Towards Computing Phase Portraits of Polygonal Differential Inclusions

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Overview of the presentation

- Motivation
- Poly onal Differential Inclusion System (SPDI)
- One cycle analysis
  - Viability and Controllability Kernels
  - Properties
- Global analysis
- Conclusions

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Motivation

- For Hybrid Systems
  - Verification (reachability, ...):
  - Qualitative behavior (Phase Portrait, ...)

- For a class of Non-deterministic systems (SPDI)
  - Verification (HSCC'01)
  - Phase Portrait (this work)

Difficulties

- In most cases: Undecidable
  - We consider planar systems

- Phase Portrait of Non-deterministic systems
  - Definition (?)
  - Objects (?)
    - What is a Limit Cycle?

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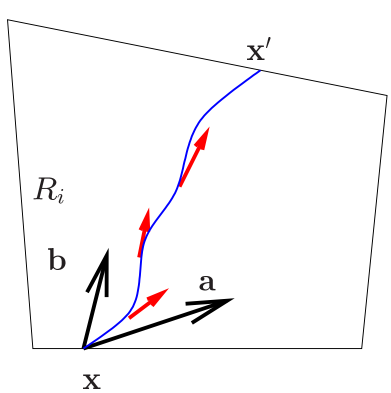
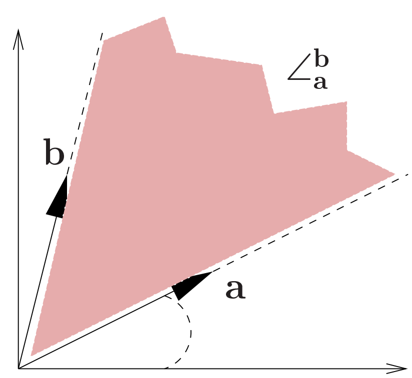
SPDI: Poly onal Differential Inclusion System

- SPDI:
  - A partition of the plane into convex poly onal re ions
  - A constant differential inclusion for each re ion

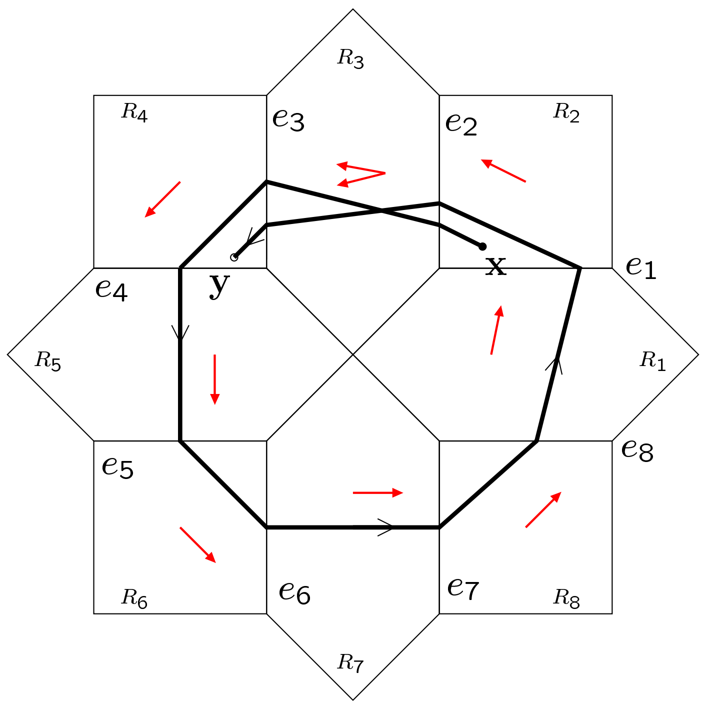
$$\dot{x} \in \angle_{\mathbf{a}_i}^{\mathbf{b}_i} \text{ if } \mathbf{x} \in R_i$$





SPDI: Poly onal Differential Inclusion System

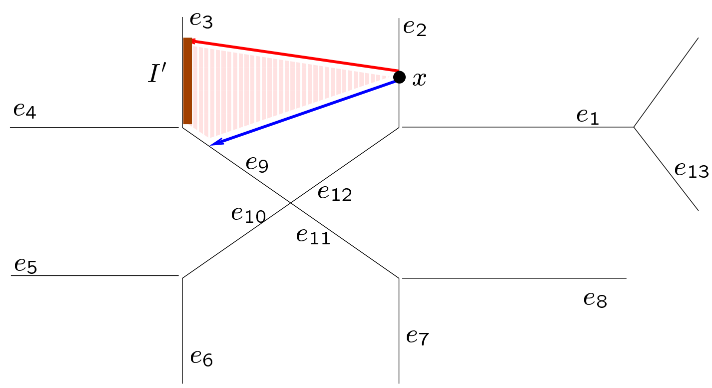
Example: An SPDI and a trajectory se ment



- A si nature σ is a sequence of traversed ed es ($\sigma = e_2, e_3, \cdots, e_8, e_1, e_2, e_3$)

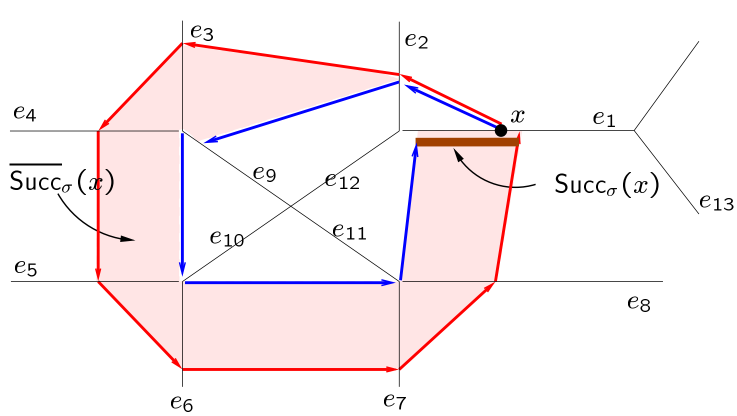
Computation of Successors (by σ)

- One step ($\sigma = e_2 e_3$)



$$I' = \mathsf{Succ}_{e_2 e_3}(x)$$

- Several steps ($\sigma = e_1 e_2 e_3 \cdots e_8 e_1$)



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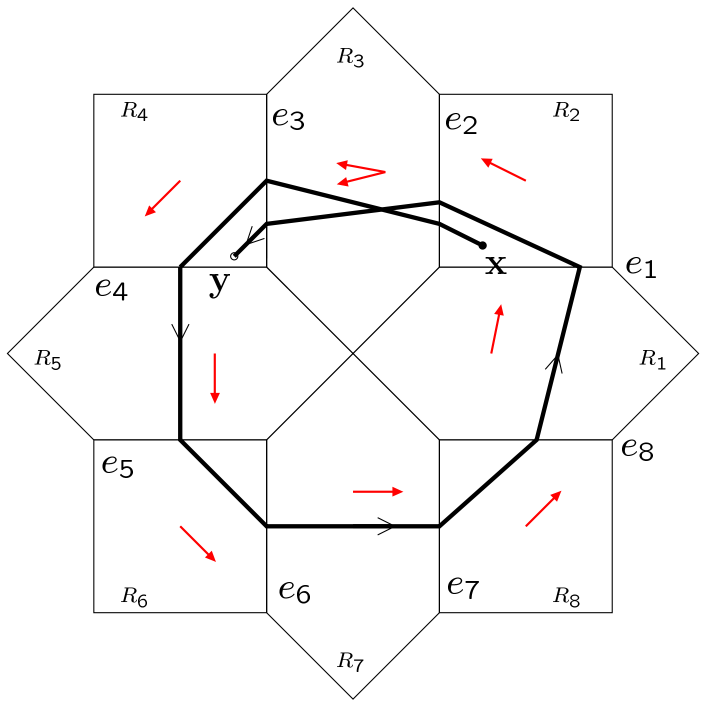
One cycle analysis

- Fix $\sigma = e_1 \cdots e_8 e_1$

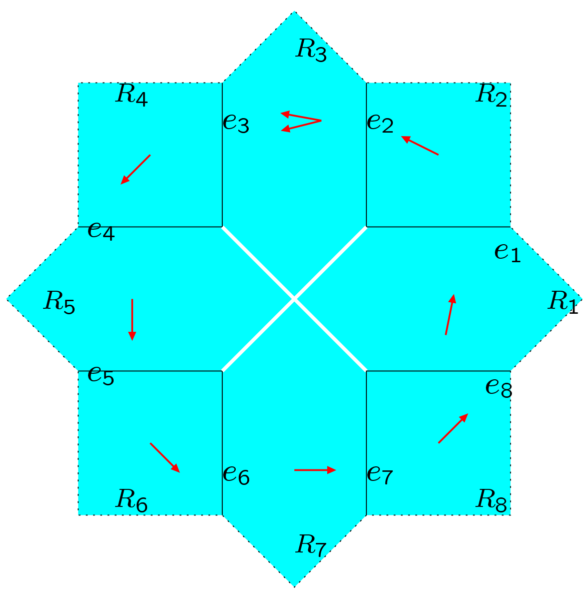


- Explore trajectories with si nature σ

One cycle analysis
Definin the set

- Given $\sigma = e_1 \cdots e_8 e_1$, take

$$K_\sigma = \bigcup_{i=1}^{k} (int(P_i) \cup e_i)$$



- Cyclin in σ $\Longrightarrow$ remainin in K_σ

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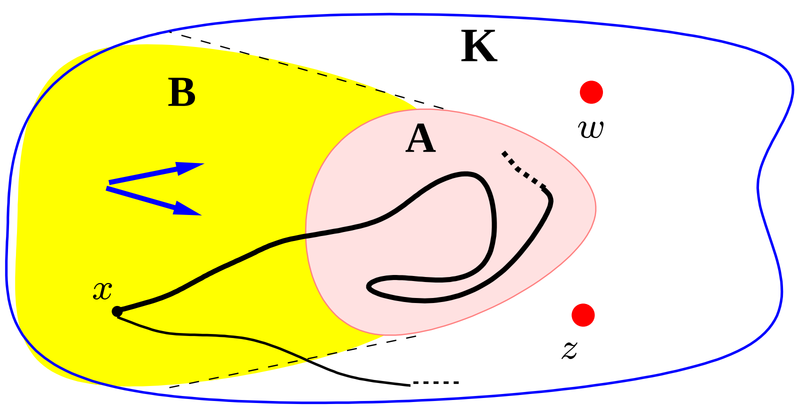
Viability Kernel

- M is a *viability domain* if $\forall \mathbf{x} \in M$, $\exists$ at least one trajectory ξ, startin in $\mathbf{x}$ and remain-in in M

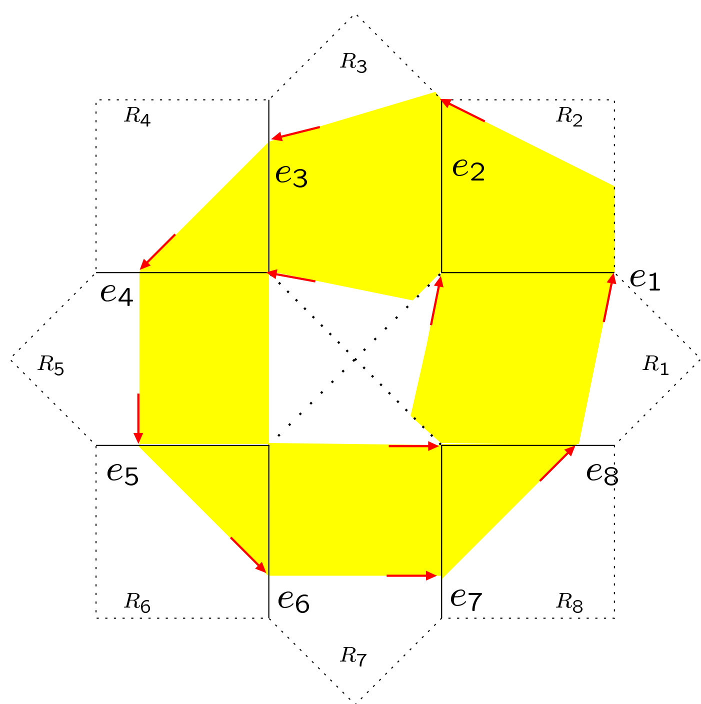
- Viab(K): *Viability kernel* of K is the lar est viability domain M contained in K



$$\text{Viab}(K) = \quad \cup B$$

One Cycle
Viability Kernel (Computation)

- We can easily compute the Viability Kernel for one cycle, which is a poly on
- **Theorem:** $\text{Viab}(K_\sigma) = \overline{\text{Pre}}_\sigma(\text{Dom}(\text{Succ}_\sigma))$



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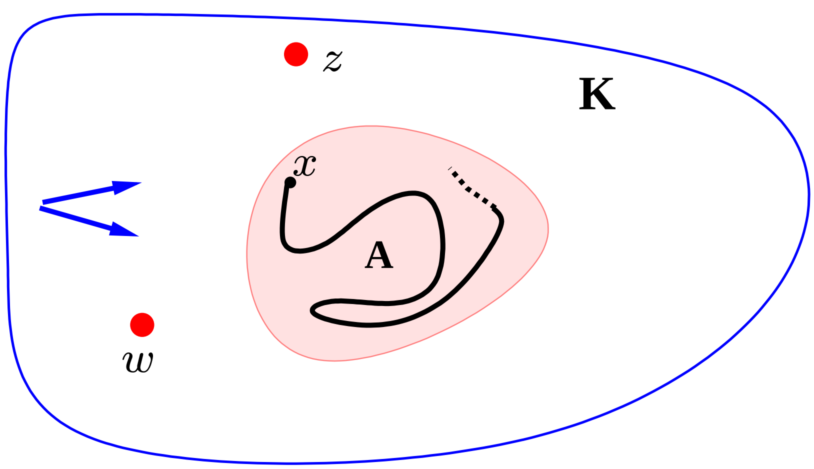
Controllability Kernel

- M is *controllable* if $\forall \mathbf{x}, \mathbf{y} \in M$, $\exists$ a trajectory se ment ξ startin   in $\mathbf{x}$ that reaches an arbitrarily small nei hborhood of $\mathbf{y}$ without leavin M

- *Controllability kernel* of K, denoted Cntr(K), is the lar est controllable subset of K

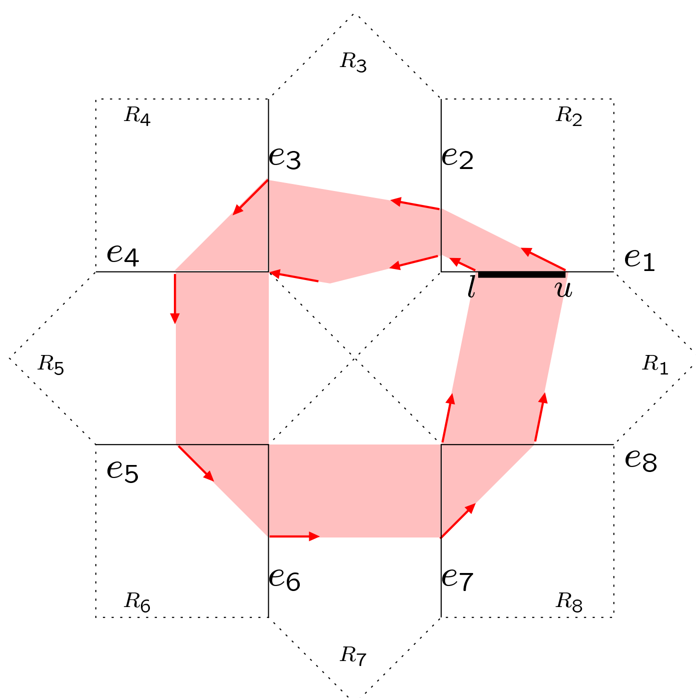


$$\text{Cntr}(K) =$$

One Cycle
Controllability Kernel (Computation)

- **Theorem:** $\mathsf{Cntr}(K_\sigma) = (\overline{\mathsf{Succ}}_\sigma \cap \overline{\mathsf{Pre}}_\sigma)(\mathcal{C}_\mathcal{D}(\sigma))$

  (We know how to compute the special interval $\mathcal{C}_\mathcal{D}(\sigma) = [l, u]$)



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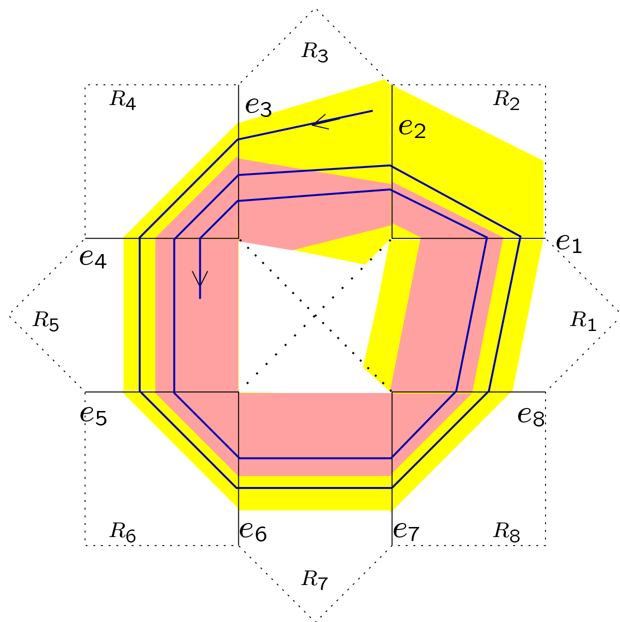
Properties

- **Theorem:** Any viable trajectory in K_σ conver es to Cntr(K_σ)



- Controllability Kernel: "Weak" analo of limit cycle

- Viability Kernel: Its "local" attraction basin

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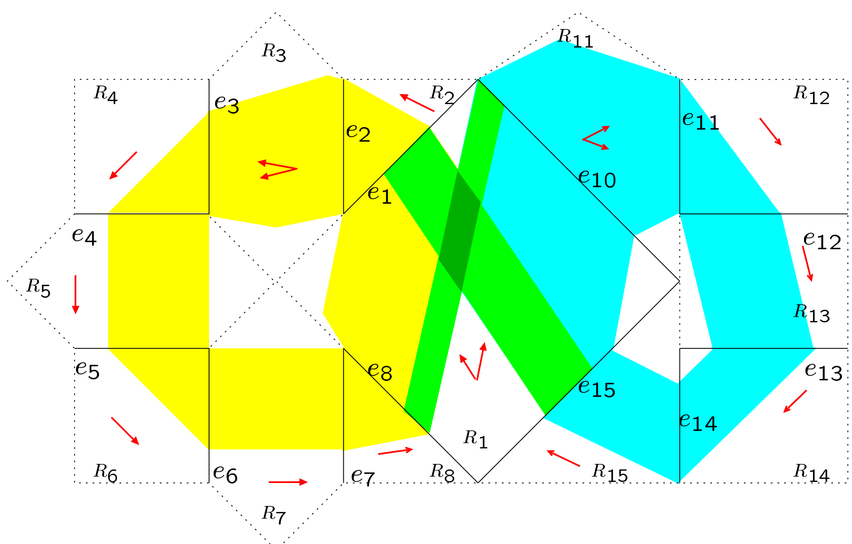

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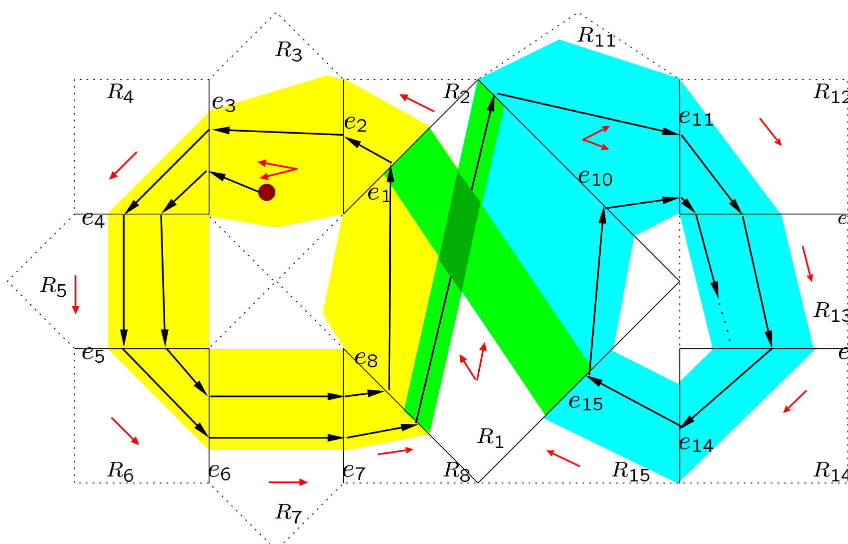

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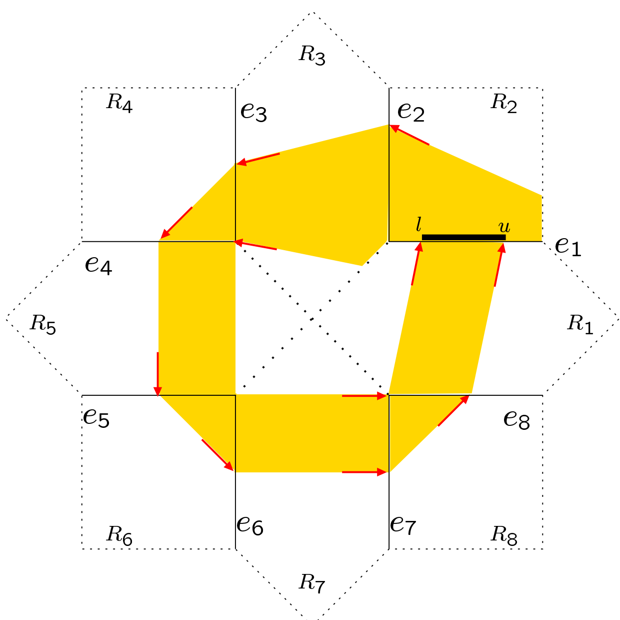

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