Reachability Analysis of Generalized Polygonal Hybrid Systems (GSPDIs)

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Reachability Analysis of GSPDIs

- Hybrid System: combines discrete and continuous dynamics
- Examples: thermostat, robot, chemical reaction

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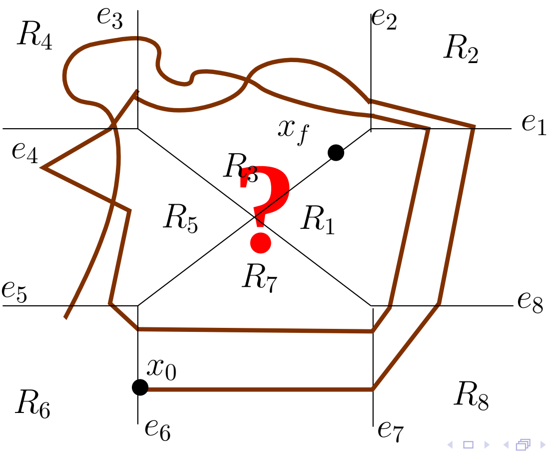

Outline

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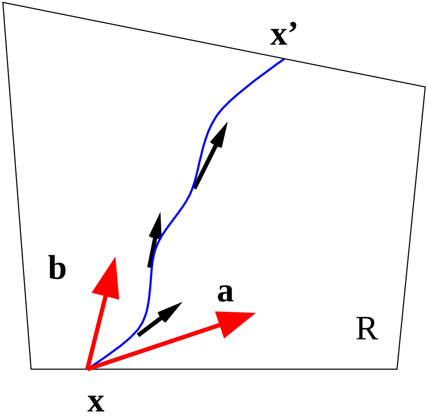
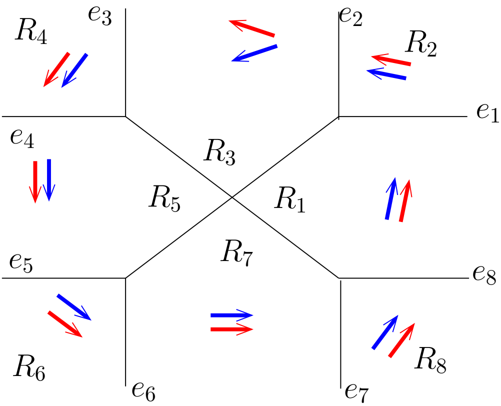
- A constant differential inclusion (angle between vectors **a** and **b**): $\dot{x} \in \angle_{\mathbf{a}}^{\mathbf{b}}$



Polygonal Hybrid Systems (SPDIs)

- A finite partition of (a subset of) the plane into convex polygonal sets (regions)
- Dynamics given by the angle determined by two vectors: $\dot{x} \in \angle_{\mathbf{a}}^{\mathbf{b}}$



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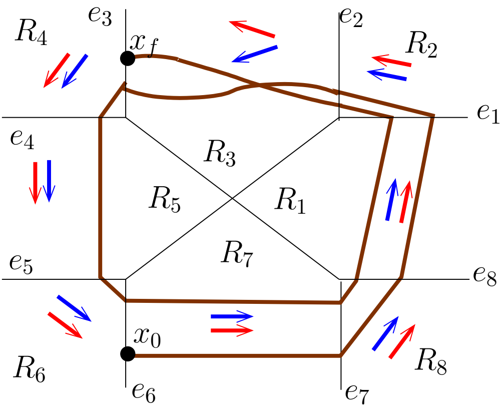

Goodness Assumption

The dynamics of an SPDI only allows trajectories traversing any edge only in one direction

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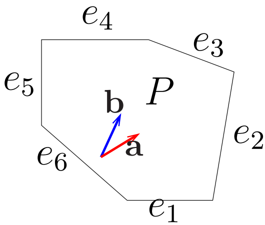
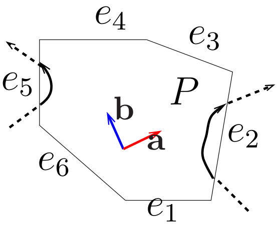


Good region



Bad region

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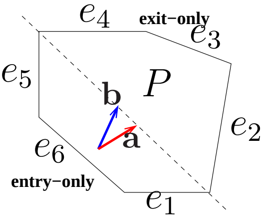

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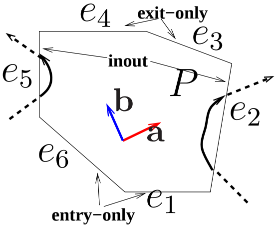

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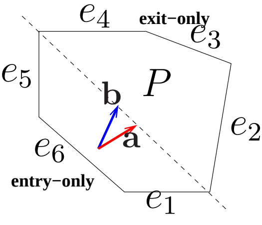

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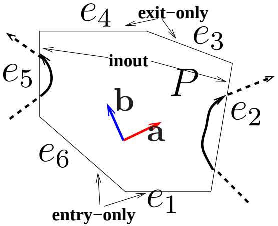

Bad region

Theorem

Under the goodness assumption, reachability for SPDIs is **decidable**

Motivation
Use of SPDIs for approximating non-linear differential equations

Example

Pendulum with friction coefficient k, mass M, pendulum length R and gravitational constant g. Behaviour: $\dot{x} = y$ and $\dot{y} = -\frac{ky}{MR^2} - \frac{g\sin(x)}{R}$

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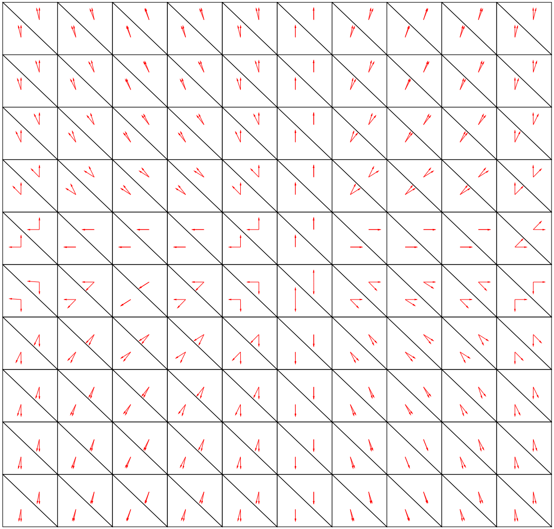
- Triangulation of the plane: Huge number of regions
- Need to reduce the complexity ... without too much overhead
  - Relax Goodness: GSPDI

Motivation
Use of SPDIs for approximating non-linear differential equations



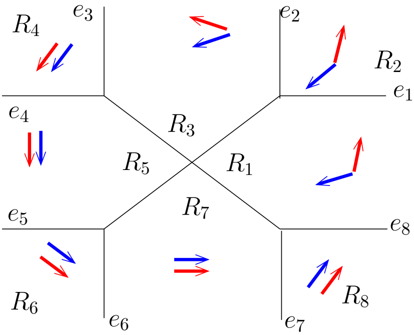
Outline

1. Polygonal Hybrid Systems (SPDIs) and Motivation
2. Generalized Polygonal Hybrid Systems (GSPDIs)
3. Reachability Analysis of GSPDIs

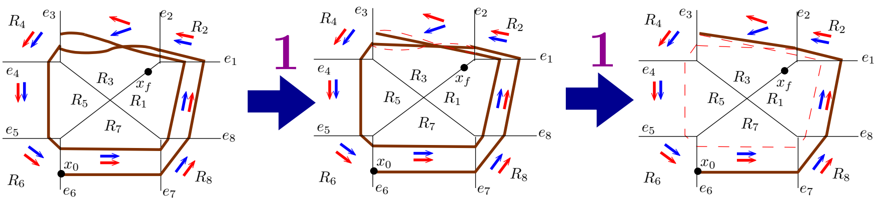
GSPDI: Generalized SPDI

Definition

An SPDI without the goodness assumption is called a GSPDI



Why Goodness is Good



2

$$e_6e_7e_8e_1e_2e_3$$
$$(e_6e_7e_8e_1e_2e_3e_4e_5)^5e_6e_7e_8e_9$$

3

$$e_6e_7e_8e_1e_2e_3$$
$$e_6e_7e_8(e_1e_2e_3e_4e_5e_6e_7e_8)^5e_9$$

4

$$e_6e_7e_8e_1e_2e_3$$
$$e_6e_7e_8(e_1e_2e_3e_4e_5e_6e_7e_8)^*e_9$$

$\Leftrightarrow$

r

$r_1\ s_1^*\ r_2$

Why Goodness is Good

Theorem

An edge-signature $\sigma = e_1 \ldots e_p$ can always be abstracted into types of signatures of the form $\sigma_{\mathcal{A}} = \mathbf{r_1 s_1^*} \ldots \mathbf{r_n}\ \mathbf{s_n^* r_{n+1}}$, where r_i is a sequence of pairwise different edges and all s_i are disjoint simple cycle.

There are **finitely many** *type of signatures.*

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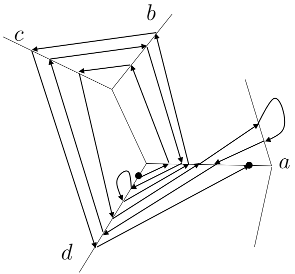
There are **finitely many** *type of signatures.*

Many proofs (decidability, soundess, completeness) depend on the goodness assumption

Problems when Relaxing Goodness

Finiteness argument for types of signature is broken for GSPDIs



Type of signature:

$$d \quad (\mathbf{abcd})^* \quad (dcba)^* \quad (\mathbf{abcd})^* \quad a$$

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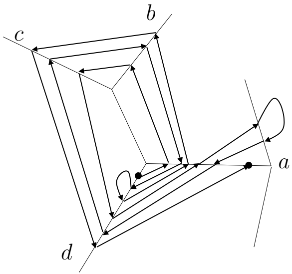

Type of signature:

$d \quad (\mathbf{abcd})^* \quad (dcba)^* \quad (\mathbf{abcd})^* \quad a$

Challenge: Reachability analysis of GSPDIs

- Reduce GSPDI reachability to SPDI reachability; or
- Provide a completely new decidability proof for GSPDI.

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Getting a Decision Algorithm for GSPDIs Based on that of SPDIs

1. It is enough to consider trajectories without self-crossing
2. It is possible to eliminate all inout edges, preserving reachability
3. It is possible to eliminate all sliding edges, preserving reachability
4. Re-state and prove some results on SPDI reachability useful to GPSDI reachability analysis
5. Prove soundness and termination

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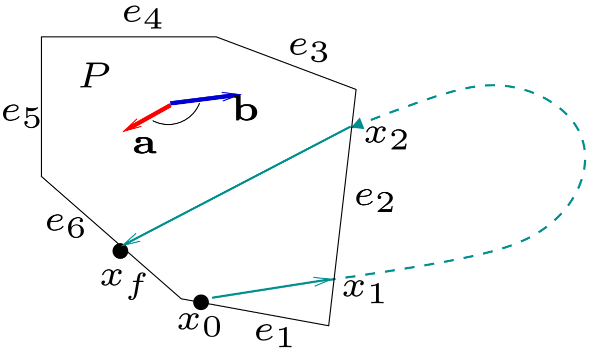
No decision algorithm for reachability of GSPDIs...

We will give a semi-test algorithm!

It is not possible to eliminate inout edges



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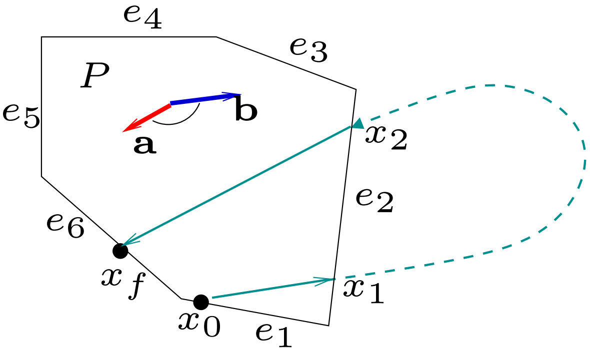

Theorem

There is **no** *structure-preserving reduction from the GSPDI reachability problem to the SPDI reachability problem.*

A Semi-Test Algorithm for GSPDIs

$\mathcal{H}_{red} = \{\mathcal{H}_1, \ldots, H_n\}$: all possible underlying SPDIs, after fixing all the inout edges of $\mathcal{H}$ as entry-only or exit-only

Algorithm

1. Detect all the inout edges;
2. Generate the set of SPDIs $\mathcal{H}_{red} = \{\mathcal{H}_1, \ldots, H_n\}$;
3. Apply the reachability algorithm for SPDIs to each $\mathcal{H}_i$ $(1 \leq i \leq n)$, $\mathbf{Reach}_{SPDI}(\mathcal{H}_i, \mathbf{x}_0, \mathbf{x}_f)$.
4. If there exists at least one SPDI $\mathcal{H}_i \in \mathcal{H}_{red}$ such that $\mathbf{Reach}_{SPDI}(\mathcal{H}_i, \mathbf{x}_0, \mathbf{x}_f) = \text{Yes}$ then $\mathbf{Reach}(\mathcal{H}, \mathbf{x}_0, \mathbf{x}_f) = \text{Yes}$, otherwise we do not know.

A Semi-Test Algorithm for GSPDIs
1. It is enough to consider trajectories without self-crossing

- Idem as for SPDIs

A Semi-Test Algorithm for GSPDIs

3. It is possible to eliminate all sliding edges, preserving reachability

Theorem

If there exists a sliding trajectory segment from points $\mathbf{x}_0 \in e_0$ *to* $\mathbf{x}_f \in e_f$ *then there always exists a non-sliding trajectory segment between them.*

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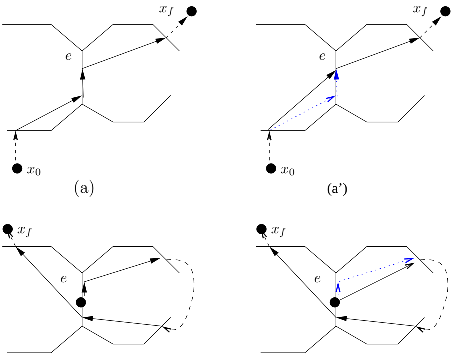

A Semi-Test Algorithm for GSPDIs

4. Re-state and prove some results on SPDI reachability useful to GPSDI reachability analysis

1. Redefine the edge-to-edge successor function
2. Rephrase topologically results using contiguity between entry-only and exit-only edges
3. Re-prove soundess of some algorithms

Theorem

Given a GSPDI $\mathcal{H}$*,* $\mathbf{Reach}(\mathcal{H}, \mathbf{x}_0, \mathbf{x}_f) = \text{Yes}$ *if* $\mathbf{Reach}_{SPDI}(\mathcal{H}_i, \mathbf{x}_0, \mathbf{x}_f) = \text{Yes}$ *for some* $\mathcal{H}_i \in \mathcal{H}_{red}$*. On the other hand,* $\mathbf{Reach}(\mathcal{H}, \mathbf{x}_0, \mathbf{x}_f)$ *is inconclusive if for all* $\mathcal{H}_i \in \mathcal{H}_{red}$*,* $\mathbf{Reach}_{SPDI}(\mathcal{H}_i, \mathbf{x}_0, \mathbf{x}_f) = \text{No}$*.*

The algorithm always terminate.

Final Remarks

News at submission to SAC-SV

- A semi-test for reachability analysis of GSPDIs (this presentation)

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Latest news at SAC-SV

- Implementation of reachability algorithm and application (current work)