one-step satisfiability.

Player Eloise wins the coalgebraic game if and only if ψ is satisfiable.

Coalgebraic One-Step Satisfiability

Set-endofunctor T, set Λ of (unary) modal operators

T-predicate lifting[1] for $\heartsuit \in \Lambda$: natural transformation

$$[\![\heartsuit]\!] : \mathcal{Q} \to \mathcal{Q} \circ T^{op}$$

Given set V, put $\Lambda(V) = \{\heartsuit a \mid \heartsuit \in \Lambda, a \in V\}$

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One-step satisfiability problem [Schröder, 2007]

Let $v \subseteq \Lambda(V)$ and $U \subseteq \mathcal{P}(V)$ with $a \neq b$ whenever $\heartsuit_1 a, \heartsuit_2 b \in v$. Put

$$[\![v]\!]_1 = \bigcap_{\heartsuit a \in v} [\![\heartsuit]\!]_U \{u \in U \mid a \in u\}$$

One-step satisfiability problem: Do we have $T(U) \cap [\![v]\!]_1 \neq \emptyset$?

Denote time to solve problem by $t(\text{size}(v), |V|)$, having $|U| \leq 2^{|V|}$.

[1] [Pattinson, 2007]

One-Step Satisfiability, example

Basic modal logic: $T = \mathcal{P}$, $\Lambda = \{\Diamond, \Box\}$; for sets X, $A \subseteq X$, put

$$[\![\Diamond]\!]_X(A) = \{B \in \mathcal{P}(X) \mid A \cap B \neq \emptyset\} \quad [\![\Box]\!]_X(A) = \{B \in \mathcal{P}(X) \mid B \subseteq A\}$$

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Example

Let $V = \{b, c, d\}$, $U = \{\{b, d\}, \{c, d\}\}$

Do we have

$$\mathcal{P}(U) \cap [\![\Diamond]\!]_U\{b\} \cap [\![\Diamond]\!]_U\{c\} \cap [\![\Box]\!]_U\{d\} \neq \emptyset \ ?$$

$v = \{\Diamond b, \Diamond c, \Box d\}$

b, d c, d

In general: $t(\mathrm{size}(v), |V|) \in \mathcal{O}(\mathrm{size}(v) \cdot 2^{|V|})$

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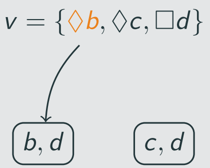

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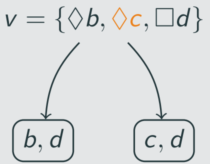

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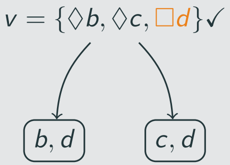

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Coalgebraic One-Step Satisfiability, example ctd.

Graded modal logic: *bag functor* $T(X) = \mathcal{B}(X) = \{\theta : X \to \mathbb{N}\}$

$\Lambda = \{\langle k \rangle, [k] \mid k \in \mathbb{N}\}$. For sets X, $A \subseteq X$, put $\theta(A) = \sum_{a \in A} \theta(a)$ and

$[\![\langle k \rangle]\!]_X(A) = \{\theta \in \mathcal{B}(X) \mid \theta(A) > k\} \quad [\![[k]]\!]_X(A) = \{\theta \in \mathcal{B}(X) \mid \theta(X \setminus A) \leq k\}$

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[Kupferman, Sattler, Vardi, 2002]: $t(\text{size}(v), |V|) \in \mathcal{O}((2^{\text{size}(v)+1} + 2)^{|V|})$

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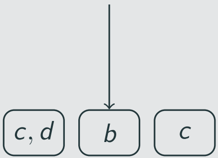

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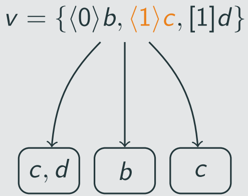

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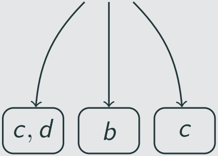

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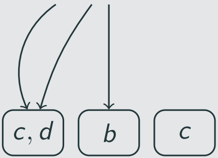

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The Coalgebraic μ-Calculus [Cirstea et al., 2009]

Assume set **V** of fixpoint variables

Syntax:

$\phi, \psi := \top \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid X \mid \heartsuit\psi \mid \mu X.\psi \mid \nu X.\psi \qquad \heartsuit \in \Lambda, X \in \mathbf{V}$

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Assume monotonicity of predicate liftings ($A \subseteq B \Rightarrow [\![\heartsuit]\!]A \subseteq [\![\heartsuit]\!]B$)

Semantics:

Models: *T-coalgebras* $(W, \xi : W \to TW)$, extension of formulas:

$$[\![X]\!]_\sigma = \sigma(X) \qquad [\![\heartsuit\psi]\!]_\sigma = \xi^{-1}[[\![\heartsuit]\!]_W [\![\psi]\!]_\sigma]$$

$$[\![\mu X.\,\psi]\!]_\sigma = \mathsf{LFP}([\![\psi]\!]_\sigma^X) \qquad [\![\nu X.\,\psi]\!]_\sigma = \mathsf{GFP}([\![\psi]\!]_\sigma^X)$$

where $\sigma : \mathbf{V} \to \mathcal{P}(W)$, where $[\![\psi]\!]_\sigma^X(A) = [\![\psi]\!]_{\sigma[X \mapsto A]}$ for $A \subseteq W$ and where $(\sigma[X \mapsto A])(X) = A$, $(\sigma[X \mapsto A])(Y) = \sigma(Y)$ for $X \neq Y$.

Observe: $\xi(x) \in TW \cap \bigcap_{\heartsuit\psi \in I(x)} [\![\heartsuit]\!]_W [\![\psi]\!]$ where $I(x) = \{\heartsuit\psi \mid x \in [\![\heartsuit\psi]\!]\}$

Instances of the Coalgebraic μ-Calculus

- Standard μ-calculus: $T = \mathcal{P}$, e.g. $\mu X.\, \psi \vee \Diamond X$
  $t(\mathsf{size}(v), |V|) \in \mathcal{O}(|v|^2 \cdot 2^{|V|})$
- Graded μ-calculus: $T = \mathcal{B}$, e.g. $\mu X.\, \psi \vee \langle 1 \rangle X$
  $t(\mathsf{size}(v), |V|) \in \mathcal{O}((2^{\mathsf{size}(v)+1} + 2)^{|V|})$ [2]
- Alternating-time μ-calculus: $T = \mathcal{G}$, e.g. $\nu X.\, \psi \wedge [D]X$
  $t(\mathsf{size}(v), |V|) \in \mathcal{O}(2^{p(\mathsf{size}(v)+|V|)})$ [3]
- (Two-valued) probabilistic μ-calculus: $T = \mathcal{D}$, e.g. $\nu X.\, \psi \wedge \langle 0.5 \rangle X$
  $t(\mathsf{size}(v), |V|) \in \mathcal{O}(2^{p(\mathsf{size}(v)+|V|)})$ [4]

[2] [Kupferman, Sattler, Vardi, 2002]
[3] [Cirstea, Kupke, Pattinson, 2009]
[4] [Cirstea, Kupke, Pattinson, 2009]

New instances of the Coalgebraic μ-Calculus

Graded μ-calculus with polynomial inequalities

$T = \mathcal{B}$, $\Lambda = \{L_{p,b}, M_{p,b} \mid b, m \in \mathbb{N}, p \in \mathbb{N}_{>0}[X_1, \dots, X_m]\}$,

$$[\![L_{p,b}]\!]_X(A_1, \dots, A_m) = \{\theta \in \mathcal{G}(X) \mid p(\theta(A_1), \dots, \theta(A_m)) > b\}$$
$$[\![M_{p,b}]\!]_X(A_1, \dots, A_m) = \{\theta \in \mathcal{G}(X) \mid p(\theta(X \setminus A_1), \dots, \theta(X \setminus A_m)) \leq b\}$$

E.g. $\mu Y.(\psi \vee L_{2X_1+(X_2)^2,2}(p \wedge Y, q \wedge Y))$

Probabilistic μ-calculus with polynomial inequalities

$T = \mathcal{D}$, $\Lambda = \{L_{p,b}, M_{p,b} \mid b, m \in \mathbb{N}, p \in \mathbb{Q}_{>0}[X_1, \dots, X_m]\}$,

$$[\![L_{p,b}]\!]_X(A_1, \dots, A_m) = \{d \in \mathcal{D}(X) \mid p(d(A_1), \dots, d(A_m)) > b\}$$
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One-step sat. problems can be solved in exponential time [Kupke et al., 2015]

Main Result

Theorem

If the one-step satisfiability problem for a coalgebraic logic can be solved in time $t(\mathsf{size}(v), |V|)$ exponential in $\mathsf{size}(v) + |V|$ for inputs $v \subseteq \Lambda(V)$, $U \subseteq \mathcal{P}(V)$, then the satisfiability problem of the μ-calculus over this logic is in $\textsc{ExpTime}$.

Some Complexity Results on Satisfiability

Previous work in the coalgebraic setting:

- [Cirstea et al. 2009]: Relying on tractable sets of one-step rules
- [Fontaine, Leal, Venema, 2010]: One-step satisfiability games

μ-calculus	one-step rules	one-step games	here
standard ($\mathcal{P}$)	ExpTime	2-ExpTime	ExpTime
alternating-time ($\mathcal{G}$)	ExpTime	2-ExpTime	ExpTime
probabilistic ($\mathcal{D}$)	ExpTime	2-ExpTime	ExpTime
graded ($\mathcal{B}$)	–	2-ExpTime	ExpTime
graded with polynomials	–	2-ExpTime	ExpTime
probabilistic with polynomials	–	2-ExpTime	ExpTime
...	...	...	...

[Kupferman, Sattler, Vardi, 2002] for graded μ-calculus: ExpTime

Tracking Automata

Fix target formula χ, let **F** denote the *Fischer-Ladner closure* of χ.

Tracking automaton for χ:

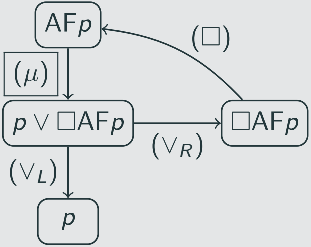
- Nondeterministic parity automaton
- State set **F**
- Transitions according to syntax graph of χ
- Priorities at *edges*, according to *alternation depth*

Tracking automata

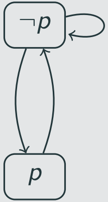
Example

Tracking automaton $\mathcal{A}_\psi$ for

$$\psi = \mathsf{AF}p = \mu X.\, p \vee \Box X:$$



Non-model for ψ:



$(\mu)(\vee_L)$ ↯

$(\mu)(\vee_R)(\Box)(\mu)(\vee_L)$ ↯

$(\mu)(\vee_R)(\Box)(\mu)(\vee_R)(\Box)\dots$

Tracking automaton A_χ accepts words that encode *bad paths* on which some least fixpoint is unfolded indefinitely; put $L(\mathsf{A}_\chi) =: \mathrm{BadPaths}$.

Determinize A_χ (e.g. through Büchi automata, using Safra/Piterman method) and complement. Obtain DPA $\mathsf{B}_\chi = (D, \Sigma, \delta, q_0, \beta)$ with

$$L(\mathsf{B}_\chi) = \overline{L(\mathsf{A}_\chi)} = \overline{\mathrm{BadPaths}} =: \mathrm{GoodPaths},$$

and $|D| \in \mathcal{O}(((nk)!)^2)$ where $n := |\chi|$ and k is alternation depth of χ and with $j := 2nk$ priorities. Define labeling function $l : D \to \mathcal{P}(\mathbf{F})$.

states $:= \{v \in D \mid l(v) \subseteq \Lambda(\mathbf{F})\}$ prestates $:= D \setminus$ states

Propagation

Given $v \in$ prestates, fix non-modal $\psi_v \in l(v)$.

One-step propagation

For sets $\mathbf{X} = X_1, \dots, X_j \subseteq D^j$, put

$$f(\mathbf{X}) = \{v \in \text{prestates} \mid \exists b \in \{0,1\}.\, \delta(v,(\psi_v,b)) \in X_{\beta(v,(\psi_v,b))}\} \cup$$
$$\{v \in \text{states} \mid l(v) \text{ is one-step satisfiable in } \bigcup_{1 \le i \le j} X_i(v)\}$$

where $\beta(v,(\psi_v,b))$ abbreviates $\beta(v,(\psi_v,b),\delta(v,(\psi_v,b)))$ and where

$$X_i(v) = \{l(u) \in X_i \mid \exists \sigma \in \text{selections}.\,(v,\sigma,u) \in \delta_i\}.$$

Propagation, ctd.

Propagation

Given sets $\mathbf{X} = X_1, \dots, X_j \subseteq D^j$, put

$$\mathbf{E} = \eta_j X_j. \dots \eta_2 X_2.\eta_1 X_1.f(\mathbf{X}) \qquad \mathbf{A} = \overline{\eta_j} X_j \dots \overline{\eta_2} X_2.\overline{\eta_1} X_1.\overline{f}(\mathbf{X}),$$

where $\eta_i = \mu$ for odd i, $\eta_i = \nu$ for even i and where $\overline{\nu} = \mu$ and $\overline{\mu} = \nu$.

Computes winning regions of *coalgebraic parity game*

Parity games vs. coalgebraic games

Parity game with d priorities	Coalgebraic game
$\psi(\mathbf{X}) = (\exists \wedge (\bigvee_{i \leq d} (P_i \wedge \Diamond X_i))) \vee (\forall \wedge (\bigvee_{i \leq d} (P_i \wedge \Box X_i)))$	$f(\mathbf{X}) = \{v \in \text{prestates} \mid \exists b \in \{0,1\}.\ \delta(v, (\psi_v, b)) \in X_{\beta(v,(\psi_v,b))}\} \cup \{v \in \text{states} \mid I(v) \text{ is one-step satisfiable in } \bigcup_{1 \leq i \leq j} X_i(v)\}$
$\text{win}_\exists = \eta_d X_d. \ldots \eta_2 X_2. \eta_1 X_1. \psi(\mathbf{X})$	$\mathbf{E} = \eta_j X_j. \ldots \eta_2 X_2. \eta_1 X_1. f(\mathbf{X})$
$\text{win}_\forall = \overline{\eta_d} X_d \ldots \overline{\eta_2} X_2. \overline{\eta_1} X_1. \neg\psi(\mathbf{X})$	$\mathbf{A} = \overline{\eta_j} X_j \ldots \overline{\eta_2} X_2. \overline{\eta_1} X_1. \overline{f}(\mathbf{X})$

Results

Theorem

We have $q_0 \in \mathbf{E}$ if and only if χ is satisfiable.

Lemma

Given target formula χ with $|\chi| = n$ and alternation depth k, $\mathbf{E}$ can be computed in time $\mathcal{O}(((nk)!)^{4nk} \cdot t(\text{size}(\chi), n))$.

Corollary

Satisfiable coalgebraic μ-calculus formulas have models of size $\mathcal{O}(((nk)!)^2)$. In all our examples, the branching degree in models is polynomial in n (polysize one-step model property).

Conclusion

Results:

- Satisfiability of a coalgebraic μ-calculus is in ExpTime if the one-step satisfiability problem of the base logic can be solved in exponential time. One-step tableau rules no longer required.
- Currently known examples of one-step satisfiability problems can be solved in exponential time. In particular: graded and probabilistic μ-calculi with polynomial inequalities
- Upper bound $\mathcal{O}(((nk)!)^2)$ on model size for *all* coalgebraic μ-calculi (implicitly also in [Cirstea, Kupke, Pattinson, 2009])

Future:

- Solving coalgebraic games in quasipolynomial time?

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