Relaxing Goodness is Still Good

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Reachability Analysis of GSPDIs

- Hybrid System: combines discrete and continuous dynamics
- Examples: thermostat, robot, chemical reaction

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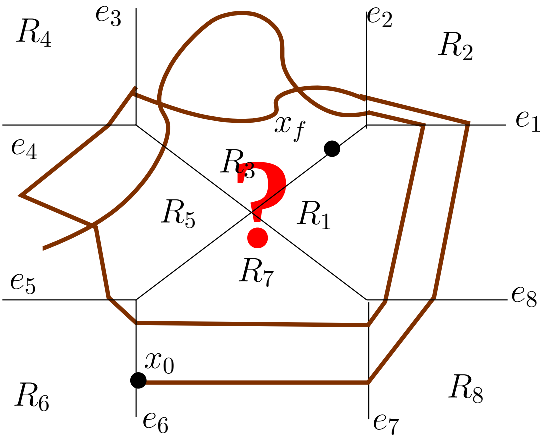

Outline

1. Polygonal Hybrid Systems (SPDIs)
2. Generalized Polygonal Hybrid Systems (GSPDIs)
3. Reachability Analysis of GSPDIs

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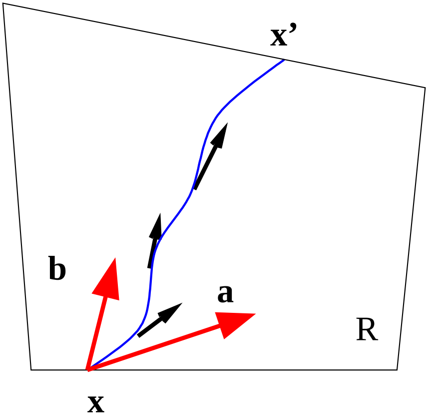
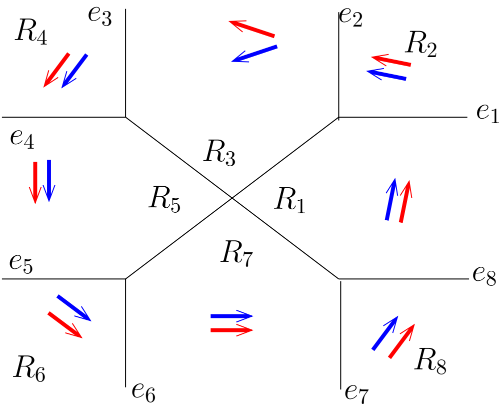
- A constant differential inclusion (angle between vectors **a** and **b**): $\dot{x} \in \angle_{\mathbf{a}}^{\mathbf{b}}$



Polygonal Hybrid Systems (SPDIs)

- A finite partition of (a subset of) the plane into convex polygonal sets (regions)
- Dynamics given by the angle determined by two vectors: $\dot{x} \in \angle_{\mathbf{a}}^{\mathbf{b}}$



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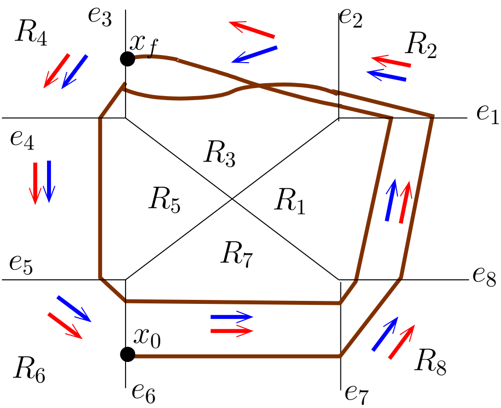

Goodness Assumption

The dynamics of an SPDI only allows trajectories traversing any edge only in one direction

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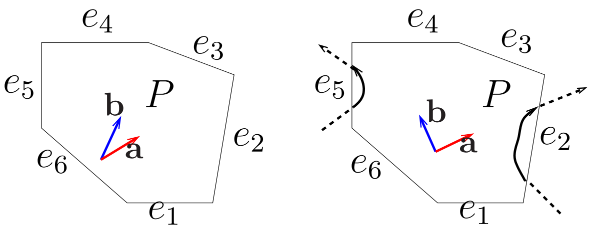

Good region

Bad region

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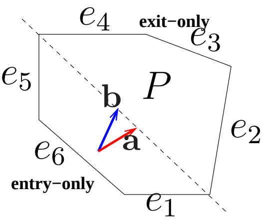

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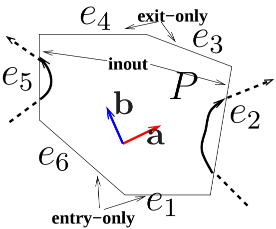

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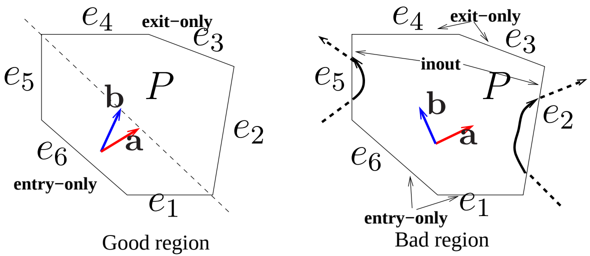


Theorem

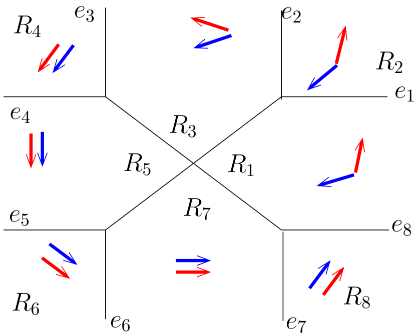
Under the goodness assumption, reachability for SPDIs is **decidable**

Outline

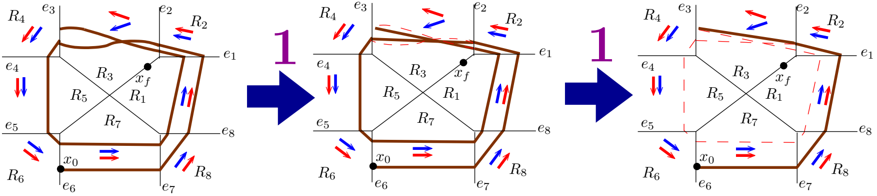
GSPDI: Generalized SPDI

Definition

An SPDI without the goodness assumption is called a GSPDI



Why Goodness is Good



2

$$(e_6e_7e_8e_1e_2e_3e_4e_5)^5e_6e_7e_8e_9$$

3

$$e_6e_7e_8(e_1e_2e_3e_4e_5e_6e_7e_8)^5e_9$$

4

$$e_6e_7e_8(e_1e_2e_3e_4e_5e_6e_7e_8)^*e_9 \quad \Longleftrightarrow \quad r_1\ s_1^*\ r_2$$

Why Goodness is Good

Lemma (Truncated Affine Multivalued Functions (TAMF))

Successors are positive TAMFs:

$$\text{Succ}(x) = F(\{x\} \cap S) \cap J$$

where $F(x) = [a_1x + b_1, a_2x + b_2]$ *with* $a_1, a_2 > 0$

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Theorem

An edge-signature $\sigma = e_1 \dots e_p$ *can always be abstracted into types of signatures of the form* $\sigma_{\mathcal{A}} = \mathbf{r_1 s_1^* \dots r_n \; s_n^* r_{n+1}}$*, where* r_i *is a sequence of pairwise different edges and all* s_i *are disjoint simple cycle.*

There are **finitely many** *type of signatures.*

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Many proofs (decidability, soundess, completeness) depend on the goodness assumption

Problems when Relaxing Goodness

- Successors are not longer guaranteed to be positive TAMFs

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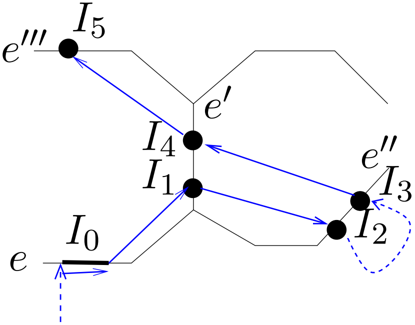
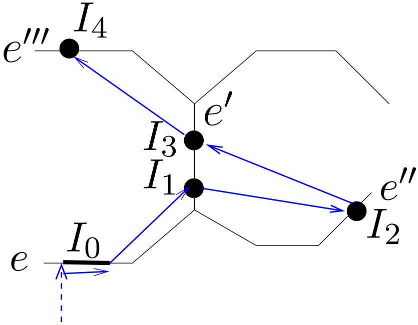
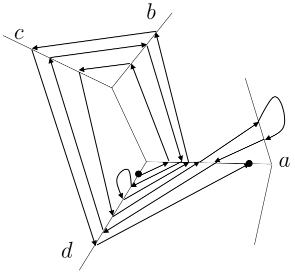
- Proper inout edges and bounces:





Problems when Relaxing Goodness

Finiteness argument for types of signature is broken for GSPDIs



Type of signature:

$d \quad (\mathbf{abcd})^* \quad (dcba)^* \quad (\mathbf{abcd})^* \quad a$

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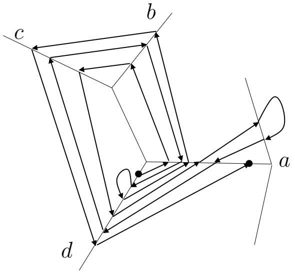

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Decidability of Reachability for GSPDIs

- Reduce GSPDI reachability to SPDI reachability (not possible: [SAC'08])
- **A decidability proof extending that of SPDI**

Outline

Decidability for GSPDI

1. Prove it is enough to consider trajectories without self-crossing
2. Redefine successors and prove them to be positive TAMFs
3. Duplicate inout edges (e and e^{-1}) s.t. each edge is traversed in one direction only
4. Prove that signatures not containing **bounces** (sub-sequences ee^{-1}) behave as for SPDIs
5. Show that we can treat signatures containing bounces
6. Prove soundness and termination

1. It is enough to consider trajectories without self-crossing

- Idem as for SPDIs

2. Successors are positive TAMFs

Lemma

In GSPDIs, successors can always be written as positive TAMFs.

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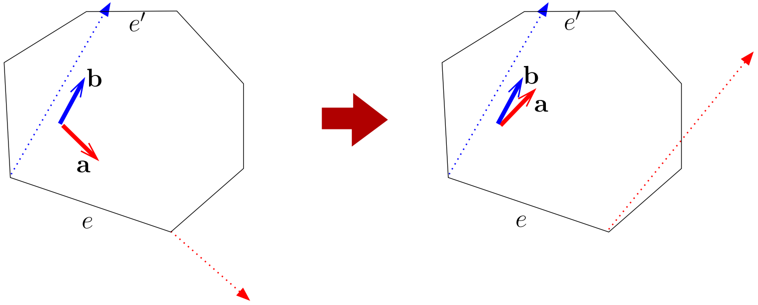
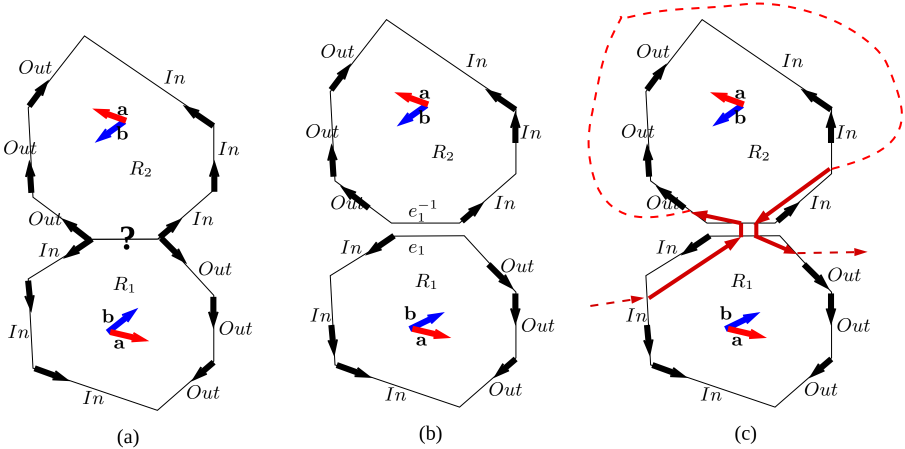
In GSPDIs, successors can always be written as positive TAMFs.

- We apply a transformation to successors of regions containing inout edges



3. Duplicate inout edges (e and e^{-1})



4. Signatures without bounces behaves as for SPDIs

Lemma

For any two edges e_0 *and* e_1, $\mathrm{Succ}_{e_0 e_1}$ *is always a positive TAMF, whenever* $e_1 \neq e_0^{-1}$. □

5. We can treat signatures containing bounces
Flip

Lemma

Let $\mathrm{Flip}[l, u] = [1 - u,\ 1 - l]$. *Then*

$$\mathrm{Succ}_{ee^{-1}} = \mathrm{Flip}$$

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Lemma

Composing Flip *with an inverted TAMF gives a positive TAMF and an inverted TAMF if we compose it with a positive TAMF.*

5. We can treat signatures containing bounces

Signatures with even number of bounces

Corollary

Any signature with an **even** *number of bounces has its behaviour characterised by a positive TAMF.*

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Lemma

Given a simple cycle σ containing an even number of bounces, its iterated behaviour can be calculated as for SPDIs.

5. We can treat signatures containing bounces

Signatures with odd number of bounces

Lemma

Given a simple cycle s with an **odd** *number of bounces, we can calculate the limit of its iterated behaviour without iterating.*

6. Soundness, Completeness and Termination

Algorithm

1. Pre-processing:
   - Redefine successors
   - Transform $\mathcal{H}$: e and e^{-1} as different edges
2. Generate the finite set of types of signatures $\Sigma = \{\sigma_0, \ldots, \sigma_n\}$
   - Simple cycles are all distinct
3. Apply $\textbf{Reach}_{\sigma_i}(\mathbf{x}_0, \mathbf{x}_f)$ for each $\sigma_i \in \Sigma$
4. $\textbf{Reach}(\mathcal{H}, \mathbf{x}_0, \mathbf{x}_f) =$ Yes iff for some $\sigma_i \in \Sigma$, $\textbf{Reach}_{\sigma_i}(\mathbf{x}_0, \mathbf{x}_f) =$ Yes.

Theorem

$\textbf{Reach}(\mathcal{H}, \mathbf{x}_0, \mathbf{x}_f)$ *is a sound and complete algorithm calculating GSPDI reachability. The algorithm always terminates.*

Final Remarks

This presentation

- A (constructive) proof that reachability is decidable for GSPDIs
  - Reuse of reachability algorithm for SPDI
  - Acceleration of simple cycles

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Current Work

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Current Work

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Future Work

- Application to analysis of nonlinear differential equations

Example

Pendulum with friction coefficient k, mass M, pendulum length R and gravitational constant g. Behaviour: $\dot{x} = y$ and $\dot{y} = -\frac{ky}{MR^2} - \frac{g\sin(x)}{R}$

Motivation

Use of SPDIs for approximating nonlinear differential equations

Example

Pendulum with friction coefficient k, mass M, pendulum length R and gravitational constant g. Behaviour: $\dot{x} = y$ and $\dot{y} = -\frac{ky}{MR^2} - \frac{g\sin(x)}{R}$

- Triangulation of the plane: Huge number of regions
- Need to reduce the complexity ... without too much overhead
  - Relax Goodness: GSPDI

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Use of SPDIs for approximating nonlinear differential equations